PROLONGATIONS OF CONNECTIONS AND SPRAYS WITH RESPECT TO WEIL FUNCTORS

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Recently, the concepts of Weil algebras and Weil functors have become actual for several reasons. One of them is that any product preserving functor with values and domain in manifolds coincides with a Weil functor on connected manifolds, which has been proved independently by [1], [2] , [6]. Moreover, there is a natural equivalence between the category of Weil functors and the category of Weil algebras, so that natural transformations between product preserving functors are completely determined by corresponding homomorphisms of Weil algebras. The present paper deals with prolongation of some geometrical objects with respect to Weil functors and [7] can be considered as our starting point. In particular, generalized or linear connections on fibred manifolds or vector bundles are prolonged canonically into generalized or linear connections, respectively, and sprays are prolonged into sprays. Moreover, the geodetic spray of a linear connection is prolonged to the geodetic spray of the prolonged connection. All considerations are in the category C^{∞}.

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1. PRELIMINARIES

In the sequel, R will denote real numbers only. Let $\mathcal{M}$ be the category of smooth manifolds and mappings and let $\mathcal{FM}$ be the category of fibred manifolds. A covariant functor $F\colon \mathcal{M} \to \mathcal{FM}$ is called a prolongation functor if the following two conditions hold: $B\circ F = \mathrm{id}_{\mathcal{M}}$, where $B\colon \mathcal{FM} \to \mathcal{M}$ is the base functor, and having an open submanifold $i\colon U \to M$, the map Fi is an embedding onto $\pi^{-1}(U)$,

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where $\pi : FM \to M$ is the image of M. A Weil algebra A is a real, finite dimensional, commutative, associative, unitary algebra of the form $A = R \oplus N$, where N is the nilradical of A. Any Weil algebra A gives rise to a prolongation functor which will also be denoted by A: $AM = \mathrm{Hom}(C^\infty M, A)$, $M \in \mathrm{Ob}\mathcal{M}$, and its value on morphisms is given by composition, see [9], [7]. In the special case of the tangent functor T corresponding to the algebra D of dual numbers we shall keep the traditional notation. The natural transformation of T into $\mathrm{id}_{\mathcal{M}}$ (defining the fibre structure) will be denoted by π. For any Weil functor A there is the following identification. Having a vector space V, any homomorphism $\varphi \in \mathrm{Hom}(C^\infty V, A)$ is determined by its values on V^*. On the other hand, any n-tuple of values $\varphi(v^i) = \alpha^{ip} a_p \in A$ on a base of V^* determines a homomorphism φ, so that $\mathrm{Hom}(C^\infty V, A) \cong V \otimes A$. For more details see [9]. Using this identification we obtaine easily the following lemma by direct computations.

Lemma 1.

(a) If V, W are vector spaces and $\psi \in \mathrm{Hom}(V,W)$, then $A\psi : V\otimes A \to W \otimes A$ is of the form $A\psi = \psi \otimes \mathrm{id}_A$.

(b) Let $i: B \to A$ be a homomorphism of Weil algebras and let C be a Weil functor. The corresponding natural transformation i of Weil functors satisfies $i_{R^n} = \mathrm{id}_{R^n} \otimes i$, $i_{CR^n} = \mathrm{id}_{R^n} \otimes \mathrm{id}_C \otimes i$.

(c) Let $Ci: C\circ B \to C \circ A$ be the natural transformation defined by applying a Weil functor C on all morphisms of a natural transformation $i: B \to A$ of Weil functors. Then the corresponding homomorphism of algebras is $i \otimes \mathrm{id}_C : B \otimes C \to A \otimes C$.

We shall also use another expression of Weil functors introduced by A. Morimoto. Let us consider a Weil algebra A. This can be obtained as a quotient algebra of the algebra $E(k)$ of germs of smooth functions on R^k at 0 by an ideal $\mathfrak{a}$ of finite codimension for some integer k. Two germs at zero of mapps $f, g \in C^\infty(R^k, M)$ are said to be A-equivalent if for any $\varphi \in C^\infty M$ $(\varphi\circ f - \varphi\circ g) \in \mathfrak{a}$. The classes of this equivalence are called A-velocities on the manifold M and A-velocity with a representative f will be denoted by $j^A f$. This gives rise to a manifold $T^A M$ of all A-velocities on M and to a map $T^A h: T^A M \to T^A N$, $T^A h(j^A f) = j^A(h \circ f)$ for any map $h: M \to N$. One can show that for any Weil algebra the functor T^A is naturally equivalent to the Weil functor A. ($j^A f(\varphi) = (\varphi\circ f \bmod \mathfrak{a}) \in A$, $\varphi \in C^\infty M$) In the sequel T^A will also be denoted by A.

