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SMOOTH STRUCTURES ON FIBRE JET SPACES

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The concept of a fibre jet, introduced by Kolář [11], seems to be a useful tool for several geometrical considerations. The fibre jet spaces are, roughly speaking, infinite dimensional analogues of classical jet spaces, and it is reasonable to endow them with a smooth structure. We introduce such a structure in the present paper. To this purpose we use some recent results by Frölicher [4], [5], [6]. Since these results are not generally known, we first give a short survey of them. Some ideas and concepts are reformulated to a form suitable for our purposes. In the second section the smooth structures on fibre jet spaces are defined. The basic properties are studied in the subsequent two sections. The infinite dimensional analogues of frame bundles and associated spaces are introduced in Section 5. (We remark that some similar objects are studied by different methods in [3].) In the next section the fibre functors are defined. This concept is a generalization of lifting functors, [2], [7], [12], [13], [14], and prolongation functors, [7], [10]. Hence as a consequence of our considerations some known results are obtained by new methods, but we also deduce some new results. In the last section a classification of certain types of fibre functors is presented. — The author is grateful to Prof. I. Kolář for suggesting the problem and for many useful discussions and valuable remarks.

1. THE CATEGORY $\mathscr{S}$ OF SMOOTH SPACES

1.1. Basic concepts, [5]. Let X be a set, S a set of maps $f: X \to \boldsymbol{R}$ and T a set of maps $c: \boldsymbol{R} \to X$. We define $D_* S = \{c: \boldsymbol{R} \to X,\ f \circ c \in C^\infty(\boldsymbol{R}, \boldsymbol{R}) \text{ for all } f \in S\}$, $D^* T = \{f: X \to \boldsymbol{R},\ f \circ c \in C^\infty(\boldsymbol{R}, \boldsymbol{R}) \text{ for all } c \in T\}$. A *smooth structure* on a set X is a pair (C, F) where $C \subset X^{\boldsymbol{R}}$, $F \subset \boldsymbol{R}^X$ such that the "duality conditions" $D_* F = C$, $D^* C = F$ hold. A *smooth space* is a triple (X, C, F) where X is a set and (C, F) is a smooth structure on X. The smooth spaces form a category $\mathscr{S}$ in which the morphisms from (X, C_X, F_X) to (Y, C_Y, F_Y) are defined as those maps $\varphi: X \to Y$ that satisfy $\varphi_*(C_X) \subset C_Y$ or, equivalently, $\varphi^*(F_Y) \subset F_X$. Having a fixed set X, the structures on X are ordered: (C, F) is called *finer* then (C', F') if the identity map of X is a morphism from (X, C, F) to (X, C', F'). For any set $C_0 \subset X^{\boldsymbol{R}}$ there is a finest structure (C, F) on X such that $C_0 \subset C$. It is called the *structure generated by* C_0 and is obtained by setting $F = D^*C_0$, $C = D_*F$. Similarly, for any set $F_0 \subset \boldsymbol{R}^X$ there is a coarsest structure (C, F) on X with $F_0 \subset F$ constructed by $C = D_*F_0$, $F = D^*C$ and called the *structure generated by* F_0. The classical smooth structure on a finite dimensional manifold can be defined by smooth functions, hence according to Boman's theorem, [1] (see also [5], [6]), the classical concept of smooth maps between smooth manifolds coincides with this more general approach. The elements of C_X are said to be *curves* in X, the elements of F_X are *functions* on X. We shall often denote an object of $\mathscr{S}$ and its underlying set by the same symbol.

1.2. Projective and inductive structures, [4]. Let $Y \in \mathrm{Ob}\mathscr{S}$, $X_i \in \mathrm{Ob}\mathscr{S}$, $\varphi_i \in \mathrm{Mor}\mathscr{S}$, $\varphi_i\colon Y \to X_i$, $i \in I$. A structure on Y is called *projective with respect to* φ_i if the following equivalence holds: For any $Z \in \mathrm{Ob}\mathscr{S}$, a map $\psi\colon Z \to Y$ is an $\mathscr{S}$ – morphism iff $\varphi_i \circ \psi$ is an $\mathscr{S}$ – morphism for all $i \in I$. We have, [4],

Proposition 1. *Let Y be a set, $X_i \in \mathrm{Ob}\mathscr{S}$ and let $\varphi_i\colon Y \to X_i$ be maps, $i \in I$. There exists a unique structure on Y projective with respect to φ_i, $i \in I$.*

This unique structure is generated by the set of functions $\{f\colon Y \to \boldsymbol{R},\ f = g \circ \varphi_i,\ g \in F_{X_i},\ i \in I\}$, hence, by definition, $C_Y = \{c\colon \boldsymbol{R} \to Y,\ \varphi_i \circ c \in C_{X_i},\ i \in I\}$.

The inductive structure is a dual concept (from the categorial point of view): Let $Y \in \mathrm{Ob}\mathscr{S}$, $X_i \in \mathrm{Ob}\mathscr{S}$, $\varphi_i \in \mathrm{Mor}\mathscr{S}$, $\varphi_i\colon X \to Y$, $i \in I$. A structure on Y is called *inductive with respect* to φ_i if for any $Z \in \mathrm{Ob}\mathscr{S}$, a map $\psi\colon Y \to Z$ is an $\mathscr{S}$ – morphism iff $\psi \circ \varphi_i \in \mathrm{Mor}\mathscr{S}$ for all $i \in I$.

Proposition 2, [4]. *Let Y be a set, $X_i \in \mathrm{Ob}\mathscr{S}$ and let $\varphi_i\colon X_i \to Y$ be maps, $i \in I$. There exists a unique structure on Y inductive with respect to φ_i, $i \in I$.*

This structure is generated by the set of curves $\{c\colon \boldsymbol{R} \to Y,\ c = \varphi_i \circ d,\ d \in C_{X_i},\ i \in I\}$, so that $F_Y = \{f\colon Y \to \boldsymbol{R},\ f \circ \varphi_i \in F_{X_i},\ i \in I\}$. Having an inductive or projective structure on Y, we have in this way explicitly described the morphisms with the domain or codomain Y, respectively.

1.3. Some properties of $\mathscr{S}$, [4], [5], [6]. The category $\mathscr{S}$ is complete and cocomplete. Limits and colimits are constructed in the category of sets and then they are endowed with the projective or inductive structure with respect to the limit or colimit cone, respectively. The category $\mathscr{S}$ is cartesian closed. The functor $H\colon \mathscr{S} \times \mathscr{S}^{\mathrm{op}} \to \mathscr{S}$ yielding the cartesian closedness may be chosen in such a way that the underlying sets of exponential objects are exactly the sets of morphisms, so that we shall write simply $\mathscr{S}(X, Y)$ instead of $H(X, Y)$. The curves of the smooth structure of $\mathscr{S}(X, Y)$ are $C_{\mathscr{S}(X,Y)} = \{c\colon \boldsymbol{R} \to \mathscr{S}(X, Y),\ \tilde{c}\colon \boldsymbol{R} \times X \to Y$ is a morphism in $\mathscr{S}\}$ where $\tilde{c}$ is defined by $\tilde{c}(t, x) = c(t)(x)$. Hence a map $\psi\colon Z \to \mathscr{S}(X, Y)$ is a morphism in $\mathscr{S}$ iff $\tilde{\psi}\colon Z \times X \to Y$, $\tilde{\psi}(z, x) = \psi(z)(x)$ is an $\mathscr{S}$ – morphism.

1.4. Subobjects in $\mathscr{S}$. $i\colon X \to Y$ is called a *subobject* of Y, if i is a monomorphism in $\mathscr{S}$ and the structure on X is the projective structure with respect to i. The map i

is also said to be an *immersion of the subobject* X into the object Y. Having a morphism $\varphi: Z \to Y$ in $\mathcal{S}$, we define the *inverse image* of the subobject $i: X \to Y$ under φ by the following pullback

$$\begin{array}{ccc} \varphi^{-1}(X) & \xrightarrow{\bar{i}} & Z \\ {\scriptstyle \varphi|X}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow{i} & Y \end{array}$$

Obviously, $\bar{i}$ is a monomorphism. Having a map $c: \boldsymbol{R} \to \varphi^{-1}(X)$ satisfying $\bar{i} \circ c \in \in \operatorname{Mor}\mathcal{S}$, we also have $i \circ \varphi \mid X \circ c \in \operatorname{Mor}\mathcal{S}$. Hence $\varphi \mid X \circ c \in \operatorname{Mor}\mathcal{S}$ and it follows from the definition of the pullback that c also is an $\mathcal{S}$ – morphism, so that we have proved that $\bar{i}: \varphi^{-1}(X) \to Z$ is a subobject of Z. The intersection of subobjects (defined by pullback) is a subobject, too. According to Proposition 1, there is a subobject structure on every subset X of any $Y \in \operatorname{Ob}\mathcal{S}$. If X is a submanifold in the classical sense, then the classical smooth structure on X coincides with that just mentioned.

1.5. Quotient objects in $\mathcal{S}$. This is the dual concept to that of a subobject: $i: X \to Y$ is a *quotien. object* of X if i is an epimorphism in $\mathcal{S}$ and the structure on Y is the inductive structure with respect to i. All quotient objects can be constructed as follows: Let $X \in \operatorname{Ob}\mathcal{S}$ and let $\sim$ be an equivalence relation on X. We define the inductive structure with respect to the canonical projection $i: X \to X \mid \sim$ on the set $X \mid \sim$. Obviously, given a morphism $\varphi: X \to Z$ satisfying the condition that $i(x) = i(y)$ implies $\varphi(x) = \varphi(y)$ for all $x, y \in X$, there exists a unique morphism $\bar{\varphi}: X \mid \sim \to Z$ such that $\bar{\varphi} \circ i = \varphi$.

1.6. The topology on the objects in $\mathcal{S}$. Let (X, C, F) be an object in $\mathcal{S}$. We define the inductive topology on X with respect to C, [6]. That means, $Y \subset X$ is an open set iff $c^{-1}(Y)$ is an open set in $\boldsymbol{R}$ for all $c \in C$. A subject $i: Y \to X$ is called *open* if $i(Y)$ is an open set in X.

