SEMIHOLONOMIC JETS AND INDUCED MODULES IN CARTAN GEOMETRY CALCULUS

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Abstract. The famous Erlangen Programme was coined by Felix Klein in 1872 as an algebraic approach allowing to incorporate fixed symmetry groups as the core ingredient for geometric analysis, seeing the chosen symmetries as intrinsic invariance of all objects and tools. This idea was broadened essentially by Elie Cartan in the beginning of the last century, and we may consider (curved) geometries as modelled over certain (flat) Klein's models.

The aim of this short survey is to explain carefully the basic concepts and algebraic tools built over several recent decades. We focus on the direct link between the jets of sections of homogeneous bundles and the associated induced modules, allowing us to understand the overall structure of invariant linear differential operators in purely algebraic terms. This allows us to extend essential parts of the concepts and procedures to the curved cases.

These notes go back to much earlier collaborative works of the authors, in particular with A. Cap and M.G. Eastwood, cf. [15, 10]. The reader may also see it as an extension of the recent notes [29] focusing on the tractor calculi and BGG machinery from a quite different perspective.

Linearized physical theories can be often viewed as complexes of linear differential operators (and the laws of Physics are then modeled as the equality of kernels and ranges of such operators), cf. [6, 12] and the references therein. The expected symmetries of the theory enforce the operators to commute with them, thus, such operators have to be *invariant*.

Our aim is to explain concepts allowing to discuss invariant linear differential operators effectively. In the homogeneous case, this will become a very algebraic story, which we then (partially) extend to the curved geometries. The main ideas for that can be traced back to [13, 2, 15]. In large extent, we adopt the language and notation from [15]. We believe that the reader will enjoy the power of the Cartan connections on our journey.

If necessary, more background on functorial geometric constructions, Klein geometries, and Cartan geometries can be found in [20, 8, 26], while the representation theory can be checked with [30]. We shall work in the category of smooth finite dimensional manifolds here.

1. The algebraic story of the Klein geometries

1.1. Klein geometries and homogenous bundles. A Klein geometry is a manifold M with a transitive smooth action of a Lie group G. Choosing a point $O \in M$, there is the isotropy subgroup H of this point and the identification $M = G/H$. Up to a choice of the origin O, all Klein geometries are such homogeneous spaces G/H.

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At the infinitesimal level, the quotient of the Lie algebras $\mathfrak{g}/\mathfrak{h}$ clearly is naturally identified with the tangent space $T_O M$ at the origin.

Notice that $G \to G/H$ is a principal H-bundle, and G comes equipped with the Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$, the prototype of Cartan connections.

Next, consider any linear representation $\mathbb{E}$ of H and the associated bundle $\mathcal{E} = G \times_H \mathbb{E}$, i.e., the classes of the equivalence relations on $G \times \mathbb{E}$ given by $(u, v) \sim (u{\cdot}h, h^{-1}{\cdot}v)$ for all $h \in H$. The tangent and cotangent bundles TM and T^*M are nice examples with the H-modules $\mathfrak{g}/\mathfrak{h}$ and $(\mathfrak{g}/\mathfrak{h})^*$, and notice how the Maurer-Cartan form provides the identifications.

In the special case when $\mathbb{E}$ happens to be a G-module (and we consider the restriction of the action to $H \subset G$), we may identify the class represented by (u, v) with the couple $(u{\cdot}H, u{\cdot}v)$. Indeed, taking another representative leads to

$$((u{\cdot}h){\cdot}H, (u{\cdot}h){\cdot}(h^{-1}{\cdot}v)) = (u{\cdot}H, u{\cdot}v).$$

Thus, we have verified that $\mathcal{E}$ is the trivial bundle $\mathcal{E} = M \times \mathbb{E}$ over $M = G/H$ for all G-modules $\mathbb{E}$.

On the other hand, *homogeneous (vector) bundles* over a Klein geometry M are the bundles with well defined G-actions by (vector) bundle morphisms. Clearly, for each such bundle, the restriction of the action to the isotropy group and the fiber over O provides the H-module $\mathbb{E}$ and the original homogeneous bundle is then identified with $\mathcal{E} = G \times_H \mathbb{E}$. Moreover, H-module morphisms lead to vector bundle morphisms between the homogenous bundles in the obvious way.

In other words, for each Klein geometry $M = G/H$, we have constructed a functor from the category of H-modules to the category of homogenous bundles over $M = G/H$ with the obvious action on morphisms.

Extending $G \to G/H$ to the principle G-bundle $\tilde{G} = G \times_H G \to G/H$, the Maurer-Cartan form ω uniquely extends to a principal connection form $\tilde{\omega}$ on $\tilde{G}$.

Finally, for G-modules $\mathbb{T}$ we can further identify $\mathcal{T}$ as the associated space $\mathcal{T} = \tilde{G} \times_G \mathbb{T} \simeq M \times \mathbb{T}$ and we see that there is the induced linear connection ∇ on all such bundles $\mathcal{T}$. These very special homogenous bundles are called *tractor bundles*.[1]

1.2. Sections of homogenous bundles and jet bundles. A global version of writing sections $\sigma \in \Gamma(\mathcal{E})$ of homogeneous bundles in coordinates views the sections as functions $\tilde{\sigma} : G \to \mathbb{E}$ (by abuse of notation, later we shall use the same letter σ for both), which have to be H-equivariant, i.e., $\tilde{\sigma}(u{\cdot}h) = h^{-1}{\cdot}\tilde{\sigma}(u)$. Indeed, such a function defines the section σ with its values $\sigma(u{\cdot}H)$ represented by $(u, \tilde{\sigma}(u))$. Obviously, this is a well defined bijection between $\Gamma(\mathcal{E})$ and $C^\infty(G, \mathbb{E})^H$.

The (left) G-action ℓ_g on the homogeneous bundles induces, of course, the action on the sections: $(g{\cdot}\sigma)(u{\cdot}H) = \ell_g \circ \sigma \circ \ell_{g^{-1}}(u{\cdot}H)$, which means that in the other picture, the action is $g{\cdot}\sigma = \sigma \circ \ell_{g^{-1}}$, which again produces H-equivariant functions on G.

Next, let us have a look at the jet-prolongations $J^k\mathcal{E}$ of our homogenous bundles $\mathcal{E} = G \times_H \mathbb{E}$. The G-action on sections projects to the G-action on $J^k\mathcal{E}$, so that they are again homogenous bundles and let us call the standard fiber $J^k\mathbb{E} = (J^k\mathcal{E})_O$ over the origin O the *k-jet prolongation* of the H-module $\mathbb{E}$.

There is the straightforward observation:

[1] These special bundles were traced back to Tracy Thomas, who introduced them when searching for generalizations of tensor bundles suitable for conformal Riemannian geometry, see [1]. In private communication with M.G. Eastwood, the authors of this note heard that the name *tractor* illustrates the fact that traction comes after tension, and also the similarity with the name of the first inventor.

1.3. Proposition. *The invariant linear differential operators $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$, of order at most k, are in bijective correspondence with the H-module homomorphisms $J^k\mathbb{E} \to \mathbb{F}$.*

Proof. Clearly, evaluating the values of an invariant differential operator D at the origin O, $D(\sigma)(O)$ depends on the k-jet $j^k_O\sigma$ only. By restricting the invariance to $H \subset G$, we obtain the requested module homomorphism by the linearity of D.

Vice versa, linear differential operators of order at most k coincide with morphisms between the homogeneous vector bundles $J^k\mathcal{E}$ and $\mathcal{F}$, and those are in bijection with the module homomorphisms. □

Although the latter observation looks promising, $J^k\mathbb{E}$ is a horrible representation of H, even if $\mathbb{E}$ was nice, e.g., irreducible. Thus, in general, we can hardly find and discuss the operators easily this way. Exceptionally, the case $k = 1$ might be discussed directly for large classes of Klein geometries, cf. [27].

1.4. Induced modules. A better way to understand invariant linear differential operators was suggested very long ago, see e.g., [19, 21], and the references therein. The point is that understanding embeddings of nice modules into complicated ones might be much easier than looking for morphisms in the original direction. Thus we look at the dual picture.

The elements X of the Lie algebra $\mathfrak{g}$ are identified with the left invariant vector fields $\omega^{-1}(X) \in \mathcal{X}(G)$. Differentiating the H-equivariant functions $\sigma : G \to \mathbb{E}$ in the direction of $X \in \mathfrak{g}$ in the unit $e \in G$ corresponds to derivatives of the sections. More precisely, if $X \in \mathfrak{h}$, then $(X{\cdot}\sigma)(u) = -X{\cdot}(\sigma(u))$ by the equivariance and, thus, the genuine differential parts are in the quotient $\mathfrak{g}/\mathfrak{h}$, thus corresponding to derivatives of the sections in directions in T_OM.

Now, consider a "word" $X_1X_2 \dots X_k$ of elements in $\mathfrak{g}$ and the corresponding differential operator $\sigma \mapsto \omega^{-1}(X_1) \circ \omega^{-1}(X_2) \circ \dots \circ \omega^{-1}(X_k){\cdot}\sigma(e)$ on the functions.

We may consider this operation as defined on the tensor algebra $T(\mathfrak{g})$ and obviously the entire ideal in $T(\mathfrak{g})$ generated by the expressions $X \otimes Y - Y \otimes X - [X, Y]$, with $X, Y \in \mathfrak{g}$ and $[X, Y]$ their Lie bracket, must act trivially.

The resulting quotient (left and right) $\mathfrak{g}$-module $\mathfrak{U}(\mathfrak{g}) = T(\mathfrak{g})/\langle X \otimes Y - Y \otimes X - [X, Y]\rangle$ is called the *universal enveloping algebra* of the Lie algebra $\mathfrak{g}$.

We would like to understand the linear forms on the jet modules $J^k\mathbb{E}$. So far we differentiate functions also in the vertical directions, and our values are in $\mathbb{E}$. Thus we should consider the tensor product

$$V(\mathbb{E}) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{h})} \mathbb{E}^*.$$

The space $V(\mathbb{E})$ clearly enjoys the structure of a $(\mathfrak{g}, H)$-module (and $(\mathfrak{U}(\mathfrak{g}), H)$-module), and it is called the *induced module* for the H-module $\mathbb{E}$.

1.5. Proposition. *The induced module $V(\mathbb{E})$ is the space of all linear forms on $J^\infty\mathbb{E}$ which factor through some $J^k\mathbb{E}$, i.e., depend on finite number of derivatives.*

Proof. The claim follows from the construction of $V(\mathbb{E})$ and the fact that choosing a complementary vector subspace to $\mathfrak{h}$ in $\mathfrak{g}$, we can decompose all letters in our words $X_1 \dots X_k$ above and, by the equalities enforced by living in the quotient by the ideal, we may "bubble" the letters in $\mathfrak{h}$ to the very right. Once there, they act algebraically and, thus, tensorizing over $\mathfrak{U}(\mathfrak{h})$ we remove just all redundancies.

The reader might consult [30] for more details. □

Obviously, $\mathbb{E}^*$ injects into $V(\mathbb{E})$ and generates this $\mathfrak{g}$-module. Now, we may enjoy a small but extremely important miracle:

1.6. Theorem (Frobenius reciprocity)**.** *For all finite dimensional representations $\mathbb{E}$ and $\mathbb{F}$ of H, there are the canonical isomorphisms*

$$\mathrm{Hom}_H(\mathbb{F}^*, V(\mathbb{E})) = \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),H)}(V(\mathbb{F}), V(\mathbb{E})).$$

Proof. If we are given a homomorphism $\Phi \in \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),H)}(V(\mathbb{F}), V(\mathbb{E}))$, we simply define $\varphi : \mathbb{F}^* \to V(\mathbb{E})$ by restriction.