2. T-NATURAL TRANSFORMATIONS OF WEIL FUNCTORS

Consider an arbitrary prolongation functor $F\colon \mathcal{M} \to \mathcal{FM}$. Having a vector field $\xi\colon M \to TM$, we obtain $F\xi\colon FM \to FTM$. On the other hand, we can prolong the flow of ξ to obtain a flow on FM, which defines a vector field $\underline{F}\xi\colon FM \to TFM$. In other words, $\mathrm{expt}(\underline{F}\xi) = F(\mathrm{expt}\,\xi)$. The following general definition is due to Kolář, [3].

Definition 1. A natural transformation $i\colon FT \to TF$ is called T-natural if the following diagram commutes for all manifolds M and vector fields ξ

$$\begin{tikzcd} TFM \arrow[r, "\pi_{FM}"] & FM \\ FM \arrow[u, "\underline{F}\xi"] \arrow[r, "F\xi"'] & FTM \arrow[u, "F\pi_M"'] \arrow[ul, "i_M"'] \end{tikzcd}$$

The aim of this section is to show that the canonical exchange homomorphism $i\colon D \otimes A \to A \otimes D$ determines a T-natural equivalence. We remark that this assertion is stated without proof in [7]. We shall use the following identifications : $A \cong T^A R$, $D \cong J^1_0(R,R)$, $T^A(J^1_0(R,R)) \cong J^1_0(R,R) \otimes T^A R$, $T(T^A R) \cong T^A R \otimes J^1_0(R,R)$. Having a map $\overline{\varphi}\colon R^k \to J^1_0(R,M)$, there is a map $\varphi\colon R^k \times R \to M$ satisfying $j^1_0(\varphi(x,-)) = \overline{\varphi}(x)$. Hence any element $j^A\overline{\varphi} \in T^A TM$ is of the form $j^A(j^1_0(\varphi(x,-)))$ and we can define $i_M\colon T^A TM \to TT^A M$, $i_M(j^A(j^1_0(\varphi(x,-)))) = j^1_0(j^A(\varphi(-,t)))$. Obviously, the map i_M form a natural equivalence. $T^A(J^1_0(R,R))$, considered as a quotient algebra of functions, is generated by elements with representatives $g = f.c\colon R^{k+1} \to R$ where $f\colon R^k \to R$, $c\colon R \to R$, but under the above identification this are the elements $j^1_0 c \otimes j^A f \in T^A TR$ and it follows that i_R is the canonical exchange homomorphism.

To prove the T-naturality of i, consider a vector field ξ on M and its flow $\varphi(t,x)$. We have $\xi(x) = j^1_0(\varphi(-,x))$, $T^A\xi(j^A g) = j^A(\xi \cdot g) = j^A(j^1_0(\varphi(-,g(x))))$. On the other hand, $T^A(\varphi(t,-))(j^A g) = j^A(\varphi(t,-)\cdot g)$, which implies $\underline{T}^A\xi(j^A g) = j^1_0(j^A(\varphi(t,-)\cdot g))$. Hence $i_M \circ T^A\xi = \underline{T}^A\xi$. The commutativity of the upper triangle in Definition 1 is obvious, so that we have proved

Proposition 1. For any Weil functor A, the natural transformation $i\colon A\circ T \to T\circ A$ determined by the canonical exchange homomorphism of

$D \otimes A$ is a T-natural equivalence.

Remark 1. According to a recent result by Kolář (private communication), a prolongation functor F admits a T-natural equivalence if and only if F is product preserving, which implies by [1], [2], [6] that F is a Weil functor. Since the transformation i from Proposition 1 is essential for all following considerations, this fact shows that our way of prolongation of some geometrical objects is applicable to Weil functors only.