Lemma 1. *Let $i_\alpha: Y_\alpha \to X$, $\alpha \in I$ be open subobjects covering X (i.e. $\bigcup_\alpha i_\alpha(Y_\alpha) = X$), $f: X \to Z$ a map. If $f \circ i_\alpha \in \operatorname{Mor}\mathcal{S}$ for all $\alpha \in I$, then $f \in \operatorname{Mor}\mathcal{S}$.*

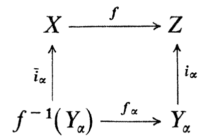
Proof. The smoothness of real functions is a local property. Hence Lemma 1 follows from the commutativity of the diagram

$$\begin{array}{ccccccc} \boldsymbol{R} & \xrightarrow{c} & X & \xrightarrow{f} & Z & \xrightarrow{g} & \boldsymbol{R} \\ \uparrow & & \uparrow{\scriptstyle i_\alpha} & \nearrow & & & \\ c^{-1}(Y_\alpha) & \xrightarrow{\bar{c}} & Y_\alpha & & & & \end{array}$$

in sets (the square is a pullback).

Let $f \in \operatorname{Mor}\mathcal{S}$, $f: X \to Z$ and let $i_\alpha: Y_\alpha \to Z$, $\alpha \in I$ be open subobjects covering Z. The subobjects $f^{-1}(Y_\alpha)$ of X are also open since we have $c^{-1}(f^{-1}(Y_\alpha)) = (f \circ c)^{-1}(Y_\alpha)$

in sets.

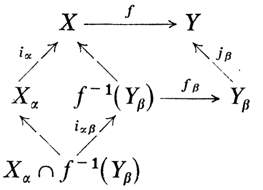


Obviously, the sets $\bar{i}_\alpha(f^{-1}(Y_\alpha))$ (see the pullback above) form an open covering of X. On the other hand, according to Lemma 1, we have

Lemma 2. *Let $f: X \to Z$ be a map and let $\bar{i}_\alpha: f^{-1}(Y_\alpha) \to X$ be open subobjects covering X. If all f_α are morphisms in $\mathscr{S}$, then $f \in \operatorname{Mor}\mathscr{S}$.*

A finite intersection of open subobjects is an open subobject. The above considerations may be expressed in the form of

Proposition 3. *Let $X, Y \in \operatorname{Ob}\mathscr{S}$ and let $f: X \to Y$ be a map. Then f is an $\mathscr{S}$ – morphism iff there exists a covering of X or Y by open subobjects $i_\alpha: X_\alpha \to X$, $\alpha \in I$ or $j_\beta: Y_\beta \to Y$, $\beta \in J$, respectively, such that the maps $f_\beta \circ i_{\alpha\beta}$ in the following diagram are morphisms in $\mathscr{S}$ for all $\alpha \in I$, $\beta \in J$.*



Remark 1. The morphisms in $\mathscr{S}$ are continuous with respect to the above defined topology. Consider $A, B \in \operatorname{Ob}\mathscr{S}$. A map $\varphi: A \to B$ is continuous iff for every curve c in A and every open set $B' \subset B$, $(\varphi \circ c)^{-1}(B')$ is open in A.

2. SMOOTH STRUCTURES ON SPACES OF FIBRE JETS

From now on, $Y \to X$ or $W \to Z$ will denote a fibred manifold with a standard fibre S or Q, respectively, $\dim X = m$, $\dim Z = n$. Any map $F: Y \to W$ is assumed to be a morphism in the category $\mathscr{FM}$ of fibred manifolds over a smooth map $f: X \to Z$. Let us recall the definition of a fibre r-jet, [11]. We say that two maps $F, G: Y \to W$ belong to the same fibre r-jet at $x \in X$ if $j^r_y F = j^r_y G$ for any $y \in Y_x$ (= the fiber over x). This will be expressed by $\boldsymbol{j}^r_x F = \boldsymbol{j}^r_x G$ and we shall denote by $\boldsymbol{J}^r(Y, W)$ the set of all fibre r-jets of local fibred manifold morphisms of Y into W. We first define a smooth structure on $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q)$. Then we show that a bijection between $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q)$ and a certain subset of $\boldsymbol{J}^r(Y, W)$ is determined by every local trivializations $\Phi: \boldsymbol{R}^m \times S \to Y$, $\Psi: \boldsymbol{R}^n \times Q \to W$, and we take the inductive structure with respect to all these bijections on $\boldsymbol{J}^r(Y, W)$.

An $\mathscr{FM}$ – morphism $F: \boldsymbol{R}^m \times S \to \boldsymbol{R}^n \times Q$ is determined by $f: \boldsymbol{R}^m \to \boldsymbol{R}^n$, $\tilde{F}: \boldsymbol{R}^m \times S \to Q$. Consider $G, F: \boldsymbol{R}^m \times S \to \boldsymbol{R}^n \times Q$ of the form $F = (f, \tilde{F})$, $G =$

$= (g, \tilde{G})$, $x \in \boldsymbol{R}^m$. Take an element $s \in S$ and a coordinate system (s^p) or (z^q) on a neighbourhood of s or $\tilde{F}(x, s)$, respectively. $\boldsymbol{j}_x^r F = \boldsymbol{j}_x^r G$ implies

$$j_x^r f = j_x^r g\,, \quad \frac{\partial^{|\alpha|} \tilde{F}^q}{\partial x^\alpha}\bigg|_{(x,s)} = \frac{\partial^{|\alpha|} \tilde{G}^q}{\partial x^\alpha}\bigg|_{(x,s)}\,, \quad 0 \leqq |\alpha| \leqq r\,,$$

where $\tilde{F}^q$ and $\tilde{G}^q$ are the coordinate expressions of $\tilde{F}$ and $\tilde{G}$. Conversely, having these equalities for every $s \in S$, we deduce $\boldsymbol{j}_x^r F = \boldsymbol{j}_x^r G$. Hence we have, [11],

Proposition 4. *For every $F, G: \boldsymbol{R}^m \times S \to \boldsymbol{R}^n \times Q$ the following conditions are equivalent*

(i) $\boldsymbol{j}_x^r F = \boldsymbol{j}_x^r G$;

(ii) $j_x^r F(-, s) = j_x^r G(-, s)$ *for every* $s \in S$;

(iii) $j_0^r(F \circ \tau_x \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a)) = j_0^r(G \circ \tau_x \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a))$ *for every* $j_0^r a \in T_m^r S$, *where* $\tau_x: \boldsymbol{R}^m \times S \to \boldsymbol{R}^m \times S$ *si the fibre translation* $(y, s) \mapsto (y + x, s)$.

We denote by const $T_m^r S$ the set of all r-jets of constant maps from $\boldsymbol{R}^m$ into S. Obviously const $T_m^r S \cong S$. We shall use the following notation. Given $F: \boldsymbol{R}^m \times S \to \boldsymbol{R}^n \times Q$, the following maps are determined:

(1) $\quad \bar{\varphi}_F: \boldsymbol{R}^m \times T_m^r S \to T_m^r(\boldsymbol{R}^n \times Q)\,, \quad (x, j_0^r a) \mapsto j_0^r(F \circ \tau_x \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a))\,,$

(2) $\quad \tilde{\varphi}_F: \boldsymbol{R}^m \times T_m^r S \to T_m^r Q\,, \quad (x, j_0^r a) \mapsto j_0^r(\tilde{F} \circ \tau_x \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a))\,,$

(3) $\quad \varphi_F = \tilde{\varphi}_F \mid \mathrm{const}\, T_m^r S: \boldsymbol{R}^m \times S \to T_m^r Q\,, \quad (x, s) \mapsto j_0^r(\tilde{F}(-, s) \circ t_x)\,.$

By Proposition 4, $\boldsymbol{j}_x^r F$ is determined by $\varphi_F(x, -)$ and $j_x^r f$. We shall show that every such pair determines a fibre r-jet of a certain map F, that is, there exists a bijection $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q) \cong J^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \times \mathscr{S}(S, T_m^r Q)$. In this way we can define a smooth structure on $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q)$ by requiring this bijection to be an isomorphism in $\mathscr{S}$. The first statement above is a consequence of the following proposition.

Proposition 5. *Let $\varphi: S \to J^r(M, N)$ be a smooth map. There exists a smooth map $\Phi: S \times M \to N$ satisfying $j_{\alpha \circ \varphi(s)}^r \Phi(s, -) = \varphi(s)$ for every $s \in S$, where α is the source projection.*

Proof. Consider some sprays θ and η on M and N. Given a map $f: M \to N$, a local map $\psi_f: TM \to TN$ is defined on a neighbourhood of the zero section as follows. Denoting by π_M, π_N the projections of TM, TN, we set $\psi_f = (\pi_N \times \exp_\eta)^{-1} \circ (f \times f) \circ (\pi_M \times \exp_\theta)$. Since $(\pi, \exp)$ is a diffeomorphism on a neighbourhood of the zero section, we have: $j_0^r(\psi_f \mid T_{\pi_M(0)} M) = j_0^r(\psi_g \mid T_{\pi_M(0)} M)$ iff $j_{\pi_M(0)}^r f = j_{\pi_M(0)}^r g$, for all $f, g: M \to N$. Owing to this fact we can construct a map ψ defined on S as follows: If $\varphi(s) = j_{\alpha \circ \varphi(s)}^r f_s$ then $\psi(s) = j_0^r(\psi_{f_s} \mid T_{\alpha \circ \varphi(s)} M) \in J^r(T_{\alpha \circ \varphi(s)} M, T_{\beta \circ \varphi(s)} N)$. That is why for every $s \in S$ there exists a unique polynomial map of degree r $\bar{\psi}_s$: $T_{\alpha \circ \varphi(s)} M \to T_{\beta \circ \varphi(s)} N$ satisfying $j_0^r \bar{\psi}_s = \psi(s)$. Hence we have constructed a map $\bar{\psi}: (S \xrightarrow{\alpha \circ \varphi} M) \oplus (TM \xrightarrow{\pi_M} M) \to TN$ which will be shown to be smooth.

Take an element $s_0 \in S$ and choose bases $(\partial/\partial x^i)(x)$ and $(\partial/\partial z^q)(z)$ of the tangent spaces on neighbourhoods U and V of $\alpha \circ \varphi(s)$ and $\beta \circ \varphi(s)$, respectively. Then the normal coordinates on a neighbourhood of every point from U and V are determined and we are also given an induced coordinate expression $(\varphi^q_\alpha(s))$ of the jet $\varphi(s)$ on a certain neighbourhood of $s_0 \in S$. The smoothness of the bases implies the smoothness of $\varphi^q_\alpha(s)$ and according to our construction of $\bar{\Psi}$ we have

$$\bar{\Psi}\left(s, \xi^i \frac{\partial}{\partial x^i}(\alpha \circ \varphi(s))\right) = \frac{1}{\alpha!} \varphi^q_\alpha(s)\, \xi^\alpha \frac{\partial}{\partial z^q}(\beta \circ \varphi(s)) . \tag{4}$$

Hence $\bar{\Psi}$ is smooth.