On the other hand, having a $\varphi \in \mathrm{Hom}_H(\mathbb{F}^*, V(\mathbb{E}))$, we first define for all $x \in \mathfrak{U}(\mathfrak{g})$ and $v \in \mathbb{F}^*$,

$$\Phi(x \otimes v) = x \otimes_{\mathfrak{U}(\mathfrak{h})} \varphi(v),$$

which extends linearly, if well defined. To check this, notice that for all $X \in \mathfrak{h}$ and $v \in \mathbb{F}^*$,

$$\Phi(X \otimes v - 1 \otimes X{\cdot}v) = X \otimes \varphi(v) - 1 \otimes \varphi(X{\cdot}v) = X \otimes \varphi(v) - 1 \otimes X{\cdot}\varphi(v),$$

which completes the proof. $\square$

2. The translation principle

2.1. Parabolic Klein models. In the rest of the paper, we shall restrict to a large class of geometries modelled over the Klein geometries G/P with G semisimple and $P \subset G$ parabolic. Let us recall that P is called parabolic if it contains a Borel subgroup in G.

We expect the reader knows the elements of the structure theory of the semisimple Lie groups, the root spaces, the Weyl group, etc. Consult [30] or [8, Chapter 2] if necessary.

At the level of Lie algebras, the parabolic subalgebras are those which contain a Borel subalgebra. The choices of all parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ correspond to graded decompositions of the semisimple Lie algebras

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k = \mathfrak{g}_- \oplus \mathfrak{p}. \tag{1}$$

Thus, the Lie brackets satisfy $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+ = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ is the decomposition of the parabolic subalgebra $\mathfrak{p}$ into the reductive Levi quotient $\mathfrak{l} = \mathfrak{g}_0$ and the nilradical $\mathfrak{p}_+$. There is also the subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \ldots \cdots \oplus \mathfrak{g}_{-1}$ complementary to $\mathfrak{p}$, which is the dual to $\mathfrak{p}_+$ with respect to the Cartan-Killing form on $\mathfrak{g}$.

For a given semisimple $\mathfrak{g}$, the above gradings with isomorphic parabolic subalgebras $\mathfrak{p} \subset \mathfrak{g}$ are given uniquely, up to conjugation in $\mathfrak{g}$. They are also uniquely determined by the *grading elements* E, i.e., E with the property $[E, X] = jX$ for all $X \in \mathfrak{g}_j$. Obviously, E is in the center of $\mathfrak{g}_0$, which is identified with $\mathfrak{l} = \mathfrak{p}/\mathfrak{p}_+$.

The closed Lie subgroups $P \subset G$ are parabolic if and only if their algebras $\mathfrak{p} = \mathrm{Lie}\, P$ are parabolic.

If G is a complex semisimple Lie group, then there is a nice geometric description: $P \subset G$ is parabolic if and only if G/P is a compact manifold (and then it is a compact Kähler projective variety), see e.g., [32, Section 1.2]. In the real setting, the so called generalized flag varieties G/P with parabolic P are always compact, too.

We talk about $|k|$-graded Klein models G/P. For the complex semisimple algebras, the Borel subalgebras are generated by Cartan subalgebras in $\mathfrak{g}$ and all simple positive co-roots α_i in $\mathfrak{g}$. The parabolic subalgebras $\mathfrak{p}$ then correspond to the subsets of the co-roots α_i, for which $-\alpha_i$ do not belong to $\mathfrak{p}$. In the language of the Dynkin diagrams, this can be nicely encoded by crossing the nodes related to negative simple co-roots in $\mathfrak{g}_-$.

In the real situation, $\mathfrak{p} \subset \mathfrak{g}$ is parabolic, if the same holds true for the complexification. The classification is more subtle here, but it can be nicely encoded by the Satake diagrams with crossed nodes being allowed only for the white ones, and if one of the nodes joined by an arrow is crossed, then the other one has to be crossed, too (see [8, Chapter 2] for detailed discussion).

The induced modules $V(\mathbb{E})$ are called (generalized) *Verma modules* and they enjoy a very rich and well understood structure theory, see e.g., [22, 3, 4].

We shall present a brief selection of tools and results from this theory, preparing our approach to invariant differential operators on curved geometries.

A panopticum of examples of geometries modelled on parabolic Klein geometries can be found in [8, Chapter 4], including projective, conformal Riemannian, CR, and many others. Similarly to the survey [29], we shall focus on a few $|1|$-graded examples here.

We shall see that the invariant linear operators appear in isomorphic patterns and the de Rham complex of differential operators on the algebra of differential forms (decomposed into irreducible components) is a prototype of all of them. We shall try to explain the general approach in a "learning by doing" way and we go through one line of examples of the geometries and patterns only.

2.2. Grassmannian examples. Let us work with the real split forms of the type A algebras, i.e., $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$. There the $|1|$-graded cases correspond to choices of just one crossed node, say the pth one, and $\mathfrak{g}_{-1} = \mathbb{R}^q \otimes (\mathbb{R}^p)^*$, with $p + q = n$. The Klein geometries are the so-called (p, q)-Grassmannians, i.e., the spaces of p-planes through origin in $\mathbb{R}^{p+q}$. The structure of the graded $\mathfrak{g}$ is nicely seen in the blockwise scheme of all trace-free matrices $X \in \mathfrak{g}$:

$$\mathfrak{g} \simeq \begin{pmatrix} 0 & 0 \\ \mathfrak{g}_{-1} & 0 \end{pmatrix} \oplus \mathfrak{z} \oplus \begin{pmatrix} \mathfrak{sl}(p, \mathbb{R}) & 0 \\ 0 & \mathfrak{sl}(q, \mathbb{R}) \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathfrak{g}_1 \\ 0 & 0 \end{pmatrix},$$

where $\mathfrak{z}$ is the one-dimensional center of the Levi factor $\mathfrak{g}_0$ (generated by the grading element E), while the rest of $\mathfrak{g}_0$ is its semisimple part. Obviously, the grading element is built of constant multiples of identity matrices in the diagonal blocks, adjusted depending on p and q.

The case $p = 1$ provides the projective spaces $\mathbb{RP}_{n-1}$. We shall always assume $1 \le p \le q$. In the small dimensions and in the language of the Dynkin diagrams, the 2-dimensional projective space is drawn as $\times\!\!-\!\!\bullet$, while our Klein geometries of, $(2, 2)$, $(2, 3)$, and $(3, 3)$ Grassmannians are encoded as

$$\bullet\!-\!\times\!-\!\bullet \qquad \bullet\!-\!\times\!-\!\bullet\!-\!\bullet \qquad \bullet\!-\!\bullet\!-\!\times\!-\!\bullet\!-\!\bullet$$

Notice, that $\mathfrak{sl}(4, \mathbb{C}) = \mathfrak{so}(6, \mathbb{C})$ and this explains how the case of $(2, 2)$-Grassmannians corresponds to the Roger Penrose's complexified model of the Universe (i.e., the four-dimensional conformal Riemannian geometry of the Minkowski space, but in the complexified form).

As well known, all representations of a semisimple Lie algebra are completely reducible and the irreducible ones can be all built from the so called fundamental weights, which form a base of the dual of the Cartan subalgebra in $\mathfrak{g}$, and we can encode them by writing one over the respective node in the Dynkin diagram and zeros on the rest. All irreducible representations are then given by the linear combinations of the fundamental ones with non-negative integral coefficients. These so-called *dominant weights* provide us with the (homogeneous) irreducible tractor bundles, i.e., those homogeneous vector bundles defined by irreducible G-modules.

2.3. The completely reducible homogenous bundles. Let us continue with our Grassmannian example. All irreducible $\mathfrak{p}$-modules are obtained as (outer) tensor products of irreducible representations of the two semisimple components in $\mathfrak{g}_0$, together with the action of the center, and the trivial action of $\mathfrak{g}_1$. The simplest nontrivial modules include those with trivial actions of the center and the second $\mathfrak{g}_0$ component, thus, defined by irreducible actions of $\mathfrak{sl}(p, \mathbb{R})$. The standard representation $\mathbb{R}^p$ leads to the homogenous bundle $\mathcal{E}^A$, while its dual defines the homogenous bundle $\mathcal{E}_A$. Similarly, if only the $\mathfrak{sl}(q,\mathbb{R})$ component acts nontrivially, the standard representation will give rise to the homogenous bundle $\mathcal{E}^{A'}$, and its dual will be $\mathcal{E}_{A'}$. The tangent bundle $\mathcal{E}_A^{A'}$ and cotangent bundle $\mathcal{E}_{A'}^A$ come from the tensor products of these bundles. Notice, this is exactly the Penrose's abstract index notation with the spinor indices, extended to general Grassmannians. Finally, the one-dimensional modules $\Lambda^p(\mathbb{R}^p) \simeq \Lambda^q(\mathbb{R}^{q*})$ are coming from actions of the center $\mathfrak{z}$ only. We call such bundles the *weight bundles* $\mathcal{E}[w]$, and we adopt the usual normalization taking

$$\mathcal{E}^{[AB\ldots C]} = \mathcal{E}[-1] = \mathcal{E}_{[A'B'\ldots C']}$$
$$\mathcal{E}_{[AB\ldots C]} = \mathcal{E}[1] = \mathcal{E}^{[A'B'\ldots C']},$$

where we put p indices on the left and q indices on the right, and [ ] or ( ) on indices mean antisymmetrization or symmetrization, respectively. The general weight bundles with integral weights are the tensor powers $\mathcal{E}[w] = \mathcal{E}[1]^w$ for positive w and $\mathcal{E}[w] = \mathcal{E}[-1]^{-w}$ for negative ones. With some care, we may extend this to real weights w.

It is easy to encode the irreducible $\mathfrak{p}$ modules by weights of the entire $\mathfrak{g}$. Indeed, the highest weights of such $\mathfrak{p}$-modules are those integral linear combinations of the fundamental weights of $\mathfrak{g}$ (recall the fundamental weights correspond to exterior forms $\Lambda^k\mathbb{R}^{p+q}$, $k = 1, \ldots, p+q-1$), whose coefficients must by non-negative, except the one over the crossed node. The coefficients over the uncrossed nodes encode the representations of the two semisimple components in $\mathfrak{g}_0$, while the coefficient over the crossed node completes the information about the action of the center $\mathfrak{z}$. We call such weights $\mathfrak{p}$-*dominant*.

In our case, all our irreducible homogeneous vector bundles live in the tensor bundles $\mathcal{E}_{E\ldots FG'\ldots H'}^{A\ldots BC'\ldots D'} \otimes \mathcal{E}[w]$ which we usually write as $\mathcal{E}_{E\ldots FG'\ldots H'}^{A\ldots BC'\ldots D'}[w]$, and there is a straightforward algorithm, how to compute the action of the grading element on the corresponding modules.[2]

First, we adopt another encoding for the weights. Writing e_i for the diagonal matrix with just one 1 entry at the ith place and zero otherwise, the dual basis to the fundamental weights corresponds to $e_i - e_{i+1}$ and we may replace the $(n-1)$-tuples $(\alpha_1, \ldots, \alpha_{n-1})$ of integers by n-tuples $(a_1, \ldots, a_n)$, so that $\alpha_i = a_i - a_{i+1}$, and we indicate the $p+q$ blockwise structure by adding a vertical bar at the proper place. Requesting the dominant weights to be non-negative integral then means $a_1 \geq a_2 \geq \cdots \geq a_n$, while for $\mathfrak{p}$-dominant weights we request this for the first p-tuple and last q-tuple separately. Of course, these longer vectors are unique for the weights, up to a common constant only. Thus, we usually normalize the choice, e.g., we may request $a_n = 0$.

Next, we write down the $\mathfrak{p}$-dominant weights corresponding to enough $\mathfrak{p}$-modules with known action of E (e.g., $\mathfrak{g}$ and the fundamental representations the two semisimple components of $\mathfrak{g}_0$), and compute which linear formula provides the right action of E on them. In our three examples displayed above, the grading elements acts on the modules determined by the coefficients a_i by the constants:

$$(2)\quad \tfrac{1}{2}(a_1+a_2-a_3-a_4) \quad \tfrac{3}{5}(a_1+a_2)-\tfrac{2}{5}(a_3+a_4+a_5) \quad \tfrac{1}{2}(a_1+a_2+a_3-a_4-a_5-a_6)$$

clearly independent of our normalization.

