ACTIONS OF JET GROUPS ON MANIFOLDS

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Abstract. In the present paper, we deduce the sharp estimates on the order r of the jet groups $G^r_{m,n}$ acting on a manifold of fixed dimension s which depend on m, n and s only. These estimates are essential for the theory of bundle functors on fibered manifolds and we find interesting that the dimensions m and n appear symmetrically in the outcome. In the case $n = 0$ we reprove the well known results by A. Zajtz.

It is well known that the elements of classical geometric objects form associated bundles with the structure group $G^r_m = \mathrm{inv}J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$, the classical *jet group* (or differential group). This is described in the theory of bundle functors on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms. But many geometric constructions are performed in other categories then $\mathcal{M}f_m$. One of the most frequent examples of general bundle functors is the fuctor of general connections defined on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and local isomorphisms. Following the classical theory we find that all finite order bundle functors on $\mathcal{FM}_{m,n}$ have their values in associated bundles with the structure groups $G^r_{m,n} = \{j^r_0 f \in G^r_{m+n}\,;\, f\colon (\mathbb{R}^{m+n} \to \mathbb{R}^m) \to (\mathbb{R}^{m+n} \to \mathbb{R}^m)$ is a morphism of fibered manifolds$\}$. We shall also call the Lie subgroups $G^r_{m,n} \subset G^r_{m+n}$ the *jet groups* (corresponding to the category $\mathcal{FM}_{m,n}$). We have shown in [2], that the possible orders of bundle functors on $\mathcal{FM}_{m,n}$ are determined by the possible orders of jet groups acting on smooth manifolds. In [3], there are such estimates obtained for the classical jet groups. In the present paper, we generalize and modify slightly Zajtz's methods to get similar estimates for the order of a jet group $G^r_{m,n}$ acting on a smooth manifold with a fixed dimension. For more detailed information and bibliography on bundle functors see [1].

Let us consider a continuous action of $G^r_{m,n}$ on an s-dimensional manifold S and for every point $y \in S$ let H_y be the isotropy subgroup at the point y. The action factorizes to an action of a group $G^k_{m,n}$ on S if and only if the kernel $G^r_{m,n}(k)$ of the jet projection $G^r_{m,n} \to G^k_{m,n}$ is a subset in H_y for all points $y \in S$. So if we assume that the order r is essential, i.e. the action does not factorize to $G^{r-1}_{m,n}$, then there is a point $y \in S$ such that H_y does not contain $G^r_{m,n}(r-1)$. Since the action is continuous H_y is closed and the homogeneous space $G^r_{m,n}/H_y$ is mapped injectively and continuously into S. Hence we have

$$\dim S \geq \dim(G^k_{m,n}/H_y)\ . \tag{1}$$

1991 *Mathematics Subject Classification.* 58 ? ??, ?? ? ??.
Key words and phrases. Jet groups, differential groups, natural bundles, bundle functors.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Therefore $\dim S$ is bounded from bellow by the smallest possible codimension of Lie subgroups in $G^r_{m,n}$ which do not contain $G^r_{m,n}(r-1)$.

We shall deduce a sharp estimate of these codimensions but since the complete proof is rather technical, we first formulate the final result concerning the actions of the groups $G^r_{m,n}$.

Theorem 1. *Let a jet group $G^r_{m,n}$, $m \geq 1$, $n \geq 0$, act continuously on a manifold S, $\dim S = s$, $s > 0$, and assume that r is essential, i.e. the action does not factorize to an action of $G^k_{m,n}$, $k < r$. Then*

$$r \leq 2s + 1\,.$$

Moreover, if m, $n > 1$, then

$$r \leq \max\{\frac{s}{m-1},\ \frac{s}{m}+1,\ \frac{s}{n-1},\ \frac{s}{n}+1\}$$

and if $m > 1$, $n = 0$, then

$$r \leq \max\{\frac{s}{m-1},\ \frac{s}{m}+1\}.$$

All these estimates are sharp for all $m \geq 1$, $n \geq 0$, $s \geq 1$.