The next lemma shows that our T-natural equivalence behaves well with respect to the linear structure on tangent bundles.

Having a vector bundle E, the multiplication by a scalar $\alpha \in R$ or the addition $E \oplus E \to E$ will be denoted by α_E or σ_E respectively.

Lemma 2. For any manifold M the following diagrams commute.

$$\begin{tikzcd} ATM \oplus ATM \arrow[r, "A\sigma_{TM}"] \arrow[d, "i_M \oplus i_M"] & ATM \arrow[d, "i_M"] \\ TAM \oplus TAM \arrow[r, "\sigma_{TAM}"] & TAM \end{tikzcd} \qquad \begin{tikzcd} ATM \arrow[r, "A\alpha_{TM}"] \arrow[d, "i_M"] & ATM \arrow[d, "i_M"] \\ TAM \arrow[r, "\alpha_{TAM}"] & TAM \end{tikzcd}$$

Proof. We may restrict ourselves to $M = R^n$ and in this case the commutativity is easily computed directly by Lemma 1.

3. APPLICATIONS TO SPRAYS

In the special case T = A, the T-natural equivalence from Proposition 1 is the canonical involution $j: TT \to TT$. Let us recall the definition of a spray, [4].

Definition 2. A spray on a manifold M is a vector field $\varsigma : TM \to TTM$ satisfying

(i) $\qquad \pi_M \circ \exp t\varsigma \circ \alpha_{TM} = \pi_M \circ \exp(\alpha t)\varsigma$

(ii) $\qquad j_M \circ \varsigma = \varsigma .$

Consider a Weil functor A.

Proposition 2. For any spray ς on M the mapping $\varsigma^A = Ti_M \circ A\varsigma \circ i_M^{-1}$ is a spray on AM.

Proof. According to (i) we have

$A\pi_M \circ i_M^{-1} i_M \circ \exp t \underline{A}\varsigma \circ i_M^{-1} i_M \circ A\alpha_{TM} \circ i_M^{-1} = A\pi_M \circ i_M^{-1} i_M \circ \exp(\alpha t)\underline{A}\varsigma \circ i_M^{-1}$

Since we have $A\pi_M \circ i_M^{-1} = \pi_{AM}$ by T-naturality and

$i_M \circ A\alpha_{TM} \circ i_M^{-1} = \alpha_{TAM}$ by Lemma 2, the latter condition implies

$$\pi_{AM} \circ \exp t\, \varsigma^A \circ \alpha_{TAM} = \pi_{AM} \circ \exp(\alpha t)\, \varsigma^A .$$

The condition (ii) for ς yields $Aj_M \circ A\varsigma = A\varsigma$. Hence

$$\varsigma^A = Ti_M \circ i_{TM} \circ A\varsigma \circ i_M^{-1} = Ti_M \circ i_{TM} \circ Aj_M \circ A\varsigma \circ i_M^{-1} .$$

By local considerations using Lemma 1 we easily obtain

$$Ti_M \circ i_{TM} \circ Aj_M \circ i_{TM}^{-1} \circ (Ti_M)^{-1} = j_{AM}$$

which completes the proof.

4. PROLONGATIONS OF GENERALIZED CONNECTIONS

We shall deal with generalized connections introduced by P. Libermann, [5], in the form of the lifting mappings.

Definition 3. A generalized connection on a fibred manifold $p: Y \to X$ is a mapping $\Gamma: TX \oplus Y \to TY$ satisfying

(i) $(Tp \oplus \pi_Y) \circ \Gamma = id_{TX \oplus Y}$

(ii) $\Gamma(-,y)$ is linear for all $y \in Y$.

Consider a Weil functor A and a fibred manifold $p: Y \to X$. Since A preserves products, $Ap: AY \to AX$ also is a fibred manifold. For the same reason the morphisms of fibred manifolds are transformed into morphisms of fibred manifolds. Local considerations show, that the fibred products of manifolds and mappings are also preserved.