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[2] In general, the action of the grading element of a parabolic subalgebra $\mathfrak{p}$ in semisimple $\mathfrak{g}$ is computed as the (sum of) scalar product(s) of the lines in the inverse Cartan matrix of $\mathfrak{g}$ corresponding to the crossed nodes, with the coefficients of the highest weight (in the expression via the fundamental weights), cf. [8, Section 3.2.12].

Moreover, there is the so-called lowest weight ρ being the sum of all the fundamental ones (i.e., with coefficient one over each node). For good reasons which we shall see below, we shall add ρ to our weights when encoding them. Thus the trivial representation will be written as $((n-1)\ (n-2)\ \dots\ (n-p) \mid (n-p-1)\ \dots\ 0)$. In particular, in our three examples we get the three vectors

$$(3\ 2 \mid 1\ 0) \qquad (4\ 3 \mid 2\ 1\ 0) \qquad (5\ 4\ 3 \mid 2\ 1\ 0).$$

Using the same formulae for the action of the center will simply add the constant $\frac{1}{2}pq$ corresponding to the action on the lowest weight.

The n-tuples for $\mathfrak{g}$-dominant weights in this encoding satisfy $a_1 > a_2 > \cdots > a_{n-1} > a_n = 0$. The $\mathfrak{p}$-dominant ones request only $a_1 > a_2 > \cdots > a_p$ and $a_{p+1} > \cdots > a_n$.

2.4. Homomorphisms between Verma modules. We continue working with semisimple $\mathfrak{g}$ and parabolic $\mathfrak{p}$. Consider two $\mathfrak{p}$-modules $\mathbb{E}$, $\mathbb{F}$ and the corresponding homogeneous bundles $\mathcal{E}$ and $\mathcal{F}$ (we shall not care about the choice of the Lie groups now).

As obvious from Propositions 1.3 and 1.5, and Theorem 1.6, all invariant linear differential operators $\Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ are uniquely defined by $(\mathfrak{U}(\mathfrak{g}), P)$-homomorphisms $\Phi : V(\mathbb{F}) \to V(\mathbb{E})$.

The Verma modules $V(\mathbb{E})$ are always equipped by an obvious filtration

$$\mathbb{R} \subset (\mathbb{R} \oplus \mathbb{E}^*) = V_1(\mathbb{E}) \subset V_2(\mathbb{E}) \subset \cdots \subset V(\mathbb{E}).$$

Since $\mathbb{F}^*$ is finite dimensional, its image under Φ is contained in $V_k(\mathbb{E})$ for some integer $k > 0$. The least k with this property is called the *order* of the morphism Φ. Clearly, this corresponds to the differential order of the corresponding linear differential operator.

We may also translate the concept of the symbol into this dual setup. By virtue of the Poincaré-Birkhoff-Witt theorem, the grading corresponding to the filtration is $\mathrm{gr}(V(\mathbb{E})) = \mathrm{gr}(\mathfrak{U}(\mathfrak{g}_-)) \otimes \mathbb{E}^* = S(\mathfrak{g}_-) \otimes_{\mathbb{R}} \mathbb{E}^*$, (see e.g. [8, Section 2.1.10]). Thus $V(\mathbb{E}) = S(\mathfrak{g}_-) \otimes_{\mathbb{R}} \mathbb{E}$ as a $\mathfrak{g}_0$-module. In particular, there are the short exact sequences

$$0 \to V_{k-1}(\mathbb{E}) \to V_k(\mathbb{E}) \to S^k \mathfrak{g}_- \otimes \mathbb{E} \to 0. \tag{3}$$

Now, the *symbol* of a homomorphism $\Phi : V(\mathbb{F}) \to V(\mathbb{E})$ is defined as

$$\sigma(\Phi) : \mathbb{F}^* \to V_k(\mathbb{E}) \to V_k(\mathbb{E})/V_{k-1}(\mathbb{E}) = S^k(\mathfrak{g}_-) \otimes \mathbb{E}^*,$$

where k is the order of Φ.

The center of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is quite well understood for all semisimple algebras $\mathfrak{g}$. For each Verma module $V(\mathbb{E})$, the restriction of the action to this center is called the *infinitesimal character* of $V(\mathbb{E})$. Clearly, the infinitesimal characters of $V(\mathbb{F})$ and $V(\mathbb{E})$ must coincide if there should be a non-zero homomorphism between them.

Now, another miracle comes: By the famous Harish-Chandra theorem, *two Verma modules $V(\mathbb{E})$ and $V(\mathbb{F})$ have got the same infinitesimal character if and only if their highest weights appear in the same orbit of the affine action of the Weyl group on the space of weights.*

Here, the Weyl group is generated by all reflections defined by the simple roots, and the affine action is this very action applied to the sums of the weights with the lowest weight ρ.

In our Grassmannian examples, all the elements of the Weyl group act just by the permutations of the coefficients in the n-tuple representing the weight. Thus, for our special cases (we add the 2-dimensional projective space), we are getting the following patterns of all $\mathfrak{p}$-dominant weights in the affine orbit of the trivial representation. Notice, we organize the columns by decreasing constant of the action of the grading element, and the order of the homomorphisms between the modules in the neighboring modules in the neighboring columns is always 1.

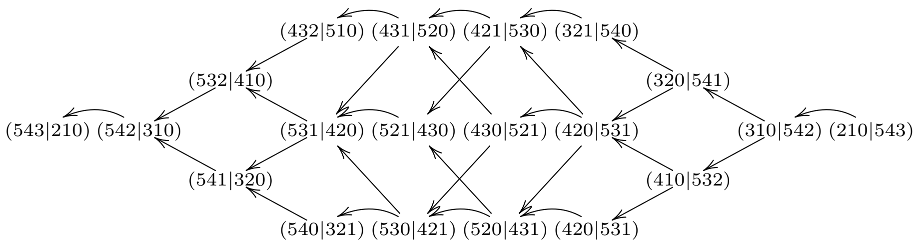
In fact the columns are giving the decompositions of $\Lambda^j(T^*M)$ into irreducible components and the arrows correspond to the restrictions and decompositions of the exterior differential d. Of course, as homomorphisms of the Verma modules, they go in the opposite directions.

$$(2|10) \longleftarrow (1|20) \longleftarrow (0|21) \tag{4}$$

$$\begin{tikzcd}[column sep=tiny]
(32|10) & & (21|30) \arrow[dl] & & (10|32) \arrow[dl] \\
& (31|20) \arrow[ul] & & (20|31) \arrow[ul] \arrow[dl] & \\
& & (30|21) \arrow[ul] & &
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}[column sep=tiny]
(43|210) & & (32|410) \arrow[dl] & & (21|430) \arrow[dl] & & (10|432) \arrow[dl] \\
& (42|310) \arrow[ul] & & (31|420) \arrow[ul] \arrow[dl] & & (20|431) \arrow[ul] \arrow[dl] & \\
& & (41|320) \arrow[ul] & & (30|421) \arrow[ul] \arrow[dl] & & \\
& & & (40|321) \arrow[ul] & & &
\end{tikzcd} \tag{6}$$

(7)



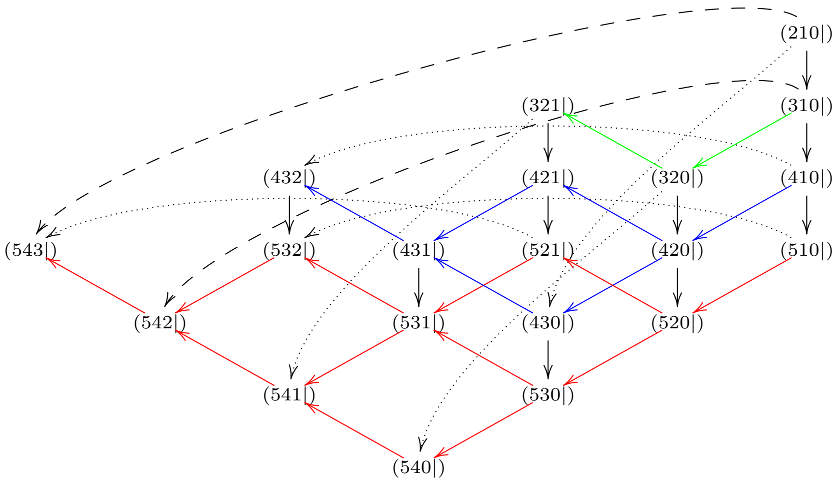
There are algorithms discovering all non-zero homomorphisms in such patterns, cf. [3, 4]. There are no other non-zero homomorphisms for the de Rham complex on the 2-dimensional projective space in (4). In (5), the central diamond (the square of arrows) displays two non-zero compositions, which equal each other up to sign (as expected, since the whole pattern must be the de Rham complex of operators). Moreover, there is the special homomorphism of fourth order joining the most right and most left modules. This corresponds to the Paneitz operator, whose symbol is the square of the Laplacian, see [15]. All other compositions of morphisms not mentioned above are zero.

In (6) and (7), the situation is the same with all the diamonds there. In (6), there are additionally two fourth order homomorphisms joining the modules in the first line. In (7), there are six such fourth order morphisms, but also two quite different ones – a morphism of order nine joining the most right and most left modules, and one of order seven joining their neighbors. We shall not go into details of this example here, there is the work in progress, [28], covering this case in detail.

Notice that the decompositions of the spaces of exterior forms $\Lambda^k((\mathbb{R}^p)^* \otimes \mathbb{R}^q)$ are easily understood by mimicking every symmetrization in $\otimes^k(\mathbb{R}^p)^*$ by identical antisymmetrization in $\otimes^k(\mathbb{R}^q)$, and vice versa. This is very nicely encoded by means of the so called Young symmetrizers and Young diagrams. The pattern (7) is drawn in this language in [8, Section 3.2.17].

Actually, another good way to understand the Grassmannians is to identify the operators coming from the lower dimensional ones (corresponding to forgetting some of the nodes in the Dynkin diagrams). The following diagram rewrites (7) this way, and completes all the long arrows (not seen in the de Rham complex directly). Viewing it as a 3-dimensional picture, we can clearly see the parts corresponding to the $(2,3)$, $(2,2)$, and $(1,2)$ Grassmannians there. Some of these are indicated by colors (we write only the first half of the weight encodings, which determines the rest):

(8)



Notice, how the higher dimensional cases inherit the exceptional 'long' operators, while adding some new ones, too. See [28] for more details about non-zero compositions of arrows in (8).

2.5. Translation principle idea. Let us come back to the general theory. The homomorphisms between Verma modules appear with striking regularity. This fact is a consequence of another quite straightforward observation:

Suppose $\mathbb{W}$ is a G-module (and P-module by restriction), and $\mathbb{E}$ a P-module. Then we may view $\mathfrak{U}(g) \otimes \mathbb{E}^* \otimes \mathbb{W}^*$, as a $\mathfrak{g}$-module, in two different ways:

$$X(x \otimes e \otimes w) = Xx \otimes e \otimes w$$
$$X(x \otimes e \otimes w) = Xx \otimes e \otimes w + x \otimes e \otimes Xw,$$

for all $X \in \mathfrak{g}$, $x \in \mathfrak{U}(\mathfrak{g})$, $e \in \mathbb{E}^*$, and $w \in \mathbb{W}^*$. The first one descends to the $\mathfrak{g}$-module (and also $(\mathfrak{U}(\mathfrak{g}), P)$-module) structure on $V(\mathbb{E} \otimes \mathbb{W})$, while the second one yields the structure of $V(\mathbb{E}) \otimes \mathbb{W}^*$.