Let U, V be the domains of $\exp_\theta$, $\exp_\eta$. Given the Riemannian metrics on M, N, let us choose smooth positive functions $\mu: M \to \boldsymbol{R}$, $\nu: N \to \boldsymbol{R}$, $\varepsilon: S \times TM \to \boldsymbol{R}$, $\delta: S \times TN \to \boldsymbol{R}$ satisfying $\{v \in TM, |v| < \mu(\pi_M(v))\} \subset U$, $\{v \in TN, |v| < \nu(\pi_N(v))\} \subset V$, $\varepsilon(s, -) \equiv 1$ on a neighbourhood of the zero section, $\varepsilon(s, v) \equiv 0$ for $|v| > 1/2\mu(\pi_M(v))$, $\delta(s, -) \equiv 1$ on a neighbourhood of the zero section, $|v|\, \delta(s, v) < \nu(\pi_N(v))$. (This can be easily done by means of a partition of unity.) Now we are able to define a smooth map $\Phi: S \times M \to N$ by

$$\Phi(s, x) = \begin{cases} \exp_\eta \circ (\delta \,.\, \varepsilon \,.\, \bar{\Psi}) \circ (\mathrm{id}_S \times (\pi_M \times \exp_\theta)^{-1})\,(s, (\alpha \circ \varphi(s), x)) \\ \qquad\qquad \text{if} \quad x \in \exp_\theta(T_{\alpha \circ \varphi(s)} M)\,, \\ \beta \circ \varphi(s) \quad \text{if} \quad x \notin \exp_\theta(T_{\alpha \circ \varphi(s)} M)\,, \end{cases}$$

where . means the usual multiplication by a real function. Then (4) implies our Proposition.

Corollary 1. *The map*

$$j^r_x F \mapsto (j^r_x f, \varphi_F(x, -)) \tag{5}$$

is a bijection $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q) \cong \boldsymbol{J}^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \times \mathscr{S}(S, T^r_m Q)$.

Proof. Given $j^r_x f \in J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$ and $\varphi \in \mathscr{S}(S, T^r_m Q)$, by Proposition 5 there exists a map $\Phi: \boldsymbol{R}^m \times S \to Q$ satisfying $j^r_0 \Phi(-, s) = \varphi(s)$ for all $s \in S$. We set $\bar{F} = \Phi \circ \tau_{-x}$, $F = (f, \bar{F})$. According to (3) we have $\varphi_F = \varphi$, hence Corollary 1 follows from Proposition 4.

Definition 1. The smooth structure on $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q)$ is the transfer of the smooth structure on $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \times \mathscr{S}(S, T^r_m Q)$ by (5).

Remark 2. A map $g: X \to \boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^m \times Q)$ is an $\mathscr{S}$ – morphism iff the induced maps $g_1: X \to J^r(\boldsymbol{R}^m, \boldsymbol{R}^n)$, $g_2: X \times S \to T^r_m Q$ are morphisms in $\mathscr{S}$. In particular, if c is a curve in $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q)$ then $c_2: \boldsymbol{R} \times S \to T^r_m Q$ is smooth and by Proposition 5 there exists a morphism $F: \boldsymbol{R} \times \boldsymbol{R}^m \times S \to Q$ satisfying $c_2(t, s) = j^r_0 F(t, -, s)$. Therefore we also have a morphism $\bar{c}_2: \boldsymbol{R} \times T^r_m S \to T^r_m Q$ defined by $\bar{c}_2(t, j^r_0 a) = j^r_0(F(t, -, -) \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a))$. Conversely, having $\bar{c}_2$ we define c_2 by restriction. In this way $\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q)$ can also be considered as a subobject of $J^r(\boldsymbol{R}^m, \boldsymbol{R}^n) \times \mathscr{S}(T^r_m S, T^r_m Q)$.

A useful consequence of Proposition 5 is

Proposition 6. *A map* $c: \mathbf{R} \to \mathbf{J}^r(\mathbf{R}^m \times S, \mathbf{R}^n \times Q)$ *is a curve iff there exist* $\mathscr{S}$ *– morphisms* $F: \mathbf{R} \times \mathbf{R}^m \times S \to \mathbf{R}^n \times Q$, $x: \mathbf{R} \to \mathbf{R}^m$ *satisfying*

(i) $F(t, -, -) \in \operatorname{Mor} \mathscr{FM}$ *for all* $t \in \mathbf{R}$,

(ii) $j^r_{x(t)} F(t, -, -) = c(t)$ *for all* $t \in \mathbf{R}$.

Proof. Given morphisms F and x satisfying (i) and (ii), the smoothness of c is obvious. Conversely, let us have a curve c. This induces two morphisms $c_1: \mathbf{R} \to \mathbf{J}^r(\mathbf{R}^m, \mathbf{R}^n)$, $c_2: \mathbf{R} \times S \to T^r_m Q)$ (see Remark 2). According to Proposition 5, there exist morphisms $f: \mathbf{R} \times \mathbf{R}^m \to \mathbf{R}^n$, $\Phi: \mathbf{R} \times \mathbf{R}^m \times S \to Q$ satisfying $j^r_{\alpha \circ c(t)} f(t, -) = c_1(t)$, $j^r_0 \Phi(t, -, s) = c_2(t, s)$. Setting $x(t) = \alpha \circ c(t)$, $F(t, z, s) = (f(t, z), \Phi \circ \tau_{-x(t)}(t, z, s))$, we prove Proposition 6.

Consider fibred manifolds $Y \to X$, $W \to Z$. Given local trivializations $\Phi: \mathbf{R}^m \times S \to Y$, $\Psi: \mathbf{R}^n \times Q \to W$, an injective map $\langle \Phi, \Psi \rangle: \mathbf{J}^r(\mathbf{R}^m \times S, \mathbf{R}^n \times Q) \to \mathbf{J}^r(Y, W)$,

$$\langle \Phi, \Psi \rangle (j^r_x F) = j^r_{\varphi(x)}(\Psi \circ F \circ \Phi^{-1}) \tag{6}$$

is determined, φ being the underlying base map of Φ.

Definition 2. The object $\mathbf{J}^r(Y, W)$ in $\mathscr{S}$ is the set $\mathbf{J}^r(Y, W)$ endowed with the inductive structure with respect to the set of maps $\{\langle \Phi, \Psi \rangle: \mathbf{J}^r(\mathbf{R}^m \times S, \mathbf{R}^n \times Q) \to \mathbf{J}^r(Y, W); \Phi, \Psi$ are local trivializations of $Y, W\}$.

A map $f: \mathbf{J}^r(Y, W) \to P$ is a morphism in $\mathscr{S}$ iff any local trivializations Φ, Ψ of Y, W satisfy $f \circ \langle \Phi, \Psi \rangle \in \operatorname{Mor} \mathscr{S}$. Cearly, the structures defined on $\mathbf{J}^r(\mathbf{R}^m \times S, \mathbf{R}^n \times Q)$ in Definition 1 and Definition 2 coincide. There exist canonical projections in $\mathscr{S}$:

$$\alpha: \mathbf{J}^r(Y, W) \to X\,, \quad j^r_x F \mapsto x \qquad \text{(source projection)}\,,$$

$$\beta: \mathbf{J}^r(Y, W) \to Z\,, \quad j^r_x F \mapsto f(x) \quad \text{(target projection)}\,.$$

Definition 3. Consider a subcategory $\mathscr{C} \subset \mathscr{FM}$. The object $\mathbf{J}^r\mathscr{C}(Y, W)$ is the subset of all fibre r-jets of local $\mathscr{C}$ – morphisms in $\mathbf{J}^r(Y, W)$ endowed with the subobject structure. In the case $\mathscr{C} = \mathscr{FM}$ we shall write only $\mathbf{J}^r(Y, W)$.

Proposition 7. *Consider a subcategory* $\mathscr{C} \subset \mathscr{FM}$ *and local trivializations* Φ, Ψ *of* Y, W. *If* $\Phi, \Phi^{-1}, \Psi, \Psi^{-1} \in \operatorname{Mor} \mathscr{C}$ *then the restriction of* (6) $\langle \Phi, \Psi \rangle: \mathbf{J}^r\mathscr{C}(\mathbf{R}^m \times S, \mathbf{R}^n \times Q) \to \mathbf{J}^r\mathscr{C}(Y, W)$ *is an open subobject.*

Proof. It follows from the properties of subobjects that it is sufficient to prove our Proposition in the case $\mathscr{C} = \mathscr{FM}$.

Consider such a map c that $\langle \Phi, \Psi \rangle \circ c$ is a curve. The proof will be complete after proving that in this situation c also is a curve. Given a function f on $\mathbf{J}^r(\mathbf{R}^m \times S, \mathbf{R}^n \times Q)$ and a point $t_0 \in \mathbf{R}$, we have to show that $f \circ c$ is smooth in t_0. Choose functions $\varepsilon_k: \mathbf{R}^k \to \mathbf{R}$, $k = m, n$, satisfying $\varepsilon_k \equiv 1$ on a neighbourhood of 0, $\varepsilon_k \equiv 0$ outside the sphere $K_k(0, 1)$. We may suppose $\alpha \circ c(t_0) = 0 = \beta \circ c(t_0)$. We set

$f_0 = (\varepsilon_m \circ \alpha) \cdot (\varepsilon_n \circ \beta) \cdot f$, where . denotes the multiplication of real functions, and define a function $\bar{f}: \boldsymbol{J}^r(Y, W) \to \boldsymbol{R}$ by

$$\bar{f}(j_x^r F) = \begin{cases} f_0 \circ \langle \Phi, \Psi \rangle^{-1} (j_x^r F) & \text{if} \quad j_x^r F \in \langle \Phi, \Psi \rangle \, (\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q))\,, \\ 0 \quad \text{in the other cases}\,. \end{cases}$$

The map $\bar{f} \circ \langle \Phi, \Psi \rangle \circ c$ is smooth and coincides with $f \circ c$ on a neighbourhood of $t_0 \in \boldsymbol{R}$. That is why c is a curve. The subobject is open because the image of $\langle \Phi, \Psi \rangle$ is determined by the bundle projection, QED.