Let Γ be a generalized connection on a fibred manifold $p: Y \to X$. Using the T-natural equivalence i, we can construct the composed map

$$TAX \oplus AY \xrightarrow{i_X^{-1} \oplus id_{AY}} ATX \oplus AY \xrightarrow{A\Gamma} ATY \xrightarrow{i_Y} TAY .$$

Proposition 3. The map $\underline{A}\Gamma = i_Y \circ A\Gamma \circ (i_X^{-1} \oplus id_{AY})$ is a generalized connection on the fibred manifold $Ap: AY \to AX$.

Proof. Since Γ is a generalized connection, we have

$$(ATp \oplus A\pi_Y) \circ A\Gamma = id_{ATX \oplus AY} .$$

To prove $(TA_p \oplus \pi_{AY})\circ \underline{A}\Gamma = id_{TAX \oplus AY}$, we need

$i_X \circ ATp \circ i_Y^{-1} = TAp$, $A\pi_Y \circ i_Y^{-1} = \pi_{AY}$, but this is obvious, since i is a T-natural equivalence.

The linearity condition (ii) is computed directly. Let η_1, $\eta_2 \in TAX$, $y \in AY$, $\pi_{AX}(\eta_1) = \pi_{AX}(\eta_2) = Ap(y)$. Choose g_1, g_2, h

in such a way that $i_X^{-1}(\eta_1) = j^A g_1$, $i_X^{-1}(\eta_2) = j^A g_2$, $y = j^A h$ and $\pi_X \circ g_1 = \pi_X \circ g_2 = p\circ h$. Using Lemma 2 we obtain $(\alpha, \beta \in R)$

$i_X^{-1}(\alpha\eta_1 + \beta\eta_2) = A\,\sigma_{TX}(A\,\alpha_{TX}(j^A g_1), A\,\beta_{TX}(j^A g_2)) = j^A(\alpha g_1 + \beta g_2)$.

Then

$A\Gamma(j^A(\alpha g_1 + \beta g_2), j^A h) = j^A\Gamma(\alpha g_1 + \beta g_2, h) =$

$= j^A(\alpha\Gamma(g_1,h) + \beta\Gamma(g_2,h)) =$

$= A\sigma_{TY}(A\alpha_{TY}\circ A\Gamma(j^A g_1, j^A h), A\beta_{TY}\circ A\Gamma(j^A g_2, j^A h))$.

By Lemma 2

$\underline{A}\Gamma(\alpha\eta_1 + \beta\eta_2, y) = \alpha\underline{A}\Gamma(\eta_1, y) + \beta\underline{A}\Gamma(\eta_2, y)$. Q.E.D.

Let us consider a generalized connection Γ on a fibred manifold $p: Y \to X$ and a vector field ξ. This is lifted to a vector field $\Gamma\xi$ on Y defined by $\Gamma\xi(y) = \Gamma(\xi\circ p(y), y)$ and called the Γ-lift of ξ.

<u>Proposition 4.</u> Let A be a Weil functor, Γ a generalized connection on a fibred manifold $p: Y \to X$. For any vector field ξ on X it holds $\underline{A}\Gamma(\underline{A}\xi) = \underline{A}(\Gamma\xi)$.

<u>Proof.</u> We have $\Gamma\xi = \Gamma\circ(\xi\circ p \oplus id_Y)$, so that $A(\Gamma\xi) = A\Gamma\circ(A\xi\circ Ap \oplus id_{AY})$. On the other hand,

$\underline{A}\Gamma(\underline{A}\xi) = \underline{A}\Gamma\circ(\underline{A}\xi\circ Ap \oplus id_{AY}) = i_Y\circ A\Gamma\circ(i_X^{-1}\oplus id_Y)\circ(\underline{A}\xi\circ Ap \oplus id_Y) =$

$= i_Y\circ A\Gamma\circ(A\xi\circ Ap \oplus id_{AY}) = i_Y\circ A(\Gamma\xi)$. Q.E.D.

The covariant differentiation ∇^Γ defined by a generalized connection Γ on a fibred manifold $p: Y \to X$ can be expressed as

$$\nabla^\Gamma_\xi s = Ts\circ\xi - \Gamma\circ(\xi \oplus s): X \to VY,$$

where VY is the vertical tangent bundle of Y.

Lemma 3. The restriction of the map i_Y to AVY has its values in the vertical tangent bundle VAY.