Clearly, there is the unique $(\mathfrak{U}(\mathfrak{g}), P)$-module homomorphism φ between the above modules, defined as identity on $1 \otimes e \otimes w$. An easy check reveals that φ descends to the isomorphism

$$V(\mathbb{E} \otimes \mathbb{W}) = V(\mathbb{E}) \otimes \mathbb{W}^*. \tag{9}$$

Next, an arbitrary non-trivial irreducible G-module $\mathbb{W}$ is never an irreducible P-module. On the contrary, the $\mathfrak{g}_0$-orbit of the action containing the highest weight vector of $\mathbb{W}$ forms the $\mathfrak{p}$-irreducible component $\mathbb{W}_\alpha$ on which the grading element acts by the biggest scalar α, and the entire $\mathbb{W}$ enjoys a composition series

$$\mathbb{W} = \mathbb{W}_{\alpha-\ell} + \mathbb{W}_{\alpha-\ell+1} + \cdots + \mathbb{W}_\alpha \tag{10}$$

where the labeling reflects the scalar action by the grading element, and the 'right ends' $W_j = \mathbb{W}_j + \cdots + \mathbb{W}_\alpha$ form $\mathfrak{p}$-submodules, i.e., we get the filtration

$$\mathbb{W}_\alpha = W_\alpha \subset W_{\alpha-1} \subset \cdots \subset W_{\alpha-\ell} = \mathbb{W}, \tag{11}$$

with $\mathbb{W}_j = W_j/W_{j+1}$. As a $\mathfrak{g}_0$-module, the composition series is a direct sum of submodules $\mathbb{W}_j$ and each of them further decomposes into $\mathfrak{g}_0$-irreducible submodules $\mathbb{W}_{j,k}$. The composition series (1) of the adjoint representation on $\mathfrak{g}$ is a good example.

Now, consider an irreducible $\mathfrak{p}$-module $\mathbb{E}$, and the module $\mathbb{W}$ as before. Then we arrive at the composition series

$$\mathbb{E} \otimes \mathbb{W} = \mathbb{E} \otimes \mathbb{W}_{\alpha-\ell} + \cdots + \mathbb{E} \otimes \mathbb{W}_\alpha,$$

and each $\mathbb{E} \otimes \mathbb{W}_i$ splits into direct sum of irreducible $\mathfrak{g}_0$-modules $\mathbb{E}_{i,j}$.

Finally, assume that one of the many modules $\mathbb{E}' = \mathbb{E}_{i,j}$ has got a distinct infinitesimal character then all the other modules in the above decomposition. Then the injection $V(\mathbb{E}') \to V(\mathbb{E} \otimes \mathbb{W})$ is defined by its image being the joint eigenspace of the infinitesimal character of $V(\mathbb{E}')$. Consequently, there is the complementary subspace defined as the generalized eigenspaces of all the other infinitesimal characters there.

Thus, under the latter assumption, $V(\mathbb{E}')$ *canonically splits off the* $V(\mathbb{E} \otimes \mathbb{W}) = V(\mathbb{E}) \otimes \mathbb{W}^*$ *as a direct summand.*

Now we are ready to tell the translation idea: If $\Phi : V(\mathbb{F}) \to V(\mathbb{E})$ is a non-trivial $(\mathfrak{U}(\mathfrak{g}), P)$-module homomorphism (so in particular, the Verma modules share the same infinitesimal character), then in view of (9), and assuming further that both $V(\mathbb{E}')$ and $V(\mathbb{F}')$ enjoy the same and unique infinitesimal character in $V(\mathbb{E} \otimes \mathbb{W})$ and $V(\mathbb{F} \otimes \mathbb{W})$, respectively, we obtain the composed homomorphism

$$V(\mathbb{F}') \to V(\mathbb{F} \otimes \mathbb{W}) = V(\mathbb{F}) \otimes W^* \to V(\mathbb{E}) \otimes \mathbb{W}^* = V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{E}'). \tag{12}$$

We talk about *twisting the homomorphism* Φ by tensoring it with the identity on $\mathfrak{g}$-module $\mathbb{W}^*$.

A difficult question remains, how to recognize whether the *translated morphism* $V(\mathbb{F}') \to V(\mathbb{E}')$ is nontrivial.

2.6. The Jantzen-Zuckermann translation principle. Before we explain why the shapes of the de Rham complexes in (4) - (7) happen to be the general patterns for all infinitesimal characters, let us focus on the action of the Weyl group $W_\mathfrak{g}$ on the weights.

For each $\mathfrak{p}$-dominant weight α, there is exactly one $s \in W_\mathfrak{g}$ such that $\alpha + \rho = s \cdot (\lambda + \rho)$ for a uniquely defined $\mathfrak{g}$-dominant weight $\lambda + \rho$, i.e., $\lambda + \rho$ sits in the closed dominant Weyl chamber. If λ itself is $\mathfrak{g}$-dominant, then we say that the infinitesimal character ξ_α is *regular*. If $\lambda + \rho$ sits in a wall (or intersection of several walls) of the dominant chamber, we call ξ_α *singular* (or more precisely k-singular, if sitting on intersection of k walls).

Starting with a $\mathfrak{g}$-dominant λ, we obtain the so called Hasse diagram of the orbit of the subgroup $W_\mathfrak{p}$ of those $s \in W_\mathfrak{g}$ with the affine action producing $\mathfrak{p}$-dominant weights. This Hasse diagram is independent of the chosen dominant weight λ, see [8, Section 3.2.18] for recipes how to get it. If λ is not dominant, but $\lambda + \rho$ is, then still the affine action of $W_\mathfrak{p}$ produces some $\mathfrak{p}$-dominant weights on its orbit, which all appear with 2^k repetitions, if the infinitesimal character is k-singular.

The length of s is defined as the least number of simple reflections composed to built s. Looking at our de Rham patterns, the lengths of such s is the number of transpositions of neighbors in the permutation of the numbers and this also labels the columns there (e.g., going from zero to nine in (7)).

For a moment, let us come back to the regular infinitesimal characters and let us write $\mathbb{E}_\alpha$ or $\mathbb{W}_\mu$ for the modules with $\mathfrak{p}$ or $\mathfrak{g}$-dominant highest weights α or μ, respectively. Following [33, 5], we define two functors on $\mathfrak{U}(\mathfrak{g})$-modules, which split into direct sums of components with respect to the infinitesimal characters. These include our (generalized) Verma modules. We shall write p_λ for the projection of such modules to the component with the infinitesimal character ξ_λ. Consider two $\mathfrak{g}$-dominant weights λ, μ, and define the *translation functors*

$$\varphi^{\lambda}_{\lambda+\mu} = p_{\lambda+\mu} \circ (- \otimes \mathbb{W}_\mu) \circ p_\lambda \tag{13}$$

$$\psi^{\lambda+\mu}_{\lambda} = p_\lambda \circ (- \otimes (\mathbb{W}_\mu)^*) \circ p_{\lambda+\mu} \tag{14}$$

where the action on morphisms is given by twisting by the identity in the tensor product.

Actually, the same construction works if μ is $\mathfrak{g}$-dominant, while λ is a $\mathfrak{p}$-dominant weight with a singular infinitesimal character. We say that the weights λ and λ' are *equi-singular* if their singular character is represented by a weight on the same (intersection of) wall(s) of the dominant Weyl chamber. In particular, all weights with regular infinitesimal character are considered equi-singular in this sense.

2.7. Theorem. *Consider a $\mathfrak{g}$-dominant weight μ and a $\mathfrak{p}$-dominant weight λ such that $\lambda + \rho$ is in the closed dominant Weyl chamber.*
(1) The functor $\psi^{\lambda+\mu}_{\lambda}$ is left adjoint to $\varphi^{\lambda}_{\lambda+\mu}$.
(2) If the weights λ and $\lambda + \mu$ are equi-singular, then

$$\psi^{\lambda+\mu}_{\lambda}(V(\mathbb{E}_{s\cdot(\lambda+\mu)})) = V(\mathbb{E}_{s\cdot\lambda}), \quad \varphi^{\lambda}_{\lambda+\mu}(V(\mathbb{E}_{s\cdot\lambda})) = V(\mathbb{E}_{s\cdot(\lambda+\mu)}),$$

whenever $s{\cdot}\lambda$ is $\mathfrak{p}$-dominant.

Proof. Since $\mathbb{W}_\mu$ is finite dimensional, there is the tautological isomorphism for all $\mathfrak{p}$-modules $\mathbb{E}$ and $\mathbb{F}$,

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}(V(\mathbb{F}) \otimes (W_\mu)^*, V(\mathbb{E})) = \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}(V(\mathbb{F}), V(\mathbb{E}) \otimes \mathbb{W}_\mu). \tag{15}$$

As we know, only the summand $p_\lambda(V(\mathbb{E}_{s\cdot(\lambda+\mu)}) \otimes (\mathbb{W}_\mu)^*)$ can contribute to $\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}(V(\mathbb{E}_{s\cdot(\lambda+\mu)}) \otimes (\mathbb{W}_\mu)^*, V(\mathbb{E}_{s'\cdot\lambda})$ and similarly only $p_{\lambda+\mu}(V(\mathbb{E}_{s'\cdot\lambda}) \otimes \mathbb{W}_\mu)$ can contribute to $\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}(V(\mathbb{E}_{s\cdot(\lambda+\mu)}), V(\mathbb{E}_{s'\cdot\lambda}) \otimes \mathbb{W}_\mu)$. Thus, we have arrived at the requested natural equivalence

$$\begin{aligned}\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}&\big(\psi^{\lambda+\mu}_{\lambda}\big(V(\mathbb{E}_{s\cdot(\lambda+\mu)})\big), V(\mathbb{E}_{s'\cdot\lambda})\big)\\ &\simeq \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}\big(V(\mathbb{E}_{s\cdot(\lambda+\mu)}), \varphi^{\lambda}_{\lambda+\mu}\big(V(\mathbb{E}_{s'\cdot\lambda})\big)\big).\end{aligned}$$

The second claim is more difficult to prove. We present a quick sketch only. As shown in [33], for equi-singular λ and $\lambda + \mu$, the functors $\psi^{\mu+\lambda}_{\lambda}$ and $\varphi^{\lambda}_{\lambda+\mu}$ are mutually inverse natural equivalences. Clearly, the $\mathfrak{p}$-dominant weights λ and $\lambda+\mu$ appear at the same position in the Hasse diagram, the infinitesimal character is shared by the entire Hasse diagram, and thus p_λ is identity on every $V(\mathbb{E}_{s\cdot\lambda})$. Next, as discussed above, the tensor product $V(\mathbb{E}_{s\cdot\lambda} \otimes \mathbb{W}_\mu)$ (as a $\mathfrak{g}_0$-module) decomposes,

$$V(\mathbb{E}_{s\cdot\lambda} \otimes \mathbb{W}_\mu) = \mathfrak{U}(\mathfrak{g}_-) \otimes ((\mathbb{E}_{s\cdot\lambda})^* \otimes \mathbb{W}_\mu) = \mathfrak{U}(\mathfrak{g}_-) \otimes (\oplus_{j=1}^{k} \mathbb{W}_{\nu_j}) = \oplus_{j=1}^{k} V(\mathbb{W}_{\nu_j}).$$

The weights ν_j in the sum appear with multiplicities which can be computed explicitly, e.g., by means of the Klimyk formula. Finally, the projection $p_{\lambda+\mu}$ selects only those with the infinitesimal character $\xi_{\mu+\lambda}$.

Summarizing, the value of $\varphi^{\lambda}_{\lambda+\mu}$ on a generalized Verma module is a sum of generalized Verma modules. Swapping $\mathbb{W}_\mu$ and λ with $(\mathbb{W}_\mu)^*$ and $\lambda + \mu$, we get the same claim for $\psi^{\lambda+\mu}_{\lambda}$. Now, we know that $\psi^{\lambda+\mu}_{\lambda} \circ \varphi^{\lambda}_{\lambda+\mu}$ is naturally equivalent to identity and, thus, the values can always consist of one Verma module only. Certainly, $\nu = s{\cdot}(\lambda + \mu)$ appears among the weights ν_j, and it must appear with multiplicity one. As a result,

$$\varphi^{\lambda}_{\lambda+\mu}(V(\mathbb{E}_{s\cdot\lambda})) = V(\mathbb{E}_{s\cdot(\lambda+\mu)}).$$

Similarly, we understand the functor $\psi^{\lambda+\mu}_{\lambda}$, replacing μ by $-\mu$ and λ by $\mu + \lambda$. $\square$

2.8. Back to examples. As a direct consequence of the theorem we understand that the de Rham pattern is copied for each $\mathfrak{g}$-dominant weight. With all non-trivial morphisms at exactly the same positions. The order of the morphisms is easily computed as the difference of the actions of the grading element, i.e., using the formulae (**??**) or (2) (possibly working with the weights $\lambda + \rho$, since constant changes do not matter).