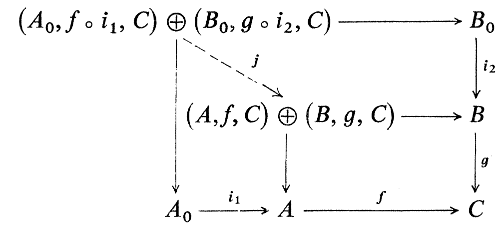
Definition 5. Consider a subcategory $\mathscr{C} \subset \mathscr{F}\mathscr{M}$. The object $\boldsymbol{J}_x^r\mathscr{C}(Y, W)$ or $\boldsymbol{J}^r\mathscr{C}(Y, W)_z$ is the subset of all fibre r-jets in $\boldsymbol{J}^r\mathscr{C}(Y, W)$ with the source or target in x or z, respectively, endowed with the subobject structure of $J^r(Y, W)$; $J_x^r\mathscr{C}(Y, W)_z$ is the subobject of all fibre r-jets in $J^r\mathscr{C}(Y, W)$ with the source in x and target in z. (In the case of $C = \mathscr{F}\mathscr{M}$ the symbol $\mathscr{F}\mathscr{M}$ will be omitted.)

3. COMPOSITION OF FIBRE JETS

Consider fibred manifolds $Y \to X$, $W \to Z$, $V \to P$ with standard fibres S, Q, M and $F, G \in \operatorname{Mor} \mathscr{F}\mathscr{M}$, $F: Y \to W$, $G: W \to V$. One sets $j_{f(x)}^r G \circ j_x^r F = j_x^r(G \circ F)$, which is obviously a correct definition.

Proposition 8. *Given a subcategory $\mathscr{C} \subset \mathscr{F}\mathscr{M}$, the composition of fibre jets determines a map from the fibred product $(\boldsymbol{J}^r\mathscr{C}(Y, W), \beta, Z) \oplus (\boldsymbol{J}^r\mathscr{C}(W, V), \alpha, Z)$ into $\boldsymbol{J}^r\mathscr{C}(Y, V)$ which is a morphism in $\mathscr{S}$. We denote this morphism by $\circ$.*

Proof. One can easily see that given $A, B, C \in \operatorname{Ob}\mathscr{S}$, subobjects $i_1: A_0 \to A$, $i_2: B_0 \to B$ and morphisms $f: A \to C$, $g: B \to C$ in $\mathscr{S}$, then $j: (A_0, f \circ i_1, C) \oplus$ $\oplus (B_0, g \circ i_2, C) \to (A, f, C) \oplus (B, g, C)$ also is a subobject, where j is the morphism uniquely determined from the diagram



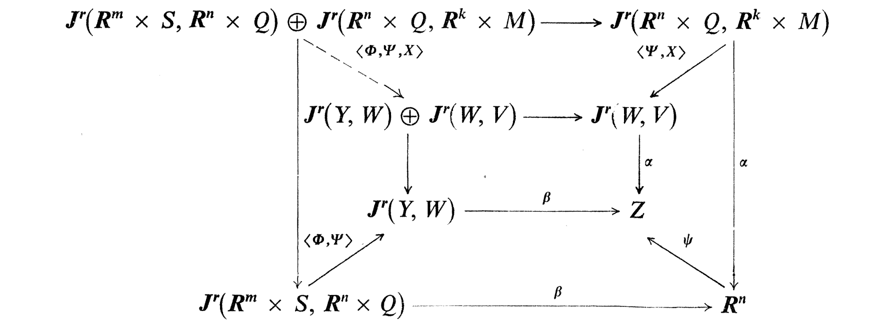
Owing to this fact, it is sufficient to prove Proposition 8 in the case $\mathscr{C} = \mathscr{F}\mathscr{M}$. First assume $Y = \boldsymbol{R}^m \times S$, $W = \boldsymbol{R}^n \times Q$, $V = \boldsymbol{R}^k \times M$ and consider two curves $c: \boldsymbol{R} \to \boldsymbol{J}^r(Y, W)$, $d: \boldsymbol{R} \to \boldsymbol{J}^r(W, V)$ satisfying $\beta \circ c = \alpha \circ d$ (an arbitrary curve in the fibred product from Proposition 8). In our case the proof will be complete after showing that $d \circ c: \boldsymbol{R} \to \boldsymbol{J}^r(Y, V)$ (defined pointwise) is also a curve. By Proposition 6 there exist morphisms $F: \boldsymbol{R} \times \boldsymbol{R}^m \times S \to \boldsymbol{R}^n \times Q$, $G: \boldsymbol{R} \times \boldsymbol{R}^n \times Q \to \boldsymbol{R}^k \times M$

satisfying $c(t) = j^r_{\alpha\circ c(t)}F(t, -, -)$, $d(t) = j^r_{\alpha\circ d(t)}G(t, -, -)$. By definition

$$(d \circ c)(t) = d(t) \circ c(t) = j^r_{\alpha\circ c(t)}(G(t, -, -) \circ F(t, -, -)) .$$

Set $H = G \circ (\mathrm{id}_{\boldsymbol{R}} \times F)$. Now $H \in \operatorname{Mor} \mathscr{S}$, $H(t, -, -) \in \operatorname{Mor} \mathscr{FM}$, $(d \circ c)(t) = j^r_{\alpha\circ c(t)}H(t, -, -)$, hence $d \circ c$ is a curve according to Proposition 6.

Given local trivializations Φ, Ψ, X of fibred manifolds Y, W, V, there exists a unique morphism $\langle \Phi, \Psi, X \rangle$ from the following commutative diagram in $\mathscr{S}$ (the squares are pullbacks, ψ is the underlying base map of Ψ).



The map $\langle \Phi, \Psi, X \rangle$ is an immersion of a subobject as one can also see from the diagram. This immersion is of the form

$$(j^r_x F, j^r_{f(x)} G) \mapsto (j^r_{\varphi(x)}(\Psi \circ F \circ \Phi^{-1}) , \quad j^r_{\psi(f(x))}(\chi \circ G \circ \Psi^{-1})) ,$$

hence the subobject is open. Now Proposition 8 follows from the above considerations, Proposition 3 and the commutativity of the following diagram:

$$\begin{array}{ccc}
(J^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^n \times Q), \beta, \boldsymbol{R}^n) \oplus (J^r(\boldsymbol{R}^n \times Q, \boldsymbol{R}^k \times M), \alpha, \boldsymbol{R}^n) & \xrightarrow{\circ} & J^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^k \times M) \\
\downarrow{\scriptstyle \langle \Phi, \Psi, X \rangle} & & \downarrow{\scriptstyle \langle \Phi, X \rangle} \\
(J^r(Y, W), \beta, Z) \oplus (J^r(W, V), \alpha, Z) & \xrightarrow{\circ} & J^r(Y, V)
\end{array}$$

Remark 3. Given $j^r_{f(x)}G$, $j^r_x F$ we also have $\bar{\varphi}_F$, $\bar{\varphi}_G$ from (1). Then we can obtain $\bar{\varphi}_{G\circ F}$ in the following way:

$$\bar{\varphi}_{G\circ F}\colon \boldsymbol{R}^m \times T^r_m S \xrightarrow{\bar{\varphi}_F} T^r_m(\boldsymbol{R}^n \times Q) \cong T^r_m(\boldsymbol{R}^n \times \text{const } T^r_n Q) \xrightarrow{T_m{}^r \bar{\varphi}_G}$$

$$\to T^r_m T^r_n(\boldsymbol{R}^k \times M) \xrightarrow{p} T^r_m(\boldsymbol{R}^k \times M) ,$$

where p is the canonical projection. This can be proved by direct evaluation.

4. INVERTIBLE FIBRE JETS

A fibre jet $A \in J^r_x(Y, W)_z$ is called invertible if there exists a fibre jet $B \in J^r_z(W, Y)_x$ satisfying $B \circ A = j^r_x \mathrm{id}_Y$, $A \circ B = j^r_z \mathrm{id}_W$.

Definition 6. The object $\mathrm{inv}\boldsymbol{J}^r(Y, W)$ in $\mathcal{S}$ is the subset of all invertible jets in $\boldsymbol{J}^r(Y, W)$ endowed with the subobject structure. Given a subcategory $\mathcal{C} \subset \mathcal{FM}$, the object $\mathrm{inv}\boldsymbol{J}^r\mathcal{C}(Y, W)$ is the subset in $\boldsymbol{J}^r(Y, W)$ of all those invertible fibre r-jets which are together with their inverses fibre r-jets of $\mathcal{C}$ – morphisms endowed with the subobject structure. In a natural way we define $\mathrm{inv}\boldsymbol{J}_x^r\mathcal{C}(Y, W)$, $\mathrm{inv}\boldsymbol{J}^r\mathcal{C}(Y, W)_z$ and $\mathrm{inv}\, \boldsymbol{J}_x^r\mathcal{C}(Y, W)_z$.