Proof. The subbundle $VY \subset TY$ is characterized by $Tp|VY \equiv 0$. Consider an arbitrary $z = j^A j_0^1 g(-,x) \in AVY$. We may assume $p \circ g(-,x) = \text{const} = h(x)$ for all $x \in R^k$. Then we have

$$TAp \circ i_Y(z) = TAp(j_0^1 j^A g(t,-)) = j_0^1 j^A h \, . \qquad \text{Q.E.D.}$$

Proposition 5. Let $p\colon Y \to X$ be a fibred manifold, Γ a generalized connection on Y. The covariant differentiation determined by the generalized connection $\underline{A}$ satisfies

$$\nabla^{\underline{A}\Gamma}_{\underline{A}\varphi} As = i_Y \circ A(\nabla^{\Gamma}_{\varphi} s)$$

for all vector fields φ on X and all local sections s of Y.

Proof. We have

$$\nabla^{\underline{A}\Gamma}_{\underline{A}\varphi} As = TAs \circ \underline{A}\varphi \quad - \quad \underline{A}\Gamma \circ (\underline{A}\varphi \oplus As) =$$

$$= TAs \circ i_X \circ A\varphi \ - \ i_Y \circ A\Gamma \circ (i_X^{-1} \oplus id_{AY}) \circ (i_X \circ A\varphi \ \oplus \ As) \ .$$

Hence Lemma 2 implies

$$\nabla^{\underline{A}\Gamma}_{\underline{A}\varphi} As = \sigma_{TAY} \circ (i_Y \circ ATs \circ A\varphi \oplus (-1)_{TAY} \circ i_Y \circ A\Gamma \circ (A\varphi \oplus As)) =$$

$$= i_Y \circ A\sigma_{TY}(ATs \circ A\varphi \oplus A(-1)_{TY} \circ A\Gamma \circ (A\varphi \oplus As)) =$$

$$= i_Y \circ A(Ts \circ \varphi \ - \Gamma \circ (\varphi \oplus s)). \qquad \text{Q.E.D.}$$

An interesting question is, whether a generalized connection on $Ap\colon AY \to AX$ is determined by its values on prolonged vector fields and local sections. An answer is given by the following considerations.

Lemma 4. Let $p\colon Y \to X$ be a fibred manifold, $\dim X \geq k$ and T_k^r be the functor of r-th order k-velocities. There is a dense subset $U \subset T_k^r X$ such that for any $j_0^r f \in U$ the fibre of $T_k^r Y$ over $j_0^r f$ is of the form

$$T_k^r Y_{j_0^r f} = \{ j_0^r(s \circ f);\ s \text{ is a local section of } Y \}.$$

Proof. We may restrict ourselves to the case $X = R^n$. Let $n \geq k$. There is a dense set $U \subset J_0^r(R^k, R^n)$ each element of which has a left inverse. Consider an arbitrary element $j_0^r f \in U$ and let $j_0^r \bar{f} \circ j_0^r f = j_0^r id_{R^k}$. Choose an arbitrary $j_0^r g \in T_k^r Y$ over $j_0^r f$.

Using local coordinates on a neighbourhood of $g(0)$, we have $j_0^r g = (j_0^r f, j_0^r \bar{g})$, where $\bar{g}\colon R^k \to R^m$ and m is the dimension of the fibres of Y. Then we set $s = (\mathrm{id}_{R^n}, \bar{g}\cdot\bar{f})$, which is the coordinate expression of a local section of Y. Moreover

$j_0^r(s\circ f) = (j_0^r f, j_0^r(\bar{g}\cdot\bar{f}\circ f)) = j_0^r g.$ Q.E.D.

<u>Lemma 5.</u> Let $A = E(k)/\mathfrak{a}$ and let $p\colon Y \to X$ be a fibred manifold. If $\dim X \geqslant k$, then $AY_u = \{As(u);\ s \text{ is a local section of } Y\}$ for a dense set U of elements u of the base AX.