To see a 1-singular example, take the $\mathfrak{p}$-dominant weight $\lambda = (21|10)$ as the left most weight in the pattern (5) (remember, the notation is describing rather $\lambda + \rho$, which sits in the wall of the dominant chamber), and apply the same permutations as in (5) again. We arrive at (weights which are not $\mathfrak{p}$-dominant are replaced by crosses)

$$\begin{tikzcd}[column sep=small] (21|10) \arrow[dr, equal] & & \times & & (10|21) \\ & (21|10) & & (10|21) \arrow[ll, dotted, bend right] \arrow[ur, equal] & \\ & & \times & & \end{tikzcd} \tag{16}$$

where the dotted homomorphism corresponds to the second order composition in the central diamond in the de Rham pattern, and it corresponds to the conformally invariant Laplacian, i.e., the Yamabe operator, on densities with the right weights.

By the results of Enright and Shelton, [16], there are bijective correspondences between the patterns for singular infinitesimal characters and patterns for regular characters in lower dimensional geometries, we shall not go into details here. For example, the pattern in (16) coincides with the regular one for one-dimensional projective geometry. There, the de Rham consists of one morphism only, exactly as seen in (16).

Similarly, the only 2-singular pattern for the $(2,2)$ Grassmannian will consists of four equal $\mathfrak{p}$-dominant weights and there are no no-trivial morphisms there. All the 2-singular patterns for the $(3,3)$ Grassmannian will be again of the same shape as the one-dimensional projective de Rham (there will be two groups of four equal $\mathfrak{p}$-dominant weights, with one non-trivial homomorphism between them).

Actually, our aim is to extend this algebraic translations to the realm of curved Cartan geometries in the next section. For this endeavor, the crucial observation in [15] was, that actually the above considerations allow for translations based on many other weights of $\mathbb{W}_\mu$ than the highest and lowest ones. We formulate this observation as two propositions:

2.9. Proposition (Proposition 9 in [15])**.** *Suppose that $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same infinitesimal character. Suppose that $V(\mathbb{E}')$ and $V(\mathbb{F}')$ have the same infinitesimal character. Let $\mathbb{W}$ be a finite-dimensional irreducible representation of G and suppose that*

- *$V(\mathbb{F}')$ occurs in the composition series for $V(\mathbb{F} \otimes \mathbb{W})$ and has distinct infinitesimal character from all other factors;*
- *$V(\mathbb{E}')$ occurs in the composition series for $V(\mathbb{E} \otimes \mathbb{W})$ and has distinct infinitesimal character from all other factors.*

It follows that $V(\mathbb{F})$ occurs in the composition series for $V(\mathbb{F}' \otimes \mathbb{W}^*)$ and that $V(\mathbb{E})$ occurs in the composition series for $V(\mathbb{E}' \otimes \mathbb{W}^*)$. We suppose further that

- *all other composition factors of $V(\mathbb{F}' \otimes \mathbb{W}^*)$ have infinitesimal character distinct from $V(\mathbb{F})$;*
- *all other composition factors of $V(\mathbb{E}' \otimes \mathbb{W}^*)$ have infinitesimal character distinct from $V(\mathbb{E})$.*

Then translation gives an isomorphism

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}(V(\mathbb{F}), V(\mathbb{E})) \simeq \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),P)}(V(\mathbb{F}'), V(\mathbb{E}'))$$

whose inverse is given by translation using $\mathbb{W}^*$*.*

Proof. Straightforward, using the tautological isomorphisms (15) and the above argumentation. □

Actually, sometimes there are also one-way translations producing non-trivial homomorphisms in the less singular patterns from the more singular ones. For example, in the case of the $(2,2)$ Grassmannian, we might start with the identity morphism in the only 2-singular pattern, produce the morphism corresponding to the first order Dirac operator on the basic spinors (which is in the other 1-singular pattern there), as well as the second order morphism from (16). But, the 4th-order morphism corresponding to the Paneitz operator in the de Rham pattern cannot be translated from anything else.

Such one-way translations are based on the following extension of the previous proposition. Recall, there is the scalar action of the grading element on irreducible $\mathfrak{p}$-modules $\mathbb{E}$. We write $\alpha(\mathbb{E})$ for this constant, now.

Notice, that the difference of these scalars determines the order of the prospective homomorphisms.

2.10. Proposition. *Let* $\Phi : V(\mathbb{E}) \to V(\mathbb{F})$ *be a nontrivial homomorphism of Verma modules, and let* $\mathbb{W}$ *be an irreducible finite dimensional* G*-module.*

Suppose that there are irreducible $\mathfrak{p}$*-modules* $\mathbb{E}_1, \mathbb{E}_2, \mathbb{F}_1, \mathbb{F}_2$ *such that:*
(i) $\mathbb{E} \otimes \mathbb{W} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}'$; $\mathbb{F} \otimes \mathbb{W} = \mathbb{F}_1 \oplus \mathbb{F}_2 \oplus \mathbb{F}'$;
(ii) Verma modules $V(\mathbb{E}_1), V(\mathbb{E}_2), V(\mathbb{F}_1), V(\mathbb{F}_2)$ *have the same infinitesimal character;*
(iii) all pieces in the composition series for $V(\mathbb{E}'), V(\mathbb{F}')$ *have different infinitesimal characters. from those in (ii);*
(iv) $\alpha(\mathbb{E}_1) < \alpha(\mathbb{E}_2)$, $\alpha(\mathbb{F}_1) > \alpha(\mathbb{F}_2)$ *and* $V(\mathbb{F})$ *splits off from* $V(\mathbb{F}_1 \otimes \mathbb{W}^*)$*.*
If there is no nontrivial homomorphism from $V(\mathbb{E}_2)$ *to* $V(\mathbb{F})$*, then the translated homomorphism*

$$\hat{\Phi} : V(\mathbb{E}_1) \to V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{F}_1)$$

is nontrivial.

Proof. Our assumptions imply that $V(\mathbb{E}_1)$ embeds to $V(\mathbb{E} \otimes \mathbb{W})$ and $V(\mathbb{F} \otimes \mathbb{W})$ projects to $V(\mathbb{F}_1)$. Hence the translated homomorphism $\hat{\Phi}$ is well defined. $V(\mathbb{F})$ splits off from $V(\mathbb{F}_1 \otimes \mathbb{W}^*)$, hence the composition

$$V(\mathbb{E}) \xrightarrow{\Phi} V(\mathbb{F}) \to V(\mathbb{F}_1 \otimes \mathbb{W}^*).$$

is nontrivial. Based on (15), this is equivalent to the fact that

$$V(\mathbb{E} \otimes \mathbb{W}) \xrightarrow{\Phi \otimes Id_{\mathbb{W}}} V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{F}_1)$$

is nontrivial.

From (iii) it follows that $V(\mathbb{E} \otimes \mathbb{W}) = V(\mathbb{E}_1 \oplus \mathbb{E}_2) \oplus V(\mathbb{E}')$. Using (ii) and (iii), it is clear that also the composition

$$V(\mathbb{E}_1 \oplus \mathbb{E}_2) \xrightarrow{\Phi \otimes Id_{\mathbb{W}}} V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{F}_1)$$

is nontrivial.

In case that the composition $V(\mathbb{E}_1) \to V(\mathbb{E}_1 \oplus \mathbb{E}_2) \to V(\mathbb{F}) \to V(\mathbb{F} \otimes \mathbb{W}) \to V(\mathbb{F}_1)$ would be trivial, it follows that there is a nontrivial homomorphism $V(\mathbb{E}_2)$ to $V(\mathbb{F}_1)$, which is a contradiction. □

3. The curved translation principle

3.1. Cartan connections. Finally, we come to the curved Cartan geometries, modeled over the Klein's homogeneous spaces G/H. This concept generalizes the affine connections on manifolds M, realized as the sum of soldering forms and principle connections on the linear frame bundle on M, i.e., the bundles of frames of TM. The affine $\mathbb{R}^n$, as the homogeneous space $\mathrm{Aff}(n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{R})$ is the relevant Klein model for the affine connections.

The reader may find all the relevant background on Cartan connections in [8, Section 1.5] or [26].

Definition. *Let G be a (finite dimensional) Lie group, H its closed subgroup. A Cartan connection of type G/H is a principal fiber bundle $\mathcal{G} \to M$ with structure group H, equipped by a 1-form $\omega \in \Omega^1(\mathcal{G},\mathfrak{g})$, satisfying all the properties of the Maurer-Cartan form on G, which make sense:*
(i) The one-form ω is H-equivariant, i.e., $(r^h)^*\omega = \mathrm{Ad}(h^{-1}) \circ \omega$ for all $h \in H$.
(ii) The one-form ω reproduces fundamental vector fields on $\mathcal{G}$, i.e., $\omega(\zeta_X(u)) = \omega^{-1}(X)(u)$, for all $u \in \mathcal{G}$, $X \in \mathfrak{h}$, $\zeta_X(u) = \frac{\partial}{\partial t}_{|t=0} u\cdot \exp tX$.
(iii) The one-form ω is an absolute parallelism, i.e., $\omega(u) : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism.

The morphisms between the Cartan connections ω and ω' on principal bundles $\mathcal{G}$ and $\mathcal{G}'$, with the same structural group H, are principle fiber bundle morphisms $\varphi : \mathcal{G} \to \mathcal{G}'$ (over the identity on the group H), satisfying $\varphi^*\omega' = \omega$.

The group of automorphisms of a given Cartan connection of type G/H is always a finite dimensional Lie group whose Lie algebra is a subalgebra in $\mathfrak{g}$. In particular, its dimension is bounded by the dimension of G, see [8, Section 1.5.11].

In practice, the geometries are mostly defined by some simpler infinitesimal data, for example a G-structure, i.e., reduction of the structure group $GL(n,\mathbb{R})$ of the linear frame bundle to a closed subgroup G. Riemannian manifolds and conformal Riemannian manifolds are typical examples. The (normalized) natural Cartan connection is then a result of a construction (a so called prolongation). The theory and many examples of important Cartan geometries are discussed in great detail in [8, Chapters 1, 4, and 5].

The existence of the automorphisms is closely related to the *curvature* of the Cartan connection ω, the two-form $K \in \Omega^2(\mathcal{G};\mathfrak{g})$, $K = d\omega + \frac{1}{2}[\omega,\omega]$.

Of course, the Maurer-Cartan equations say that $K = 0$ if ω is the Maurer-Cartan form on the Lie group G.

There is the well known theorem that the Cartan geometry is locally isomorphic to its Klein's model, if and only if K vanishes, see [8, Section 1.5.2].

3.2. Natural bundles. The construction of the homogeneous bundles $\mathcal{E}$ from the H-modules $\mathbb{E} = \mathcal{E}_O$ from 1.1, extends directly to a functor on the category of principal bundles and their morphisms, by the very same construction of the associated bundles. We shall now write $\mathcal{E}$ for the functor mapping the principal bundles $\mathcal{G} \to M$ to the associated bundle $\mathcal{G} \times_H \mathbb{E} \to M$, with the obvious action on morphisms.

Any H-module homomorphism provides a natural transformation of the corresponding functors.

Notice that exactly as in 1.2, the sections of the natural bundles are identified with the H-equivariant functions in $C^\infty(\mathcal{G},\mathbb{E})$.