Proposition 9. *A fibre jet $\boldsymbol{j}_x^r F \in \boldsymbol{J}^r(Y, W)$ is invertible iff $j_x^r f$ is invertible and the restriction $F \mid Y_x\colon Y_x \to W_{f(x)}$ is a diffeomorphism.*

Proof. It is sufficient to prove Proposition 9 in the case $Y = \boldsymbol{R}^m \times S = W$, $x = 0, f(0) = 0$. Necessity is clear, we prove sufficiency. The restriction $F \mid \{0\} \times S$: $\{0\} \times S \to \{0\} \times S$ is a diffeomorphism, so it is a regular map on a certain neighbourhood of $\{0\} \times S$. It follows from the inverse function theorem that $\tilde{\varphi}_F(0, -)$: $T_m^r S \to T_m^r S$ is a bijection (see (2)) and the inverse map to $\tilde{\varphi}_F(0, -)$ is also smooth. The map F can be expressed as $F = H \circ G$, where $H = f \times \mathrm{id}_S$, $G = \mathrm{id}_{\boldsymbol{R}^m} \times \tilde{F}$. Since f^{-1} exists on a neighbourhood U of 0, H^{-1} also exists on $U \times S$. Hence we may assume, without loss of generality, that F is a morphism over identity. In this case, given another morphism of fibred manifolds $G\colon \boldsymbol{R}^m \times S \to \boldsymbol{R}^m \times S$, we have (see (2))

$$\tilde{\varphi}_G(0, -) \circ \tilde{\varphi}_F(0, -)\colon j_0^r a \mapsto j_0^r(\tilde{F} \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a)) \mapsto$$
$$\mapsto j_0^r(\tilde{G} \circ (\mathrm{id}_{\boldsymbol{R}^m} \times (\tilde{F} \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a)))) = j_0^r(\widetilde{G \circ F} \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a))\,.$$

Hence $\tilde{\varphi}_G(0, -) \circ \tilde{\varphi}_F(0, -) = \tilde{\varphi}_{G \circ F}(0, -)$. Let us choose $\varphi_G(0, -)$ as the restriction of the inverse map of $\tilde{\varphi}_F(0, -)$ to const $T_m^r S$ and let $\boldsymbol{j}_0^r G$ be the fibre jet determined by $(j_0^r \mathrm{id}_{\boldsymbol{R}^m}, \varphi_G(0, -))$. We may assume that G is a map over the identity. Since $j_0^r(\tilde{F} \circ G \circ (\mathrm{id}_{\boldsymbol{R}^m} \times a)) = \tilde{\varphi}_F(0, -) \circ \tilde{\varphi}_G(0, -)\,(j_0^r a) = j_0^r a$ for all constant maps $a\colon \boldsymbol{R}^m \to S$, by Proposition 4 we have $\boldsymbol{j}_0^r F \circ \boldsymbol{j}_0^r G = \boldsymbol{j}_0^r \mathrm{id}_{\boldsymbol{R}^m \times S}$. In a similar way we can find the right inverse $\boldsymbol{j}_0^r H$ to $\boldsymbol{j}_0^r G$. Since $\boldsymbol{j}_0^r F = \boldsymbol{j}_0^r F \circ \boldsymbol{j}_0^r G \circ \boldsymbol{j}_0^r H = \boldsymbol{j}_0^r H$, Proposition 9 is proved.

In particular, $\mathrm{inv}\boldsymbol{J}^r(Y, W)$ is a nonempty set iff the bases of Y and W are of the same dimension and the standard fibres are diffeomorphic.

Remark 4. The set $\mathrm{inv}\, \boldsymbol{J}^r(Y, W)$ need not be open in $\boldsymbol{J}^r(Y, W)$. Indeed, set $Y = W = \boldsymbol{R} \times (0, 1)$, $F\colon \boldsymbol{R} \times (0, 1) \to \boldsymbol{R} \times (0, 1)$, $F(x, y) = (x, (1 - x^2)\, y)$. Given a curve $c(t)\colon \boldsymbol{R} \to \boldsymbol{J}^r(Y, W)$ by $c(t) = \boldsymbol{j}_t^r F$, we have $c^{-1}(\mathrm{inv}\boldsymbol{J}^r(Y, W)) = \{0\}$.

Proposition 10. *For every subcategory $\mathcal{C} \subset \mathcal{FM}$, the map ${}^{-1}$: $\mathrm{inv}\, \boldsymbol{J}^r\mathcal{C}(Y, W) \to \mathrm{inv}\boldsymbol{J}^r\mathcal{C}(W, Y)$ is an $\mathcal{S}$ – morphism.*

Proof. Since $\mathrm{inv}\boldsymbol{J}^r\mathcal{C}(Y, W)$ is a subobject in $\mathrm{inv}\boldsymbol{J}^r(Y, W)$, it is sufficient to prove our Proposition in the case $\mathcal{C} = \mathcal{FM}$. Given local trivializations Φ, Ψ of fibred manifolds Y, W, the restriction of $\langle \Phi, \Psi \rangle$ to $\mathrm{inv}\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^m \times Q)$ is an immersion of an open subobject in $\mathrm{inv}\boldsymbol{J}^r(Y, W)$. Since ${}^{-1} \circ \langle \Phi, \Psi \rangle = \langle \Psi, \Phi \rangle \circ {}^{-1}$, it is sufficient to prove Proposition 10 in the case $Y = \boldsymbol{R}^m \times S = W$ (see Proposition

3 and Definition 2). Consider a curve c in inv$\boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^m \times S)$. The curve c can be expressed in the form $c = c_1 \circ c_2$ in a similar way as the map F in the proof of Proposition 9. Hence according to Proposition 6 we may assume the existence of maps $F, G: \boldsymbol{R} \times \boldsymbol{R}^m \times S \to \boldsymbol{R}^m \times S$ satisfying

$$F(t, -, s) = \mathrm{id}_{\boldsymbol{R}^m}\,, \quad j^r_{\alpha \circ c_2(t)} F(t, -, -) = c_2(t)\,, \quad G(t, x, -) = \mathrm{id}_S\,,$$

$j^r_{\alpha \circ c_1(t)} G(t, -, -) = c_1(t)$. In addition, G can be chosen in such a way that $\mathrm{id}_{\boldsymbol{R}} \times G$: $\boldsymbol{R} \times \boldsymbol{R}^m \times S \to \boldsymbol{R} \times \boldsymbol{R}^m \times S$ is an isomorphism in $\mathscr{S}$. Hence there exists $(\mathrm{id}_{\boldsymbol{R}} \times G)^{-1}$ of the form $\mathrm{id}_{\boldsymbol{R}} \times \bar{G}$. That is why $j^r_{\beta \circ c_1(t)} \bar{G}(t, -, -) = (c_1(t))^{-1}$ and $^{-1} \circ c_1$ is a curve by Proposition 6. Owing to this fact, we may assume $c = c_2$, without loss of generality. The curve c determines a curve $d_1: \boldsymbol{R} \to \mathscr{S}(T^r_m S, T^r_m S)$, $d_1(t) = \tilde{\varphi}_{F(t,-,-)}(\alpha \circ c(t), -)$, see Remark 2. By Proposition 9 and the inverse function theorem, the induced map $\mathrm{id}_{\boldsymbol{R}} \times \tilde{d}_1: \boldsymbol{R} \times T^r_m S \to \boldsymbol{R} \times T^r_m S$ is an isomorphism in $\mathscr{S}$. The inverse morphism is of the form $\mathrm{id}_{\boldsymbol{R}} \times \tilde{d}_2: \boldsymbol{R} \times T^r_m S \to \boldsymbol{R} \times T^r_m S$ and a curve $d_2: \boldsymbol{R} \to \mathscr{S}(S, T^r_m S)$ is defined by a restriction of $\tilde{d}_2$. The curve d_2 together with the constant map $e(t) = j^r_0 \mathrm{id}_{\boldsymbol{R}^m}$ determines a curve $d: \boldsymbol{R} \to \boldsymbol{J}^r(\boldsymbol{R}^m \times S, \boldsymbol{R}^m \times S)$. By the construction of d, the equality $d = {}^{-1} \circ c$ holds, QED.

5. $\boldsymbol{H}^r\mathscr{C}Y$ AND ASSOCIATED SMOOTH SPACES

Definition 7. A subcategory $\mathscr{C} \subset \mathscr{FM}$ is said to be *rich* if the following conditions are fulfilled:

(i) if $\boldsymbol{R}^m \times S \to \boldsymbol{R}^m \in \mathrm{Ob}\mathscr{C}$, $X, Z \in \mathrm{Ob}\mathscr{M}_m$, $f: X \to Z \in \mathrm{Mor}\mathscr{M}_m$ then $X \times S$, $Z \times S \in \mathrm{Ob}\mathscr{C}$ and $f \times \mathrm{id}_S \in \mathrm{Mor}\mathscr{C}$.

(ii) if $(\pi: Y \to X) \in \mathrm{Ob}\mathscr{C}$ and $U \subset X$ is an open subset, then $\pi^{-1}(U) \in \mathrm{Ob}\mathscr{C}$ and the inclusion $i: \pi^{-1}(U) \to Y$ is a $\mathscr{C}$-morphism. Moreover, if S is a standard fibre of Y, then $\boldsymbol{R}^m \times S \in \mathrm{Ob}\mathscr{C}$ and there exist an open covering U_i, $i \in I$, of X and $\mathscr{C}$-isomorphisms $\Psi_i: \boldsymbol{R}^m \times S \to \pi^{-1}(U_i)$.

Definition 8. Consider a rich subcategory $\mathscr{C} \subset \mathscr{FM}$. We define $\boldsymbol{H}^r\mathscr{C}Y \in \mathrm{Ob}\mathscr{S}$ as inv$\boldsymbol{J}^r_0\mathscr{C}(\boldsymbol{R}^m \times S, Y)$, S being the standard fibre of fibred manifold Y with an m-dimensional base. We define $\boldsymbol{L}^r\mathscr{C}(S, m) \in \mathrm{Ob}\mathscr{S}$ as inv$\boldsymbol{J}^r_0\mathscr{C}(\boldsymbol{R}^m \times S, \boldsymbol{R}^m \times S)_0$. As usual, the symbol $\mathscr{C}$ will be omitted in the case $\mathscr{C} = \mathscr{FM}$.

Remark 5. If we choose another standard fibre S of Y, we obtain an m-isomorphic object by virtue of Definition 7. More exactly, we could define $\boldsymbol{H}^r\mathscr{C}Y$ as a class of $\mathscr{S}$-isomorphic objects inv$\boldsymbol{J}^r_0\mathscr{C}(\boldsymbol{R}^m \times S, Y)$ where S is an arbitrary standard fibre of Y. Nor will be the later use of this concept quite precise from the categorial point of view. For example, if we consider the category of principal bundles and their morphisms, we should, more precisely, speak about a category $\mathscr{C}$ with a forgetful functor into $\mathscr{FM}$. Then conditions (i), (ii) of Definition 7 would be modified in an obvious way.

We have two projections $\pi: Y \to X$, $\beta: \boldsymbol{H}^r\mathscr{C}Y \to X$. Owing to the following proposition, $H^r\mathscr{C}Y$ may be viewed as a "principal fibre bundle" over X with the structure group $L^r\mathscr{C}(S, m)$.