<u>Proof.</u> Any Weil algebra is a quotient of some $J_0^r(R^k, R)$. Hence there is a surjective natural transformation $j\colon T_k^r \to A$ for some integers r and k. First of all we show that the restriction of j_Y to a fibre over $v \in T_k^r X$ is a map onto the fibre AY_u over $u = j_X(v) \in AX$. Consider an arbitrary homomorphism $u_1 \in \mathrm{Hom}(C^\infty Y, A)$ over u. This homomorphism depends only on r-jets of functions in a point $y \in Y$. Using local coordinates, we have $y \in (R^n \times R^m)$, $u_1 = (u, \bar{u}) \in AR^n \times AR^m$. Since j is surjective, there is $v_1 = (v, \bar{v}) \in T_k^r R^n \times T_k^r R^m$ satisfying $j_{R^m}(\bar{v}) = \bar{u}$, i.e.

$j_{R^{m+n}}(v_1) = u_1$. Hence we have proved $j_Y((T_k^r Y)_v = AY_{j_X(v)}$.

Further, it is clear that a surjection transforms dense sets into dense sets. Let $V \subset T_k^r X$ be the dense set from Lemma 4. We set $U = j_X(V)$ and we have

$$AY_u = \{j_Y \circ T_k^r s(v);\ s \text{ is a local section of } Y\} =$$
$$= \{As(u);\ s \text{ is a local section of } Y\}$$

for any $u = j_X(v) \in U$. Q.E.D.

<u>Proposition 6.</u> Let $A = E(k)/\mathfrak{a}$ be a Weil algebra and let $p\colon Y \to X$ be a fibred manifold. If $\dim X \geqslant k$, then any generalized connection Γ on AY is determined by its values on prolonged vector fields and local sections.

<u>Proof.</u> This is a direct consequence of Lemma 5.

<u>Remark 2.</u> Consider $A = T_2^1$ and take $\pi_R\colon TR \to R$ for $p\colon Y \to X$. We have $T_2^1\varphi(x, \eta, \Theta) = (x, \eta, \Theta, \varphi, d\varphi/dx\cdot\eta, d\varphi/dx\cdot\Theta)$, so that the assumption $\dim X \geqslant k$ in Proposition 6 is essential.

Remark 3. Let $A = E(k)/\mathfrak{a}$ be a Weil algebra. The equality in Proposition 5 can be used for an equivalent definition of a prolongation of covariant differentiation, if the dimension of the base is greater then k.

Remark 4. Another approach to prolongations of connections was introduced by Z. Pogoda, [8]. He prolongs connections on principal fibre bundles using the canonical form of a principal connection.

5. THE LINEAR CASE

Consider a vector bundle $p\colon E \to X$. There are operations $A\sigma_E$, $A\alpha_E$ on the fibred bundle $Ap\colon AE \to AX$. Since the properties of vector bundles can be expressed by commutative diagrams, $Ap\colon AE \to AX$ is a vector bundle with operations $\sigma_{AE} = A\sigma_E$, $\alpha_{AE} = A\alpha_E$, so that $\alpha_{AE}(j^A g) = j^A(\alpha_E \circ g)$. If $(j^A g_1, j^A g_2) \in AE \oplus AE$, then we may assume $p\circ g_1 = p\circ g_2$ and then $g = (g_1, g_2) \in C^\infty(R^k, E\oplus E)$. In this way we identify $AE \oplus AE = A(E \oplus E)$ and we have $\sigma_{AE}(j^A g_1, j^A g_2) = j^A(\sigma_E \circ g)$. In particular, for $A = T$ we obtain the well known linear structure on $Tp\colon TE \to TX$. The functoriality also implies that the morphisms of vector bundles are transformed into morphisms of vector bundles.

Let us recall the well known concept of a linear connection, which is defined as a linear section $\Gamma\colon E \to J^1E$. One can easily see, that in our setting this is equivalent to the linearity of a generalized connection $\Gamma\colon TX \oplus E \to TE$ on E with respect to the linear structure on $Tp\colon TE \to TX$. In other words, the Γ-lift of any vector field on X is linear.

Proposition 7. If Γ is a linear connection on $p\colon E \to X$, then $\underline{A}\Gamma$ is a linear connection on $Ap\colon AE \to AX$.