The most classical examples of Cartan connections are the affine connection on a manifolds. Due to the reductive structure of the Klein's model (i.e., G is the affine group in dimension n, $H = \mathrm{GL}(n,\mathbb{R})$ and the horizontal directions $\mathbb{R}^n$ form an H-submodule), the natural vector bundles are essentially only components of tensor bundles over the underlying manifolds. The Cartan connection then splits as $\theta+\gamma$, the soldering form and the linear connection form, while its curvature $K = T + R$ splits naturally into the torsion and curvature of the affine connection. Finally, there is the well known Schouten's reduction theorems saying that all invariant differential operators are in this setting obtained via covariant derivatives of the arguments, the curvature and torsion, and invariant algebraic operations (see [20, Chapter 28]).

We should also recall that viewing the sections of natural bundles as functions in $C^\infty(\mathcal{G}, \mathbb{E})_{\mathrm{GL}(n,\mathbb{R})}$, the covariant derivative with respect to an affine connection is simply given by differentiating in the direction of the constant fields $\omega^{-1}(X)$ for $X \in \mathbb{R}^n$ (notice, at a fixed frame u in the linear frame bundle $\mathcal{G}$ of the base manifold M, X is then identified with a tangent vector on M).

For general Cartan geometries, the existence of natural covariant derivatives on natural bundles is a subtle, but completely algebraic, question with answers completely inherited from the Klein's models, see [8, Section 1.5.6]. The Riemannian geometry (with the three different possible homogeneous models - the Euclidean, hyperbolic and spherical spaceforms) is a special example of a geometry with reductive model ensuring the unique normalized connection, the Levi Civita connection.

Just in the case of G-modules, i.e., dealing with *tractor bundles*, there is always the natural linear connection induced by the Cartan connection itself on all of them. This follows from exactly the same arguments as in 1.1, see also [8, Section 1.5.7].

The *adjoint tractor bundle* $\mathcal{A} = \mathcal{G}\times_H \mathfrak{g}$ provides an extremely important example. The short exact sequence of H-modules $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ gives rise to short exact sequence of natural bundles

$$0 \to \mathcal{G} \times_H \mathfrak{h} \to \mathcal{A} \to TM \to 0.$$

A straightforward check reveals that the curvature of the Cartan connection descends to a two-form in $\Omega^2(M; \mathcal{A})$. See [8, Section 1.5.7] for further properties and details.

3.3. Grassmannian geometries. The Cartan geometries modeled over the (m, n)-Grassmannians are examples of G-structures defined by the reduction of the structure group $S(GL(m, \mathbb{R}) \times GL(n, \mathbb{R})) \subset GL(m + n, \mathbb{R})$, as explained in 2.2. We call them almost Grassmannian geometries and they are equivalently defined by identifying TM with tensor product of the two auxiliary vector bundles EM and F^*M of dimensions m and n, respectively, together with the identification of the top degree forms on E and F^*, again as discussed in 2.2.

As before, we may assume $m \leq n$. The projective geometries correspond to $m = 1$ and the Cartan curvature has got values in $\mathfrak{p}$, i.e., there is no torsion. If $m = n = 2$, we deal with the split signature conformal Riemannian geometries in dimension 4, and there are two curvature components there (again no torsion).

If $m = 2 < n$, then there is one torsion and one curvature there and the geometries without the torsion are higher dimensional analogues of the self-adjoint 4-dimensional conformal structures. Dealing with the quaternionic real form of the same complexified algebras we arrive at the (almost) quaternionic geometries.

If $2 < m \leq n$, the almost Grassmannian geometries come with two torsion components. The special cases of $m = n$ are of special interest, since they are another promising generalization of the 4-dimensional conformal geometries.

In all the above cases, the identification of the top degree forms on EM and FM implies that the classical results from the tensorial invariant calculus may be employed for the two sets of abstract in indices describing fields in $\mathcal{E}^{A\dots BC'\dots D'}_{E\dots FG'\dots H'}[w]$.

3.4. Invariant operators and jet prolongations. In the Klein's world of geometric analysis, the invariant operators can be viewed as natural transformations between the relevant jet prolongations of the homogeneous bundles. This does not make sense now, because the existence of curvature excludes or reduces the existence of (auto)morphisms of the Cartan geometries.

It seems there are two options to move forward: either to exploit the concept of the so called gauge-natural operators, see [20, Chapter 12], i.e., we would work over the category of principal bundles and gauge-natural bundles, and add the Cartan connection ω to the arguments of the operators, or we rather seek ways, how to extend the operators at the Klein's model to the general cases.

In the first case, we mostly drastically reduce the choice of the natural bundles. Thus, we shall focus on the second approach only.

Perhaps the first idea should be to exploit the equivalence between the invariant linear operators and H-module morphisms shown in Proposition 1.3. This happens to be a bit tricky, though.

Fortunately, we may mimic the idea of expressing the directional derivatives via the actions of the constant fields $\omega^{-1}(X)$ on the H-equivariant functions. Indeed, for each Cartan connection ω we obtain the so called *fundamental derivative* D^ω, which is an operation

$$C^\infty(\mathcal{G},\mathbb{E})_H \ni \sigma \mapsto D^\omega(\sigma) = (X \mapsto \omega^{-1}(X)\cdot\sigma) \in C^\infty(\mathcal{G},\mathfrak{g}^*\otimes\mathbb{E})_H, \tag{17}$$

for all $X \in \mathfrak{g}$. Thus, the fundamental derivative is a natural differential operation mapping sections of $\mathcal{E}$ to sections of $\mathcal{A}^*\otimes\mathcal{E}$, which may be iterated. See [8, Section 1.5.8] for details and further properties.

Of course, there is a lot of redundancy there, since the derivatives in the directions of $\omega^{-1}(X)$, with $X \in \mathfrak{h}$, act algebraically. Let us look at the situation at the level of H-modules. The value of the fundamental derivative $D^\omega\sigma$, together with the value σ, can be understood as a couple $(v,\varphi) \in \mathbb{E}\oplus\mathfrak{g}^*\otimes\mathbb{E}$. The action of $h \in \mathfrak{h}$ is

$$h\cdot(v,\varphi) = \big(h\cdot v, X \mapsto h\cdot\varphi(\operatorname{Ad}_{h^{-1}} X)\big).$$

Comparing this with the natural action on the first jet prolongation $J^1\mathbb{E}$, we can see that actually $J^1\mathbb{E}$ is naturally embedded in $\mathbb{E}\oplus\mathfrak{g}^*\otimes\mathbb{E}$ as the H-submodule consisting of couples (v,φ) with $\varphi(Z) = -Z\cdot v$, for all $Z \in \mathfrak{h}$ (which perfectly mimics the fact that $D^\omega\sigma(Z) = -Z\cdot\sigma$ for such Z).

Thus, the fundamental derivative provides the universal first jet prolongation of sections. If we choose a complementary subspace $\mathfrak{g}_-$ identified with the quotient $\mathfrak{g}/\mathfrak{h}$, we can restrict D^ω to $\mathfrak{g}_-$. Then, exactly as for the Klein's model, we arrive at

$$J^1\mathcal{E} \simeq \mathcal{G}\times J^1\mathbb{E},$$

and the universal differential operator $\sigma \mapsto j^1\sigma$ defined by the restriction of the fundamental derivative to $\mathfrak{g}_-$.

In particular, we have verified that each H-module homomorphism defining an invariant first order linear operator on the Klein's model G/H directly extends to an invariant linear differential operator on the category of Cartan connections of the type G/H.

3.5. Higher order jets. We might repeat the argumentation from the previous paragraph and find the second order prolongation module

$$J^2\mathbb{E} = \mathbb{E} + ((\mathfrak{g}/\mathfrak{h})^*\otimes\mathbb{E}) + (S^2(\mathfrak{g}/\mathfrak{h})^*\otimes\mathbb{E})$$

naturally embedded in the module

$$\mathbb{E}\oplus(\mathfrak{g}^*\otimes\mathbb{E})\oplus(\mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathbb{E})$$

via the iterated action of the fundamental derivative (notice, the first module is given as composition series, while the second one is a direct sum). Thus, we arrived again at the identification $J^2\mathcal{E} = \mathcal{G}\times_H J^2\mathbb{E}$ for all H-modules $\mathbb{E}$. Now, every second order invariant (linear) operator on the Klein's model G/H corresponds to a $\mathfrak{h}$-module homomorphism Φ on the relevant jet prolongations $J^{\mathbb{E}}$. Thus, each such operator extends to an invariant operator between the corresponding natural bundles in the entire category of Cartan connections of this type, by exploiting the iterated fundamental derivative and using the same Φ.

However, this fails for orders bigger than two. The reason is explained in [8, Section 1.5.10] – the iterated fundamental derivative D^ω always defines and injective universal operator

$$J^r\mathcal{E} \to \oplus_{j=0}^r S^j\mathcal{A}^* \otimes \mathcal{E},$$

but for $r \geq 3$, $J^r\mathcal{E}$ is not naturally identified with the associated bundle $\mathcal{G} \times_H J^r\mathbb{E}$.

We may iterate $J^1(\dots J^1(J^1\mathcal{E}))$, and although this involves unnecessary redundancies, these can be removed by the following well known categorical construction.

Definition. *The semiholonomic jets $\bar{J}^k\mathcal{E}$ are inductively defined as the equalizer of all the natural projections $J^1(\bar{J}^{r-1}\mathcal{E}) \to \bar{J}^{r-1}\mathcal{E}$, starting with $\bar{J}^1\mathcal{E} = J^1\mathcal{E}$.*

For every H-module $\mathbb{E}$, we define its semiholonomic jet prolongation as the H-module $\bar{J}^r(G \times_H \mathbb{E})_O$, i.e., the fiber over origin of the semiholonomic prolongation of $\mathcal{E}$ over $M = G/H$.

By the very construction, $\bar{J}^r\mathcal{E} \simeq \mathcal{G} \times_H \bar{J}^r$, and the iterated fundamental derivatives define the universal differential operator $\mathcal{E} \to \bar{J}^r\mathcal{E}$.

In particular, $\bar{J}^2\mathcal{E}$ is the equalizer of the two natural projections appearing as the value of the functor J^1 on the projection $J^1E \to \mathcal{E}$, and the projection $J^1(J^1\mathcal{E}) \to J^1\mathcal{E}$.

The modules defining the semiholonomic jet prolongations as natural bundles are

$$\bar{J}^r\mathbb{E} = \mathbb{E} + (\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{E} + \dots + \otimes^r(\mathfrak{g}/\mathfrak{h})^* \otimes \mathbb{E}$$

which is a composition series (the right ends are H-submodules) with quite wild action of H.

Obviously we have got a one-way analogy of the Proposition 1.3:

Proposition. *Each non-zero H-module homomorphism $\Phi : \bar{J}^k\mathbb{E} \to \mathbb{F}$ defines invariant linear differential operators $D : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{F})$ of order at most k.*

Notice that the opposite implication fails in general because the image of the universal operator $\mathcal{E} \to \bar{J}^k\mathcal{E}$ is an algebraic subvariety in the target and there are counterexamples of operators defined by a morphism on the image of the universal operator (restricted to the fiber over the origin in the model), but not extending to a genuine H-module morphism on the entire $\bar{J}^k\mathbb{E}$. We shall comment more on this phenomenon later.

3.6. Semiholonomic induced modules. Exactly as in the Klein's model case, we better look at the dual picture. Here we follow [15], where the basic concepts were defined first. Although only the conformal Riemannian structures and the relevant operators and (semiholonomic) Verma modules were discussed in [15], many steps can be employed in general, without any modification.

As we have seen, the role of the left invariant vector fields are for Cartan geometries played by the constant vector fields $\omega^{-1}(X) \in \mathcal{X}(\mathcal{G})$, $X \in \mathfrak{g}$. Differentiating the H-equivariant functions $\sigma : \mathcal{G} \to \mathbb{E}$ in the direction of $\omega^{-1}(X)$ yields the fundamental derivative of the sections. More precisely, if $X \in \mathfrak{h}$, then $(\omega^{-1}(X)\cdot\sigma)(u) = -X\cdot(\sigma(u))$ by the equivariance and, thus, the genuine differential parts are again in the quotient $\mathfrak{g}/\mathfrak{h}$, thus corresponding to derivatives of the sections in directions in T_OM.