Proposition 11. *For every subcategory $\mathscr{C} \subset \mathscr{FM}$, $L^r\mathscr{C}(S, m)$ is a smooth group and the right action of $L^r\mathscr{C}(S, m)$ on $H^r\mathscr{C}Y$ is smooth. This action is invariant and effecttve with respect to the projection β. Moreover, if $\mathscr{C}$ is rich then $H^r\mathscr{C}Y$ is locally isomorphic to $R^m \times L^r\mathscr{C}(S, m)$ in $\mathscr{S}$, i.e. there exists an open covering U_i, $i \in I$ of X such that $\beta^{-1}(U_i)$ is isomorphic to $R^m \times L^r\mathscr{C}(S, m)$ in $\mathscr{S}$. These isomorphisms can be chosen in such a way that they commute with the action of $L^r\mathscr{C}(S, m)$.*

Proof. The group operation and the action are defined by the composition of fibre jets. Hence the smoothness of the group structure and of the action is a consequence of Propositions 8 and 10. The invariance and effectivity with respect to β is clear. Let Ψ_i be a local trivialization by $\mathscr{C}$ — morphisms of Y. The map $\tau\colon R^m \times L^r\mathscr{C}(S, m) \to H^r\mathscr{C}(R^m \times S)$, defined by $(x, j_0^r\Phi) \mapsto j_0^r\tau_x \circ j_0^r\Phi$ is an isomorphism in $\mathscr{S}$. The restrictions of $\langle \mathrm{id}_{R^m \times S}, \Psi_i \rangle$ to $H^r\mathscr{C}(R^m \times S)$ are immersions of open subobjects in $H^r\mathscr{C}Y$. The maps $\langle \mathrm{id}_{R^m \times S}, \Psi_i \rangle \circ \tau$ are the required isomorphisms, QED.

Definition 9. Consider $H^r\mathscr{C}Y = \mathrm{inv} J_0^r\mathscr{C}(R^m \times S, Y)$ and an $\mathscr{S}$-object P with a smooth left action of the group $L^r\mathscr{C}(S, m)$. There is a canonical right action of $L^r\mathscr{C}(S, m)$ on $H^r\mathscr{C}Y \times P$. ($A \in H^r\mathscr{C}Y$, $p \in P$, $B \in L^r\mathscr{C}(S, m)$, $(A, p) \circ B = (A \circ B, B^{-1} \circ p)$.) This action determines an equivalence $\sim$ as follows: $(A_1, p_1) \sim (A_2, p_2)$ iff there exists $B \in L^r\mathscr{C}(S, m)$ such that $A_2 = A_1 \circ B$, $p_2 = B^{-1} \circ p_1$. We define the associated space $H^r\mathscr{C}Y(P)$ as $(H^r\mathscr{C}Y \times P)|\sim$. Hence a map with the domain $H^r\mathscr{C}Y(P)$ is smooth iff the induced map from $H^r\mathscr{C}Y \times P$ is smooth. The target projection $\beta\colon H^r\mathscr{C}Y \to X$ is also defined on $H^r\mathscr{C}Y(P)$ in a natural way, we will denote it also by β.

Proposition 12. *Consider a rich subcategory $\mathscr{C} \subset \mathscr{FM}$ and $Y \in \mathrm{Ob}\mathscr{C}$. The object $H^r\mathscr{C}Y(P)$ is locally isomorphic to $R^m \times P$ in $\mathscr{S}$.*

Proof. There exists an $\mathscr{S}$ — isomorphism χ, $\chi\colon (R^m \times L^r\mathscr{C}(S, m) \times P) \mid \sim \to R^m \times P$, defined by $i(x, A, p) \mapsto (x, A \circ p)$, where i is the canonical projection to the quotient object. There also exists an open covering U_i, $i \in I$ of X and isomorphisms $\varphi_i\colon R^m \times L^r\mathscr{C}(S, m) \to \beta^{-1}(U_i)$. Hence we obtain open subobjects $\bar{\varphi}_i = \varphi_i \times \mathrm{id}_P\colon R^m \times L^r\mathscr{C}(S, m) \times P \to H^r\mathscr{C}Y \times P$. Since the maps $\bar{\varphi}_i$ commute with the action of $L^r\mathscr{C}(S, m)$ by Proposition 11, they determine morphisms $\bar{\varphi}_i\colon (R^m \times L^r\mathscr{C}(S, m) \times P)|\sim \to H^r\mathscr{C}Y(P)$. Similarly to the proof of Proposition 7 one can easily show that $\bar{\varphi}_i$ are immersions of open subobjects. Clearly, the image of $\bar{\varphi}_i$ is $\beta^{-1}(U_i)$. The required isomorphisms are $\bar{\varphi}_i \circ \chi^{-1}$, QED.

Example 1. Let $\mathscr{PB}(G)$ denote the category of principal fibre bundles and morphisms of principal fibre bundles with identity on G. Let $P \in \mathrm{Ob}\mathscr{PB}(G)$ have an m-dimensional base. The morphisms of $\mathscr{PB}(G)$ map fibres to fibres diffeomorphically, hence the set $H^r\mathscr{PB}(G)\,P$ is formed by fibre r-jets of local isomorphisms commuting with the action of G. It is clear that $j_0^rF = j_0^rG$ holds on $H^r\mathscr{PB}(G)\,P$ iff $j_{(0,e)}^rF = j_{(0,e)}^rG$. In this way we have constructed a bijective map between $H^r\mathscr{PB}(G)\,P$

and the spaces W^rP defined by Kolář [9]. Given a curve $j_0^r F_t$ in $H^r\mathscr{PB}(G)\,P$, there exists a smooth map $G(t, -, -)$ satisfying $j_0^r F_t = j_0^r G(t, -, -)$, so that $j^r_{(0,e)}F_t$ also is a curve in W^rP. Conversely, given a curve $j^r_{(0,e)}F_t$ in W^rP, then using local coordinates we deduce from the smoothness of the action of G that also $j_0^r F_t$ is a curve in $H^r\mathscr{PB}(G)\,P$. Thus we have proved that $H^r\mathscr{PB}(G)\,P$ and W^rP are isomorphic.

Example 2. Denote by $\mathscr{VB}$ the category of finite-dimensional smooth vector bundles and smooth linear morphisms. Consider a vector bundle $E \to X$ with a standard fibre V. The linear morphisms are locally of the form $F^j(x, c_i e^i) = c_i F^j(x, e^i)$, hence precisely the jets of local isomorphisms are in $H^r\mathscr{VB}E$. Since $j_0^r F = j_0^r G$ iff $j^r_{(0,e_i)}F = j^r_{(0,e_i)}G$ for $i = 1, \ldots, k$, we have a bijection between the set $H^r\mathscr{VB}E$ and a submanifold of $\bigtimes_{i=1}^{k} J^r_{(0,e_i)}(R^m \times V, E)$. Using local coordinates one easily proves that this bijection is an isomorphism in $\mathscr{S}$. Hence the smooth structure on $H^r\mathscr{VB}E$ in our sense coincides with the classical structure of a finite dimensional manifold.

6. PROLONGATION AND FIBRE FUNCTORS

Definition 10. Let $\mathscr{C}$ be a subcategory of $\mathscr{FM}$. A fibre functor on $\mathscr{C}$ is a covariant functor $F\colon \mathscr{C} \to \mathscr{FM}$ such that the following conditions are fulfilled:

(i) (the prolongation condition) $B \circ F = B$, where $B\colon \mathscr{FM} \to \mathscr{M}$ is the base functor;

(ii) (the localization condition) let $\pi\colon Y \to X$ be a fibred manifold in $\mathscr{C}$, $U \subset X$ an open set and $i\colon \pi^{-1}(U) \to Y$ the inclusion. Then $Fi\colon F(\pi^{-1}(U)) \to FY$ is the inclusion of $\mathscr{F}\pi^{-1}(U)$ into FY, where $\mathscr{F}\pi\colon FY \to X$ is the image of $\pi\colon Y \to X$

(iii) (the regularity condition) if $f\colon Y \times P \to W$ is a smooth map such that $f(-, p) \in \operatorname{Mor}\mathscr{FM}$ for all $p \in P$, then $\widetilde{Ff}\colon FY \times P \to FW$, defined by $\widetilde{Ff}(-, p) = F(f(-, p))$, $p \in P$, is also smooth.

Remark 6. It follows directly from Definition 10 that a composition $G \circ F$ of two fibre functors $F\colon \mathscr{C} \to \mathscr{D} \subset \mathscr{FM}$, $G\colon \mathscr{D} \to \mathscr{FM}$ also is a fibre functor.

Example 3. Considering principal bundles with m-dimensional bases and morphisms over local diffeomorphisms only, we obtain a subcategory $\mathscr{PB}_m$ of $\mathscr{PB}$. There is a fibre functor $Q^1\colon \mathscr{PB}_m \to \mathscr{FM}$ which maps $P \in \operatorname{Ob}\mathscr{PB}_m$ into the bundle Q^1P of the 1-st order elements of connections. This functor can be constructed as follows: the bundle Q^1P is defined as J^1P/G, where G is the structure group of P. Given another principal bundle $\bar{P}$ with a structure group $\bar{G}$ and morphism $f\colon P \to \bar{P}$ with a group homomorphism $f_1\colon G \to \bar{G}$, the map $J^1f\colon J^1P \to J^1\bar{P}$ is f_1-equivariant with respect to the induced actions of the groups G, $\bar{G}$ on J^1P and $J^1\bar{P}$, see [12]. Hence this map factorises to a map $Q^1f\colon J^1P/G \to J^1\bar{P}/\bar{G}$.

Remark 7. One can also construct fibre functors $Q^r\colon \mathscr{PB}_m \to \mathscr{FM}$ which map

$P \in \mathrm{Ob}\mathscr{P}\mathscr{B}_m$ to the bundle $Q^r P$ of the r-th order elements of connections in the sense of Ehresmann, [8].

Remark 8. If we identify a manifold $X \in \mathscr{M}$ with the fibred manifold $\mathrm{id}_X: X \to X$ and a morphism $f: X \to Y, f \in \mathrm{Mor}\,\mathscr{M}$ with $(f, f): (X \to X) \to (Y \to Y)$, the category $\mathscr{M}$ can be considered as a subcategory of $\mathscr{F}\mathscr{M}$. Then the concept of the fibre functor is a generalization of lifting functors and prolongation functors in the sense of Kolář, see [2], [7], [10], [13], [14].