Proof. Consider any elements $\eta \in TAX$, $y_1, y_2 \in AE$, $Ap(y_1) = Ap(y_2) = \pi_{AX}(\eta)$. Let $y_1 = j^A f_1$, $y_2 = j^A f_2$, $i_X^{-1}\eta = j^A\gamma$, $p\circ f_1 = p\circ f_2 = \pi_X \circ \gamma$ and let $\alpha, \beta \in R$. We can find $\underline{A}\Gamma(\eta, \alpha y_1 + \beta y_2)$ by the following computation.

$$(\eta, \alpha y_1 + \beta y_2) = (\eta, j^A(\alpha f_1 + \beta f_2)) \xmapsto{i_X^{-1} \oplus \mathrm{id}_E}$$

$$\longmapsto (j^A\gamma, j^A(\alpha f_1 + \beta f_2)) \xmapsto{A\Gamma} j^A(\Gamma(\gamma, \alpha f_1 + \beta f_2)) =$$

$$= j^A(T\sigma_E(T\alpha_E \circ \Gamma(\gamma, f_1), T\beta_E \circ \Gamma(\gamma, f_2)) \longmapsto$$

$$\xmapsto{i_E} i_E \circ AT\sigma_E(AT\alpha_E \circ A\Gamma(j^A\gamma, j^A f_1), AT\beta_E \circ A\Gamma(j^A\gamma, j^A f_2) =$$

$$= TA\sigma_E \circ i_{E\oplus E}(AT\alpha_E \circ A\Gamma(j^A\gamma, j^Af_1), AT\beta_E \circ A\Gamma(j^A\gamma, j^Af_2) =$$

$$= T\sigma_{AE}(T\alpha_{AE} \circ \underline{A}\Gamma(\eta, y_1), T\beta_{AE} \circ \underline{A}\Gamma(\eta, y_2)) \qquad \text{Q.E.D.}$$

Having a vector bundle $p: E \to X$, there is a canonical identification of any V_yE with $E_{p(y)}$, so that there is a canonical morphism of vector bundles $\varkappa_E: VE \to E$. Let us consider a covariant differentiation ∇ on $p: E \to X$. We define $\tilde{\nabla}$ by

$$\tilde{\nabla}_\xi s = \varkappa_E \circ \nabla_\xi s: X \to E.$$

We remark that for a linear connection Γ, $\tilde{\nabla}^\Gamma$ is the usual covariant differentiation determined by Γ.

Lemma 6. For any vector bundle $p: E \to X$ it holds

$$\varkappa_{AE} \circ i_E = A\varkappa_E.$$

Proof. We may restrict ourselves to $E = R^n \times R^m$. In this case $\varkappa_E: R^n \times R^m \times \{0\} \times R^m \to R^n \times R^m$ is the projection to the first and the last factor. Hence

$A\varkappa_E: (R^n \otimes A) \times (R^m \otimes A) \times \{0\} \times (R^m \otimes A) \longrightarrow (R^n \otimes A) \times (R^m \otimes A)$

is also such a projection. On the other hand, we can similarly locally write $VAE = (R^n \otimes A) \times (R^m \otimes A) \times \{0\} \times (R^m \otimes A)$, where $\varkappa_{AE}$ is also the above projection, and the corresponding coordinate expression of i_E is the identity. Q.E.D.

Using this lemma and Proposition 5 we obtain

Proposition 8. For any linear connection Γ on a vector bundle $p: E \to X$, any vector field ξ on X and any section s of E we have $\tilde{\nabla}^{\underline{A}\Gamma}_{\underline{A}\xi} As = A(\tilde{\nabla}^\Gamma_\xi s)$.

6. APPLICATIONS TO THE CLASSICAL LINEAR CONNECTIONS

A linear connection $\Gamma: TM \oplus TM \to TTM$ on the tangent bundle TM of a manifold M is called a linear connection on M. By Proposition 7, there is a linear connection $\underline{A}\Gamma: TAM \oplus ATM \to TATM$ on $A\pi_M: ATM \to AM$. Using the T-natural equivalence i, we can construct the map

$$\Gamma^A = Ti_M \circ \underline{A}\Gamma \circ (id_{TM} \oplus i_M^{-1}) : TAM \oplus TAM \to TTAM.$$

Proposition 9. The map Γ^A is a linear connection on AM for any linear connection Γ on M.