Next, consider again a 'word' $X_1X_2\dots X_k$ of elements in $\mathfrak{g}$ and the corresponding differential operator $\sigma\mapsto\omega^{-1}(X_1)\circ\omega^{-1}(X_2)\circ\dots\circ\omega^{-1}(X_k)\cdot\sigma(u)$ on the functions, evaluated in a frame $u\in\mathcal{G}$.

We may consider this operation as defined on the tensor algebra $T(\mathfrak{g})$ and again, there is the ideal $\mathcal{I}$ in $T(\mathfrak{g})$ generated by the expressions $X\otimes Y-Y\otimes X-[X,Y]$, with $X,Y\in\mathfrak{g}$, but at least one of them in $\mathfrak{h}$, which acts trivially. This is the consequence of the fact, that the curvature of the Cartan connection ω vanishes if one of the arguments is vertical.

The resulting quotient (left and right) $\mathfrak{g}$-module $\bar{\mathfrak{U}}(\mathfrak{g})=T(\mathfrak{g})/\mathcal{I}$ is called the *semiholonomic universal enveloping algebra* of the Lie algebra $\mathfrak{g}$.

Next we want to understand the linear forms in the dual of the semiholonomic jet modules $(\bar{J}^k\mathbb{E})^*$. Exactly as with the induced modules, we differentiate functions also in the vertical directions, and our values are in $\mathbb{E}$. Thus we consider the tensor product

$$\bar{V}(\mathbb{E})=\bar{\mathfrak{U}}(\mathfrak{g})\otimes_{\mathfrak{U}(\mathfrak{h})}\mathbb{E}^*.$$

The space $\bar{V}(\mathbb{E})$ clearly enjoys the structure of a $(\mathfrak{g},H)$-module (and $(\bar{\mathfrak{U}}(\mathfrak{g}),H)$-module), and it is called the *semiholonomic induced module* for the H-module $\mathbb{E}$.

3.7. Proposition. *The induced module $\bar{V}(\mathbb{E})$ is the space of all linear forms on $\bar{J}^\infty\mathbb{E}$ which factor through some $\bar{J}^k\mathbb{E}$, i.e., depend on finite number of derivatives. There is the natural surjection $\bar{V}(\mathbb{E})\to V(\mathbb{E})$.*

Proof. As in the induced modules case, the claim follows from the construction of $\bar{V}(\mathbb{E})$ and the fact that choosing a complementary vector subspace to $\mathfrak{h}$ in $\mathfrak{g}$, we can decompose all letters in our words $X_1\dots X_k$ above and, by the equalities enforced by living in the quotient by the ideal, we may "bubble" the letters in $\mathfrak{h}$ to the very right. Once there, they act algebraically and, thus, tensorizing over $\mathfrak{U}(\mathfrak{h})$ we remove just all redundancies. □

Obviously again, $\mathbb{E}^*$ injects into $\bar{V}(\mathbb{E})$, generates this $\mathfrak{g}$-module, and there is the natural filtration

$$\mathbb{E}^*=\bar{V}_0(\mathbb{E})\subset\bar{V}_1(\mathbb{E})\subset\dots\subset\bar{V}_k(\mathbb{E})\subset\dots\subset\bar{V}(\mathbb{E})$$

inherited from the filtration on $T(\mathfrak{g})$.

Next, assume that there is a fixed complementary subalgebra $\mathfrak{g}_-\simeq\mathfrak{g}/\mathfrak{h}$ to $\mathfrak{h}\subset\mathfrak{g}$. Then the Poincaré-Birkhoff-Witt procedure reveals that the graded semiholonomic universal algebra equals $\operatorname{gr}\bar{\mathcal{U}}(\mathfrak{g})=T(\mathfrak{g}_-)$ as vector space, while the graded semiholonomic induced module $\bar{V}$ is then, as a vector space, isomorphic to $\bar{\mathcal{U}}(\mathfrak{g}_-)\otimes_{\mathbb{R}}\mathbb{E}^*$. Moreover, there is the the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{V}_{k-1}(\mathbb{E}) & \longrightarrow & \bar{V}_k(\mathbb{E}) & \longrightarrow & \otimes^k(\mathfrak{g}/\mathfrak{h})\otimes\mathbb{E}^* & \longrightarrow & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V_{k-1}(\mathbb{E}) & \longrightarrow & V_k(\mathbb{E}) & \longrightarrow & S^k(\mathfrak{g}/\mathfrak{h})\otimes\mathbb{E}^* & \longrightarrow & 0 \end{array}\tag{18}$$

where the most right vertical arrow is given by the symmetrization.

Next, notice the Frobenius reciprocity 1.6 works without any change in the proof.

3.8. Proposition (Frobenius reciprocity, [15])**.** *For all finite dimensional representations $\mathbb{E}$ and $\mathbb{F}$ of H, there are the canonical isomorphisms*

$$\operatorname{Hom}_H(\mathbb{F}^*,\bar{V}(\mathbb{E}))=\operatorname{Hom}_{(\bar{\mathfrak{U}}(\mathfrak{g}),H)}(\bar{V}(\mathbb{F}),\bar{V}(\mathbb{E})).$$

Proof. If we are given a homomorphism $\Phi\in\operatorname{Hom}_{(\bar{\mathfrak{U}}(\mathfrak{g}),H)}(\bar{V}(\mathbb{F}),\bar{V}(\mathbb{E}))$, we define $\varphi:\mathbb{F}^*\to\bar{V}(\mathbb{E})$ by restriction.

On the other hand, having a $\varphi \in \mathrm{Hom}_H(\mathbb{F}^*, \bar{V}(\mathbb{E}))$, we define for all $x \in \bar{\mathfrak{U}}(\mathfrak{g})$ and $v \in \mathbb{F}^*$,

$$\Phi(x \otimes v) = x \otimes_{\bar{\mathfrak{U}}(\mathfrak{h})} \varphi(v),$$

which extends linearly, if well defined. This is again checked by noticing that for all $X \in \mathfrak{h}$ and $v \in \mathbb{F}^*$,

$$\Phi(X \otimes v - 1 \otimes X{\cdot}v) = X \otimes \varphi(v) - 1 \otimes \varphi(X{\cdot}v) = X \otimes \varphi(v) - 1 \otimes X{\cdot}\varphi(v),$$

which completes the proof. □

3.9. Lifting homomorphisms. Let us restrict ourselves to the parabolic geometries, i.e., semisimple Lie groups G with parabolic subalgebras P, the generalized Verma modules, and their semiholonomic versions.

While the homomorphisms between the generalized Verma modules,

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}),H)}(V(\mathbb{F}), V(\mathbb{E}))$$

are often very well understood, very little is known about the spaces

$$\mathrm{Hom}_{(\bar{\mathfrak{U}}(\mathfrak{g}),H)}(\bar{V}(\mathbb{F}), \bar{V}(\mathbb{E}))$$

which we are interested in, now.

The strategy proposed in [15] is to discuss the possible liftings of the existing homomorphisms $V(\mathbb{F}) \to V(\mathbb{E})$ to morphisms $\bar{V}(\mathbb{F}) \to \bar{V}(\mathbb{E})$ with respect to the canonical projection. Moreover, due to the Frobenius reciprocity, this is equivalent to the search for the dashed H-module morphisms in the following commutative diagram:

$$\begin{tikzcd} & \bar{V}(\mathbb{E}) \arrow[d] \\ \mathbb{F}^* \arrow[ur, dashed] \arrow[r] & V(\mathbb{E}) \end{tikzcd} \tag{19}$$

In turn, for irreducible modules $\mathbb{E}$, $\mathbb{F}$, this is equivalent to finding a highest weight vector in $\bar{V}(\mathbb{E})$ covering the relevant highest weight vector in $V(\mathbb{E})$.

Next, recall from 2.4 that the order of homomorphism $\Phi : V(\mathbb{F}) \to V(\mathbb{E})$ is the lowest k such that Φ maps $\mathbb{F}^*$ into $V_k(\mathbb{E})$. The order of homomorphisms Φ between the semiholonomic Verma modules is defined in the same way.

Then, again following the Klein's model case, the symbol of Φ is

$$\sigma(\Phi) : \mathbb{F}^* \to \bar{V}_k(\mathbb{E}) \to \bar{V}_k(\mathbb{E})/\bar{V}_{k-1}(\mathbb{E}) = \otimes^k(\mathfrak{g}_-) \otimes \mathbb{E}^*.$$

3.10. Proposition ([15]). *A homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ of order at most two always lifts to a homomorphism $\bar{V}(\mathbb{F}) \to \bar{V}(\mathbb{E})$.*

Proof. The claim is equivalent to the existence of an H-equivariant splitting of the canonical projection $\bar{V}_2(\mathbb{E}) \to V_2(\mathbb{E})$. Following [15], we define such a splitting by identity on $V_1\mathbb{E} = \bar{V}_1\mathbb{E}$, and we determine it completely by mapping

$$V_2(\mathbb{E}) \ni XYe \mapsto \frac{1}{2}(XY + YX + [X,Y])e \in \bar{V}_2(\mathbb{E})$$

for all $e \in \mathbb{E}^*$, $X, Y \in \mathfrak{g}$. Checking its equivariance is straightforward. □

This simple proposition implies again that all first and second order linear invariant operators extend canonically from the Klein's models to the entire category of the corresponding Cartan geometries.

Let us now restrict to $|1|$-graded parabolic geometries, which is the case with all our examples. Then our definition of symbol is compatible with the homogeneities in terms of the actions of the grading elements E in $\mathfrak{g}_0$. Here we also enjoy the following proposition. In general, we could also think about the finer filtering of the induced modules governed by the action of E, as we are doing in the filtering of the $\mathfrak{g}$-modules when viewed as $\mathfrak{p}$-modules.

3.11. Proposition ([15]). *For all $|1|$-graded parabolic geometries and irreducible P-modules $\mathbb{E}$, $\mathbb{F}$, the homomorphisms $V(\mathbb{F}) \to V(\mathbb{E})$, or $\bar{V}(\mathbb{F}) \to \bar{V}(\mathbb{E})$, are determined by their symbols.*

If the homomorphism $\bar{\Phi}$ of the semiholonomic Verma modules covers Φ, then the symbol of Φ is obtained by symmetrization of the symbol of $\bar{\Phi}$.

The order of the homomorphism is given as the difference of the actions of the grading element $E \in \mathfrak{g}_0$ on $\mathbb{F}$ and $\mathbb{E}$.

Proof. The existence of the grading element in $\mathfrak{g}_0$ defining the grading of $\mathfrak{g}$ shows that the highest weight vector determining a kth order operator must sit in $\otimes^k(\mathfrak{g}/\mathfrak{p})\otimes \mathbb{E}^*$. Since $\mathbb{F}^*$ generates both $V(\mathbb{F})$ and $\bar{V}(\mathbb{F})$, the homomorphisms are uniquely determined by the embeddings $\mathbb{F}^*$.

The next claim is obvious from the commutative diagram (18).

The final observation is clear since we deal with $|1|$-graded geometries, so the degree k of $S^k(\mathfrak{g}_-) \otimes \mathbb{E}$ must be just the mentioned difference. □

3.12. Curved translation principle. We are going to show, that the translation principal extends to some extent from the homogeneous case to the general curved Cartan geometries. The main idea (introduced in [15]) is to show, that many translations of morphisms $\Phi : V(\mathbb{F}) \to V(\mathbb{E})$ which can be covered by $\bar{\Phi} : \bar{V}(\mathbb{F}) \to \bar{V}(\mathbb{E})$ lead to results which again can be covered.