Let $F_x Y$ be the fibre of FY over $x \in X$, $M_m := \boldsymbol{R}^m \times M$, $S(M, m) := F_0(M_m)$. Consider a subcategory $\mathscr{C} \subset \mathscr{F}\mathscr{M}$. The subobject $\mathscr{S}\mathscr{C}(Y, W)$ of $\mathscr{S}(Y, W)$ is the subset of all morphisms of $\mathscr{C}$ endowed with the subobject structure. The subobject of those $\mathscr{C}$ – morphisms which map the fibre over $x \in X$ into the fibre over $z \in Z$ will be denoted by $\mathscr{S}_x\mathscr{C}(Y, W)_z$. For any $Y, W \in \mathrm{Ob}\mathscr{C}$ the fibre functor F on $\mathscr{C}$ determines a map $F_{Y,W}: \mathscr{S}\mathscr{C}(Y, W) \times FY \to FW, (f, y) \mapsto Ff(y)$, which will be called the generalized associated map of the functor F. By restriction of F_{M_m, N_n} we get a map $\Lambda_{(M,m;N,n)}: \mathscr{S}_0\mathscr{C}(M_m, N_n)_0 \times S(M, m) \to S(N, n)$. We have

$$F_{W,V}(f, F_{Y,W}(g, y)) = F(f \circ g)(y) = F_{Y,V}(f \circ g, y)\,. \tag{7}$$

Remark 9. The generalized associated map can be constructed to any functor on a category $\mathscr{C} \subset \mathscr{F}\mathscr{M}$ satisfying (i) and (ii) of Definition 10. It follows directly from the definition of the smooth structure on $\mathscr{S}\mathscr{C}(Y, W)$ that the smoothness of the generalized associated maps for all $Y, W \in \mathrm{Ob}\mathscr{C}$ is equivalent to the condition (iii) of Definition 10. We also obtain directly from the regularity condition that, given a fibre functor F on $\mathscr{C}$, the image of $M_m \in \mathrm{Ob}\mathscr{C}$ is isomorphic to $\boldsymbol{R}^m \times S(M, m)$, if all fibre translations τ_x are $\mathscr{C}$ – morphisms. (We set $\boldsymbol{R}^m \times S(M, m) \ni (x, y) \mapsto$ $\mapsto F\,\tau_x(y) \in F_x M_m$.) Moreover, it follows from (iii) that the group homomorphism

$$\lambda_{(M,m)}: \boldsymbol{R}^m \to \mathrm{Diff}\,(FM_m, FM_m)\,, \quad \text{defined by} \quad x \mapsto F\tau_x\,, \quad \text{is smooth}\,.$$

These facts can be verified also without the assumption of regularity if the domain of the functor F is large enough, as the following proposition shows. Note that we use only the condition (i) of Definition 8.

Proposition 13. *Given a rich subcategory $\mathscr{C} \subset \mathscr{F}\mathscr{M}$, $M_m \in \mathrm{Ob}\mathscr{C}$ and a functor $F: \mathscr{C} \to \mathscr{F}\mathscr{M}$ satisfying (i) and (ii) of Definition 10, FM_m is isomorphic to $\boldsymbol{R}^m \times$ $\times\, S(M, m)$ in $\mathscr{F}\mathscr{M}$ and the fibre translations determine a smooth group homomorphism $\lambda_{(M,m)}: \boldsymbol{R}^m \to \mathrm{Diff}\,(FM_m, FM_m)$ defined by $x \mapsto F\tau_x$.*

Proof. The category $\mathscr{C}$ is rich, hence we can define a functor $G: \mathscr{M}_m \to \mathscr{F}\mathscr{M}$ in the following way: $GX = F(X \times M)$, $Gf = F(f \times \mathrm{id}_M)$. Since G is a lifting functor, we know [2] that $G\boldsymbol{R}^m$ is isomorphic to $\boldsymbol{R}^m \times G_0\boldsymbol{R}^m = \boldsymbol{R}^m \times S(M, m)$ and the map

$$\tilde{\lambda}: \boldsymbol{R}^m \times G\boldsymbol{R}^m \to G\boldsymbol{R}^m\,, \quad \tilde{\lambda}(x, y) = G\,t_x(y) = F\,\tau_x(y) \quad \text{is smooth}\,, \quad \text{QED}\,.$$

Remark 10. The identification $\varphi: \boldsymbol{R}^m \times S(M, m) \to FM_m$, $(x, y) \mapsto F\,\tau_x(y)$ from the proposition and remark above keeps the fibre translations, that is, under this identification we have

$$(8) \qquad FM_m = \mathbf{R}^m \times S(M, m)\,, \quad F\,\tau_x(z, s) = (x + z, s)\,.$$

Consider a functor F on $\mathscr{C}$ satisfying (i) and (ii) of Definition 10 and (8). Given a morphism $f: M_m \to N_n$ in $\mathscr{C}$ over $f_0: \mathbf{R}^m \to \mathbf{R}^n$, we have

$$(9) \qquad Ff(x, s) = F\tau_{f_0(x)} \circ F(\tau_{-f_0(x)} \circ f \circ \tau_x) \circ F\tau_{-x}(x, s) =$$
$$= (f_0(x), \Lambda_{(M,m;N,n)}(\tau_{-f_0(x)} \circ f \circ \tau_x, s))\,.$$

Proposition 14. *If a functor $F: \mathscr{FM} \to \mathscr{FM}$ satisfies the conditions* (i) *and* (ii) *of Definition* 10, *then the regularity condition* (iii) *is also satisfied.* (*That means that the regularity condition of a fibre functor on $\mathscr{FM}$ is a consequence of the prolongation condition and the localization condition.*)

Proof. Since the smoothness has a local character, we may assume $Y = M_m$, $P = \mathbf{R}^p$. The map $\tilde{Ff}$ is smooth iff the map $\mathbf{R}^p \ni p \mapsto F(f(-, p))$ is an $\mathscr{S}$ – morphism. Set $i: \mathbf{R}^m \times M \to \mathbf{R}^p \times \mathbf{R}^m \times M$, $(x, s) \overset{i}{\mapsto} (0, x, s)$, $\eta_p: \mathbf{R}^p \times \mathbf{R}^m \times M \to \mathbf{R}^p \times \mathbf{R}^m \times M$, $(z, x, s) \overset{\eta_p}{\mapsto} (p + z, x, s)$. We have $f(-, p) = f \circ \eta_p \circ i$, so that $\tilde{Ff}(-, p) = Ff \circ F\eta_p \circ Fi$. Hence the proof will be complete after proving the following

Lemma 3. *The map $p \mapsto F\eta_p$ is an $\mathscr{S}$ – morphism.*

Proof. The lemma is a consequence of well known results about lifting functors [2]. We define a functor $G: \mathscr{M}_{m+p} \to \mathscr{FM}$ as follows: $GX = F(X \times M)$, $Gf = F(f \times \mathrm{id}_M)$. It is clear that G is a lifting functor, hence $p \mapsto F\eta_p$ is smooth.

Corollary 2. *The regularity condition of prolongation functors,* [7], [10], *is a consequence of the prolongation condition and of the localization condition.*

Proof. Given a functor $F: \mathscr{M} \subset \mathscr{FM} \to \mathscr{FM}$ (see Remark 8) satisfying (i) and (ii) of Definition 10, we can construct a functor $G: \mathscr{FM} \to \mathscr{FM}$ in such a way that the restriction of G to $\mathscr{M}$ is F: Given $\pi: Y \to X \in \mathrm{Ob}\mathscr{FM}$, $f \in \mathrm{Mor}\mathscr{FM}$ over $f_0 \in \mathrm{Mor}\mathscr{M}$, we put $GY = FX$, $Gf = Ff_0$. Obviously G satisfies (i) and (ii) of Definition 10, hence the regularity condition for F follows from Proposition 14.

Note that the proof is correct also for some categories other than $\mathscr{FM}$, for example our proposition also holds for functors on $\mathscr{PB}(G)$.

Definition 11. A fibre functor on $\mathscr{C}$ is said to be of order r if for any $\mathscr{C}$ – morphisms $f, g: Y \to W$, $j^r_x f = j^r_x g$ implies $Ff \mid F_x Y = Fg \mid F_x Y$.

Remark 11. Consider a subcategory $\mathscr{C} \subset \mathscr{FM}$. Given a positive integer r, an equivalence $\sim_r$ on $\mathscr{SC}(Y, W) \times FY$ is defined as follows: $(f, y) \sim_r (g, z)$ iff $y = z$, $j^r_{\mathscr{F}\pi(y)} f = j^r_{\mathscr{F}\pi(y)} g$. There is a quotient object $j^r: \mathscr{SC}(Y, W) \times FY \to (\mathscr{SC}(Y, W) \times FY)|\sim_r$ in $\mathscr{S}$. Given an r-th order fibre functor F on $\mathscr{C}$, the generalized associated map can be factorized by the equivalence $\sim_r$, hence we obtain the induced maps

$\bar{F}_{Y,W}: (\mathcal{S}\mathcal{C}(Y, W) \times FY)|\sim_r \to FW$. By Remark 9 and by the properties of quotient objects, these maps are smooth. By Remark 10, given a functor F which satisfies the assumptions of Proposition 13, F is of order r iff the maps $\Lambda_{(M,m;N,n)}$ can be factorized by $\sim_r$.

Remark 12. There is a bijection χ between $(\mathcal{S}\mathcal{C}(Y, W) \times FY)|\sim_r$ and $(J^r\mathcal{C}(Y, W), \alpha, X) \oplus (FY, \mathcal{F}\pi, X)$ defined by $\chi(j^r_{\mathcal{F}\pi(y)}f, y) = j^r(f, y)$. Hence we also have the induced maps $F_{Y,W}: (J^r\mathcal{C}(Y, W), \alpha, X) \oplus (FY, \mathcal{F}\pi, X) \to FW$, defined by $F_{Y,W} = \bar{F}_{Y,W} \circ \chi$.

Definition 12. Given a fibre functor F of order r on $\mathcal{C}$, the maps $F_{Y,W}: (J^r\mathcal{C}(Y, W), \alpha, X) \oplus (FY, \mathcal{F}\pi, X) \to FW$ defined in Remark 12 will be called *the associated maps of the r-th order fibre functor F.*

Definition 13. The subcategory $\mathcal{C} \subset \mathcal{F}\mathcal{M}$ is called *r-admissible* if for all $Y, W \in \mathrm{Ob}\mathcal{C}$ and for every curve c in $J^r\mathcal{C}(Y, W)$ there locally exists a curve d in $\mathcal{S}\mathcal{C}(Y, W)$ such that $c(t) = j^r_{\alpha\circ c(t)}d(t)$ holds.