Proof. Since Ti_M is a linear mapping, the linearity condition (ii) of Definition 3 holds. The condition (i) is easily verified by local computations by Lemma 1. The linearity of the generalized connection Γ^A follows from the definition of Γ^A, Lemma 2 and Proposition 7. Q.E.D.

Any linear connection Γ on a manifold M determines the geodetic spray ξ_Γ on M by the composition

$$\xi_\Gamma : TM \xrightarrow{diag} TM \oplus TM \xrightarrow{\Gamma} TTM.$$

The following proposition is obtained directly by comparing the construction of Γ^A with the construction of the prolonged sprays.

Proposition 10. Let A be a Weil functor and let Γ be a linear connection on a manifold M. The geodetic spray of the linear connection Γ^A on the manifold AM coincides with the prolongation of the geodetic spray of the connection Γ with respect to A, i.e. $(\xi_\Gamma)^A = \xi_{\Gamma^A}$.

Lemma 7. It holds

$$i_M \circ \varkappa_{ATM} = \varkappa_{TAM} \circ Ti_M .$$

Proof. This can be proved by direct computations in local coordinates similarly to the proof of Lemma 6.

Proposition 11. For any linear connection Γ on a manifold M and any vector fields ξ, η on M we have

$$\tilde{\nabla}^{\Gamma^A}_{\underline{A}\xi} \underline{A}\eta = \underline{A}(\tilde{\nabla}^{\Gamma}_{\xi} \eta).$$

Proof.

$$\tilde{\nabla}^{\Gamma^A}_{\underline{A}\xi} \underline{A}\eta = \varkappa_{TAM} \circ \nabla^{\Gamma^A}_{\underline{A}\xi} \underline{A}\eta = \varkappa_{TAM} \circ Ti_M \circ \nabla^{\underline{A}\Gamma}_{\underline{A}\xi} \underline{A}\eta =$$

$$= \varkappa_{TAM} \circ Ti_M \circ i_{TM} \circ A(\nabla^{\Gamma}_{\xi} \eta).$$

By Lemma 7 and Lemma 6

$$\tilde{\nabla}^{\Gamma^A}_{\underline{A}} \underline{A}\eta = i_M \circ \varkappa_{ATM} \circ i_{TM} \circ A(\nabla^{\Gamma}_{\xi}\eta) = i_M \circ A\varkappa_{TM} \circ A(\nabla^{\Gamma}_{\xi}\eta) =$$
$$= i_M \circ A(\tilde{\nabla}^{\Gamma}_{\xi}\eta) . \qquad \text{Q.E.D.}$$

Remark 5. We give the expression of the prolonged connection in local coordinates. Consider a linear connection Γ on R^n, i.e.

$$\Gamma : R^n \times R^n \times R^n \longrightarrow R^n \times R^n \times R^n \times R^n,$$

$$(x^i, y^i, z^i) \longmapsto (x^i, z^i, y^i, \Gamma^k_{ij} y^i z^i) , \quad \text{where } \Gamma^k_{ij} \in C^\infty R^n.$$

The multiplication $\mu : R \times R^n \longrightarrow R^n$ is prolonged into $A\mu : A \times AR^n \longrightarrow AR^n$ and defines an A-module structure on AR^n. The module structure defined in this way is studied in [7], similar considerations are also possible in our setting. Let us denote by $x^{i,\nu}$ the coordinates on AR^n defined by $Ax^i = x^{i,\nu} e_\nu$, where e_ν is a base of A. Then direct computations give:

$$\Gamma^A(x^{i,\nu}, y^{i,\nu}, z^{i,\nu}) = (x^{i,\nu}, z^{i,\nu}, y^{i,\nu}, A\Gamma^k_{ij} * (y^{i,\nu}) * (z^{i,\nu}))$$

where $*$ denotes the above mentioned module multiplication. Taken into account

$$\tilde{\nabla}^{\Gamma}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = - \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

one verifies that the Morimoto's prolonged connection [7] coincides with our one. This fact also follows directly from Proposition 6 and Proposition 11 if $\dim M \geqslant k$, provided A is a quotient of E(k).

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