A special case of the invariant operators are the so called splitting operators used in (12). There, the embeddings $V(\mathbb{F}') \to V(\mathbb{F} \otimes \mathbb{W})$ and projections $V(\mathbb{E} \otimes \mathbb{W}) \to V(\mathbb{E}')$ are morphisms, whose orders are given by the relevant position of the dashed modules in the filtration of $\mathbb{W}$. Thus, if the difference from the top or bottom, respectively, is at most two, we can be sure that the necessary splitting will exist in the semiholonomic version as well.

In the semiholonomic case, we either can assume that the filtration of $\mathbb{W}$ is of length at most two, or we can restrict ourselves only to submodules which are at most two steps from the highest component in the filtration for the embeddings, and at most two steps from the bottom for the projections.

Then, we can cover the translation from the Klein's model in the curved case as seen in the next diagram:

$$\begin{tikzcd}
\bar{V}(\mathbb{F}') \arrow[r] \arrow[d] \arrow[rrrrr, dashed, bend left, "\bar{D}'"] & \bar{V}(\mathbb{F}\otimes\mathbb{W}) \arrow[r, equal] \arrow[d] & \bar{V}(\mathbb{F})\otimes\mathbb{W}^* \arrow[r, "\bar{D}\otimes 1"] \arrow[d] & \bar{V}(\mathbb{E})\otimes\mathbb{W}^* \arrow[r, equal] \arrow[d] & \bar{V}(\mathbb{E}\otimes\mathbb{W}) \arrow[r] \arrow[d] & \bar{V}(\mathbb{E}') \arrow[d] \\
V(\mathbb{F}') \arrow[r] \arrow[rrrrr, dashed, bend right, "D'"'] & V(\mathbb{F}\otimes\mathbb{W}) \arrow[r] & V(\mathbb{F})\otimes\mathbb{W}^* \arrow[r, "D\otimes 1"] & V(\mathbb{E})\otimes\mathbb{W}^* \arrow[r] & V(\mathbb{E}\otimes\mathbb{W}) \arrow[r] & V(\mathbb{E}')
\end{tikzcd}$$

Summarizing, all homomorphism of Verma modules which can be obtained from 1st or 2nd order morphisms by means of translation described in propositions 2.9 and 2.10, using only splitting operators of order at most two, admit the covering by homomorphisms of semiholonomic Verma modules. Moreover, the symbols of the covered homomorphisms are obtained by symmetrizations of symbols of those covering ones.

3.13. Non-existence. The (non)existence of homomorphisms of semi-holonomic Verma modules can be sometimes seen directly from the semiholonomic jet module picture, as discussed in 3.5.

In general, the action of $\mathfrak{g}_1$ on $\bar{J}^k\mathbb{E}$ is horrible, but the restriction of this action to $\otimes^{k-1}\mathfrak{g}_-^* \otimes \mathbb{E} \subset \bar{J}^k\mathbb{E}$ is relatively simple. For all $Z \in \mathfrak{g}_1$, $\varphi \in \otimes^{k-1}\mathfrak{g}_-^* \otimes \mathbb{E} \subset \bar{J}^k\mathbb{E}$, and $X_1, \dots, X_k \in \mathfrak{g}_{-1}$ we obtain

$$\begin{aligned}(Z\cdot\varphi)(X_1,\dots,X_k) = \sum_{i=1}^{k}\Big(& [X_i,Z]\cdot\varphi(X_1,\dots^\wedge, X_k) \\ & - \sum_{j=1}^{i-1}\varphi(X_1,\dots,[[X_i,Z],X_j],\dots^\wedge,X_k)\Big)\end{aligned} \tag{20}$$

where the wedges indicate the relevant omission of arguments.

Now, if there is a $\mathfrak{p}$-homomorphism $\Phi : \bar{J}^k\mathbb{E} \to \mathbb{F}$, and both $\mathbb{E}$ and $\mathbb{F}$ are irreducible, then Φ must vanish on the image of the $\mathfrak{p}_+$ action. This simple observation led to complete description of all first order operators in [27], for all parabolic geometries.

The first step was very simple there: fixing the action of the semisimple part of $\mathfrak{g}_0$, and leaving the so called weight (i.e., the action of the center $\mathfrak{z} \subset \mathfrak{g}_0$) free, the above condition restricted to the action $\mathfrak{g}_1$ on $\mathbb{E} \subset \bar{J}^1\mathbb{E}$ reveals that only the derivatives in the direction of the smallest distributions corresponding to $\mathfrak{g}_{-1}$ are feasible and it provides one linear constraint on the weight. Then we can show that the operators exist for all such weights.

In the one graded case, this recovers the earlier known fact from conformal Riemannian geometry, that fixing the action of the semisimple part of $\mathfrak{g}_0$ for $\mathbb{E}$ and $\mathbb{F}$, then each invariant projection of $\mathfrak{g}_1 \otimes \mathbb{E}$ to $\mathbb{F}$ yields an invariant operator for a unique weight of $\mathbb{E}$, cf. [17].

The same approach gets very much more complicated for higher orders. We shall illustrate the procedures on our simplest (2|2)-Grassmannian example, which will conclude our survey.

3.14. 4-dimensional conformal geometries. As exploited in [15], all fundamental representations $\mathbb{W}$ in the conformal Riemannian geometry, i.e., for the algebra $\mathfrak{so}(n+1,1)$, are of length at most two and so we obtain no restriction when using them in order to move from the trivial representation to any other one.

A particular case is our $(2,2)$-Grassmannian example. Dealing with the de Rham pattern for the regular infinitesimal characters there (see (5) and notice it contains only operators of order one or their nontrivial compositions, except the mysterious fourth order Paneitz operator not depicted there), we immediately see, that the pattern remains the same for all regular characters. In particular, this shows that our description of all Verma module homomorphisms for the regular infinitesimal characters in the paragraph after (5) extend to the curved 4-dimensional conformal geometries, except the Paneitz operator. Let us discuss this closer now.

We shall use the usual Penrose abstract index notation, as started in subsection 2.3. Thus let write $X = X_A^{A'}$, $Y = Y_A^{A'} \in \mathfrak{g}_{-1}$, $Z = Z_{A'}^{A} \in \mathfrak{g}_1$, and consider the trivial representation with weight w.

We shall first recover the second order Yamabe operator from (16). Then we need to compute the action of Z on $\varphi = \varphi_{A'}^{A} \in \mathfrak{g}_{-1}^*[w]$. In this case, (20) becomes:

$$(Z\cdot\varphi)(X,Y) = [X,Z]\cdot\varphi(Y) + [Y,Z]\cdot\varphi(X) - \varphi([X,Z],Y)$$

and, since we deal with trivial representation of the semisimple part of $\mathfrak{g}_0$,

$$[X,Z]\cdot\varphi(Y) = wZ(X)\varphi(Y).$$

Indeed, the grading element E acts on X_A by $1/2$, thus it acts on $X_{[A,B]}$ by 1, as anticipated, while the central part, i.e., the coefficient at the grading element is obtained by the evaluation.

Writing down the action of $\mathfrak{g}_1$ on the densities with weight w by means of the abstract indices, we arrive at $\varphi([[-,Z],-])^{AB}_{A'B'} = \varphi^B_{A'}Z^A_{B'} + \varphi^A_{B'}Z^B_{A'}$, and so the action gets the following shape:

$$Z^A_{A'}\cdot u^B_{B'} = wZ^A_{A'}u^B_{B'} + wZ^B_{B'}u^A_{A'} - u^B_{A'}Z^A_{B'} - u^A_{B'}Z^B_{A'}. \tag{21}$$

The potential second order operator valued again in densities is obtained by antisymmetrization in both upper and lower indices. The condition that the morphism must vanish on the entire image of the action says

$$2(w+1)Z^{[A}_{[A'}u^{B]}_{B']} = 0$$

and thus we obtain the value $w = -1$ as the right weight for the densities. This homomorphism yields the famous Yamabe operator, the conformally invariant version of the Laplace operator. Such natural operators can be expressed by a universal formula in terms of any of the metrics in the conformal class, and notice that even in the case of the flat conformal sphere S^4, the Yamabe operator involves additional lower order correction term to the Laplacian in the symbol. The reader can find a detailed explanation of such phenomena for all parabolic geometries in [9].

3.15. The Paneitz operator. In principle, we should be able to continue along the same line of arguments, look at image of the action of $\mathfrak{g}_1$ on $\otimes^3\mathfrak{g}_1$ inside $\bar{J}^4(\mathbb{E})$ for the trivial module $\mathbb{E} = \mathbb{R}$. But such an endeavor gets pretty complicated.

Thus, it is time to switch to the dual picture. In the holonomic Verma modules, we are looking for the singular vectors with trivial weight, in the module induced by the top-rank exterior forms $\mathbb{F}$, and we seek for them in the top layer of $V_4(\mathbb{F})$. Recall that we identify $\mathfrak{g}_{-1}$ with the matrices with 2 rows and columns, and let us write them as $(y_{11}, y_{12}, y_{21}, y_{22})$.

A direct check reveals that the relevant singular vector for the Yamabe operator is the determinant $y_{11}y_{22} - y_{12}y_{21}$, understood as an element in $S^2(\mathfrak{g}_{-1}) \otimes \mathbb{R}$ (with the right density weight). In our fourth order case, the only singular vector of the right trivial weight is the square of the determinant (up to constant multiple, of course). The computations are tedious but straightforward, and they can be nicely done, e.g., using Maple.

The situation gets much more complicated in the curved Cartan's worlds. We want to cover the known singular vector. Since the y_{ij}'s do not commute any more, we have to modify the formula for the determinant appropriately. As with the jets, we simply consider complete antisymmetrizations of both upper and lower indices. This gives us more terms than in the symmetric case:

$$\frac{1}{2}(y_{11}y_{22} - y_{21}y_{12} - y_{12}y_{21} + y_{22}y_{11}) \tag{22}$$

and taking the second power, we obtain 16 terms (instead of 4 in the holonomic case).

Now, the crucial observation is that there are actually three independent options how to perform the two antisymmetrizations over 4 indices - we choose the first couple and then continue with the remaining one. We have to consider their linear combinations with the sum of coefficients equal to one, in order to cover the symmetric singular vector. These are the only $\mathfrak{g}_0$-highest weight vectors which project onto the fixed symmetric singular vector.

Next, as a matter of fact (again quite straightforwardly computed, e.g., in Maple) the action of the generator z_{21} of $\mathfrak{g}_1$ is nontrivial and identical on all three of the options for the second power of determinant. Thus this action will never vanish on any of our coverings of the symmetric singular vector.

This shows that there cannot be any semi-holonomic Verma module homomorphism providing the Paneitz operator.

3.16. Final remarks. Analogous result was proved for all the longest operators in the de Rham pattern in conformal geometries of all even dimensions $n \geq 4$ in [15]. The semi-holonomic Verma module technique was first developed there.

Notice that actually there are the very exceptional invariant operators extending the flat ones in the de Rham even in the curved case. This can happen due to the fact that the image of the universal semi-holonomic jet operator is an algebraic subvariety in the total jet module, thanks to the Bianchi and Ricci identities for Cartan connections and their differential consequences.

E.g., in the case of the original Paneitz operator, we simply use the same formula as in the flat case and the restriction of the corresponding $\mathfrak{g}_0$-module homomorphism to the image of the universal jet operator happens to be equivariant.

All the other operators in the $(2,2)$-Grassmannian pattern, singular or regular can be obtained from the most simple ones: first order Dirac and first order exterior differential. At the same time, all the fundamental G-representations are coming with filtrations of length two or one, so we can directly use Jantzen-Zuckermann procedure to get all patterns also in the curved case (except the longest arrows). In higher conformal dimensions, the Propositions 2.9, 2.10 were necessary to built all the other long arrows between the forms in the de Rham, see [15].

The fourth order generalized Paneitz operators for the quaternionic-like geometries are studied in detail in [23, 24]. Again, in the de Rham, there are still invariant operators extending the flat case, but they need a much more careful approach, cf. [23, 24].

The case of the $(3,3)$-Grassmannians is discussed in great detail in the parallel paper by the same authors, [28].

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