Proposition 15. *Given an r-th order fibre functor F on an r-admissible category $\mathcal{C} \subset \mathcal{F}\mathcal{M}$, the associated maps of the functor F are smooth.*

Proof. Since $\mathcal{C}$ is r-admissible, the bijection χ defined in Remark 12 is an $\mathcal{S}$ – isomorphism. Actually, by Proposition 6 and Definition 13, curves are prolonged to curves by χ. Since the map $(f, y) \mapsto (j^r_{\mathcal{F}\pi(y)}f, y)$ is smooth (see Proposition 6), the map χ^{-1} is smooth as well. Proposition 15 follows from Remark 11.

Remark 13. By Proposition 6, the whole category $\mathcal{F}\mathcal{M}$ is r-admissible for all $r \in N$, hence the associated maps of an r-th order fibre functor on $\mathcal{F}\mathcal{M}$ are smooth. Since the subcategories $\mathcal{M} \subset \mathcal{F}\mathcal{M}$, $\mathcal{M}_m \subset \mathcal{F}\mathcal{M}$ (see Remark 8) are obviously r-admissible for all $r \in N$, we obtain as a consequence of the above considerations the well known results that the associated maps of r-th order lifting functors and prolongation functors are smooth, see [2], [7]. There are also other categories which are r-admissible for all $r \in N$. For example, the categories $\mathcal{P}\mathcal{B}(G)$, $\mathcal{P}\mathcal{B}_m(G)$, $\mathcal{V}\mathcal{B}$, $\mathcal{V}\mathcal{B}_m$, $\mathcal{V}\mathcal{B}(V)$, $\mathcal{V}\mathcal{B}_m(V)$. This can be proved by methods similar to those in Example 1 and Example 2.

7. A DESCRIPTION OF FIBRE FUNCTORS

Let $\mathcal{F}\mathcal{M}(S, m)$ denote the category where the objects are fibred manifolds with a standard fibre S and an m-dimensional base, and the morphisms are such morphisms from $\mathcal{F}\mathcal{M}_m$ the restriction of which to the individual fibres are diffeomorphisms. We shall write $\mathcal{F}\mathcal{M}(S, m) = \mathcal{C}$ in the sequel. The following construction is a generalization of the description of all lifting functors [2], [7], [14], (the classical result corresponds to the case when the standard fibre is a one-element set).

First consider an r-th order fibre functor F on $\mathcal{C}$, the associated map of which is smooth. This associated map of F defines a smooth action φ of $L^r\mathcal{C}(S, m)$ on

$M := F_0(\boldsymbol{R}^m \times S)$ by $\varphi(j_0^r\Phi, y) = F\,\Phi(y)$. Conversely, given a manifold M with a smooth left action of $\boldsymbol{L}^r\mathscr{C}(S, m)$, we can define an r-th order fibre functor F on $\mathscr{C}$ as follows. Given $Y, W \in \mathrm{Ob}\mathscr{C}$, $f\colon Y \to W \in \mathrm{Mor}\mathscr{C}$, we set $\boldsymbol{H}^r\mathscr{C}f\colon \boldsymbol{H}^r\mathscr{C}Y \to \boldsymbol{H}^r\mathscr{C}W$, $\boldsymbol{H}^r\mathscr{C}Y \ni j_0^r\Phi \mapsto j_0^r(f \circ \Phi) \in \boldsymbol{H}^r\mathscr{C}W$. Now we set $FY = \boldsymbol{H}^r\mathscr{C}Y(M)$, $Ff = (\boldsymbol{H}^r\mathscr{C}f, \mathrm{id}_M)\colon \boldsymbol{H}^r\mathscr{C}Y(M) \to \boldsymbol{H}^r\mathscr{C}W(M)$.

Proposition 16. *There is a natural bijective correspondence between the set of all left smooth actions of $\boldsymbol{L}^r\mathscr{C}(S, m)$ on smooth manifolds and the set of all r-th order fibre functors on $\mathscr{C}$, the associated map of which is smooth. More exactly, if φ is an action of $\boldsymbol{L}^r\mathscr{C}(S, m)$ on $F_0(\boldsymbol{R}^m \times S)$ induced by a fibre functor F and G is the fibre functor obtained from this action by the above construction then F and G are naturally equivalent.*

Proof. We only have to find such isomorphisms ψ_Y for all $Y \in \mathrm{Ob}\mathscr{C}$ that the diagram commutes for all $Y, W \in \mathrm{Ob}\mathscr{C}$, $f\colon Y \to W \in \mathrm{Mor}\mathscr{C}$.

$$\begin{array}{ccc} FY & \xleftarrow{\psi_Y} & \boldsymbol{H}^r\mathscr{C}Y(M) \\ \downarrow{\scriptstyle Ff} & & \downarrow{\scriptstyle (\boldsymbol{H}^r\mathscr{C}f,\, \mathrm{id}_M)} \\ FW & \xleftarrow{\psi_W} & \boldsymbol{H}^r\mathscr{C}W(M) \end{array}$$

Consider an object $Y \in \mathrm{Ob}\mathscr{C}$. Given $A = j_0^r\varphi \in \boldsymbol{H}^r\mathscr{C}Y$, we have a map $FA\colon M \to F_{\beta(A)}Y$, $FA(y) = F\varphi(y)$. Moreover, since $A^{-1} = j_0^r\bar{\varphi}$, we have $F(\bar{\varphi} \circ \varphi) \mid M = \mathrm{id}_M$, $F(\varphi \circ \bar{\varphi}) \mid F_{\beta(A)}Y = \mathrm{id}_{F_{\beta(A)}Y}$. Hence FA is a bijection. Since $FA(y) = FA(FB \circ FB^{-1}(y)) = F(A \circ B)(FB^{-1}(y))$, there is a map $\psi_Y\colon \boldsymbol{H}^r\mathscr{C}Y(M) \to FY$, $(A, y) \mapsto FA(y)$. Suppose $FA_1(y_1) = FA_2(y_2)$. Then $\beta(A_1) = \beta(A_2)$, so that there exists a $B \in \boldsymbol{L}^r\mathscr{C}(S, m)$, $A_1 = A_2 \circ B$. We have $F(A_2 \circ B)(y_1) = FA_2(FB(y)) = FA_2(y_2)$, hence $y_1 = B^{-1} \circ y_2$. That is why the map $\bar{\psi}_Y\colon FY \to \boldsymbol{H}^r\mathscr{C}Y(M)$, $FA(y) \mapsto (A, y)$ is well defined and $\bar{\psi}_Y = \psi_Y^{-1}$, so that ψ_Y is a bijection. It can be easily seen that ψ_Y is an $\mathscr{S}$ – isomorphism. The commutativity of the diagram is obvious, QED.

Remark 14. The above construction and Proposition 16 also apply to some categories other than $\mathscr{FM}_m(S, m)$. For example, they are also correct for $\mathscr{PB}_m(G)$ and $\mathscr{VB}_m(V)$ (vector bundles with a standard fibre V, an m-dimensional base and morphisms are over $\mathscr{M}_m$ – morphisms and are diffeomorphic on individual fibres). Since these two categories are admissible, the smoothness of the associated maps in this case follows from Proposition 15.

Remark 15. A description of all r-th order fibre functors on $\mathscr{FM}$ is also possible. This can be done in a similar way as for the prolongation functors, see [7] and the description above. We must choose one representative M of every class $[M]$ of diffeomorphic manifolds. Then we can define a category $\mathscr{L}^r$, the objects of which are the pairs m, $[M]$, $m \in \boldsymbol{N}$, $\mathrm{Mor}\mathscr{L}^r(m, M; n, N) = \boldsymbol{J}_0^r(\boldsymbol{R}^m \times M, \boldsymbol{R}^n \times N)_0$, and the composition is given by the composition of fibre jets. An action of the category $\mathscr{L}^r$ on a system of manifolds $S(M, m)$ for all $m \in \boldsymbol{N}$ and all representatives M means a system of smooth maps $A_{(M,m;N,n)}\colon \boldsymbol{J}_0^r(M_m, N_n)_0 \times S(M, m) \to S(N, n)$ satisfying

$\Lambda_{(M,m;N,n)}(A, \Lambda_{(P,p;M,m)}(B, s)) = \Lambda_{(P,p;N,n)}(A \circ B, s)$. There is a bijective correspondence between the fibre functors on $\mathcal{FM}$ and the systems of manifolds with an action of $\mathcal{L}^r$, if we identify naturally isomorphic functors.

References

[1] *J. Boman:* Differentiability of a function and of its composition with functions of one variable, Math. Scand. *20*, 1967, p. 249—268.

[2] *D. B. A. Epstein, W. P. Thurston:* Transformation groups and natural bundles, Proc. London Math. Soc. III, *38* (1979), 219—238.

[3] *A. Dekret:* Prolognation of natural bundles, to appear.

[4] *A. Frölicher:* Durch Monoide erzeugte kartesisch abgeschlossene Kategorien, Seminarber. Fachbereich Math. Fernuniv. Hagen *5* (1979), pp. 7—48.

[5] *A. Frölicher:* Smooth structures, Category theory (Gummersbach, 1981), pp. 69—81, Lecture Notes in Math., 962, Springer, Berlin—New York, 1982.

[6] *A. Frölicher:* Cartesian closed categories and analysis of smooth maps, Proceedings of the workshop on categories and foundation of continuum physics, Buffalo, May 17—21, 1982, to appear.

[7] *J. Janyška:* Geometrical properties of prolongation functors, Čas. pěst. mat., *110* (1985), 77—86.

[8] *J. Janyška:* Geometrical prolongation functors and natural operations with connections, (in Czech), Ph. D. thesis.

[9] *I. Kolář:* Canonical forms on the prolongations of principal fibre bundles, Rev. Roumaine math. Pures Appl., 1971, *16*, 1091—1106.

[10] *I. Kolář:* Prolongations of generalized connections, Colloquia mathematica societatis János Bolyai, 31. Differential Geometry, Budapest 1979, pp. 317—325.

[11] *I. Kolář:* Higher order absolute differentiation with respect to generalized connections, Differential geometry, Banach center publications, Vol. *12*, Warsaw 1984, pp. 153—161.

[12] *D. Krupka:* Elementary theory of differential invariants, Arch. Math. (Brno), *XIV* (1978), 207—214.

[13] *A. Nijenhuis:* Natural bundles and their general properties, Diff. Geom. in honour of K. Yano, Tokyo 1972, 317—334.

[14] *R. S. Palais, C. L. Terng:* Natural bundles have finite order, Topology *16* (1977), 271—277.

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