Proof of the estimate $r \leq 2s+1$. Let H_y be the isotropy group in $y \in S$ which does not contain the normal closed subgroup $G^r_{m,n}(r-1)$. We denote $\mathfrak{g}^r_{m,n}$, $\mathfrak{g}^r_{m,n}(r-1)$ and $\mathfrak{h}$ the Lie algebras of $G^r_{m,n}$, $G^r_{m,n}(r-1)$ and H_y, respectively. Since $G^r_{m,n}(r-1) \subset G^r_{m+n}(r-1)$ is a connected and simply connected nilpotent Lie group, the exponential map is a global diffeomorphism of $\mathfrak{g}^r_{m,n}(r-1)$ onto $G^r_{m,n}(r-1)$ and therefore h does not contain $\mathfrak{g}^r_{m,n}(r-1)$. So we have to find a lower bound of the codimensions of such subalgebras of $\mathfrak{g}^r_{m,n}$ that do not contain the whole $\mathfrak{g}^r_{m,n}(r-1)$.

The Lie algebra $\mathfrak{g}^r_{m+n}$ is formed by r-jets at the origin of vector fields $X \in C^\infty(T\mathbb{R}^{m+n})$, $X(0) = 0$, and the Lie subalgebra $\mathfrak{g}^r_{m,n}$ consists just of jets of projectable vector fields. That is why we have to deal with polynomial vector fields of the form $x \mapsto a^i_j x^\mu \frac{\partial}{\partial x_i}$, $i = 1, \dots, m+n$, $0 < |\mu| \leq r$, where $a^i_\mu = 0$ if $i \leq m$ and $\mu_j > 0$ for some $m < j \leq m+n$. The formula for the Lie bracket is

$$\begin{aligned}
&\Big[b^j_\mu x^\mu \frac{\partial}{\partial x_j}, a^j_\lambda x^\lambda \frac{\partial}{\partial x_j}\Big] = c^i_\gamma x^\gamma \frac{\partial}{\partial x_i} \qquad \text{where}\\
&c^i_\gamma = \sum_{\substack{1 \leq j \leq m+n\\ \mu+\lambda-1_j=\gamma}} \Big(\lambda_j b^j_\mu a^i_\lambda - \mu_j a^j_\lambda b^i_\mu\Big)
\end{aligned} \tag{2}$$

Note there is no implicit summation in the brackets and 1_j denotes the multiindex $(\alpha_1, \dots, \alpha_{m+n})$ with $\alpha_i = \delta^i_j$.

Let us denote $\mathfrak{g}_p$, $0 \leq p \leq r-1$ the space of all homogeneous polynomial vector fields of degree $p+1$ in $\mathfrak{g}^r_{m,n}$, i.e $a^i_\lambda = 0$ for all $|\lambda| \neq p+1$. Since $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$, there is the grading $\mathfrak{g}^r_{m,n} = \mathfrak{g}_0 \oplus \dots \oplus \mathfrak{g}_{r-1}$. This is also obtained from the obvious filtration of $\mathfrak{g}^r_{m,n}$ $\mathfrak{g}^r_{m,n} = \mathfrak{g}^r_{m,n}(0) \supset \mathfrak{g}^r_{m,n}(1) \supset \dots \supset \mathfrak{g}^r_{m,n}(r-1) \supset \mathfrak{g}^r_{m,n}(r) = 0$. If we consider the intersections, then we get the filtration $\mathfrak{h} = \mathfrak{h}_0 \supset \mathfrak{h}^1 \supset \dots \supset \mathfrak{h}^{r-1} \supset 0$ and the quotient spaces $\mathfrak{h}_p = \mathfrak{h}^p/\mathfrak{h}^{p+1}$ are subalgebras in $\mathfrak{g}_p$. Therefore we can construct a new algebra $\tilde{\mathfrak{h}} = \mathfrak{h}_0 \oplus \dots \oplus \mathfrak{h}_{r-1}$ with grading and both the algebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ have the same codimension. By the construction, $\mathfrak{g}^r_{m,n}(r-1) \not\subset \mathfrak{h}$ if and only if $\mathfrak{h}_{r-1} \neq \mathfrak{g}_{r-1}$, so that $\tilde{\mathfrak{h}}$ does not contain $\mathfrak{g}^r_{m,n}(r-1)$ as well. That is why in the proof of Theorem 1 we may restrict ourselves to Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}^r_{m,n}$ with grading $\mathfrak{h} = \mathfrak{h}_0 \oplus \cdots \oplus \mathfrak{h}_{r-1}$ satisfying $\mathfrak{h}_i \subset \mathfrak{g}_i$, for all $0 \leq i \leq r-1$, and $\mathfrak{h}_{r-1} \neq \mathfrak{g}_{r-1}$.

Now the proof of the estimate $r \leq 2m+1$ becomes rather easy. To see this, let us evaluate $\left[x^\alpha \frac{\partial}{\partial x_i}, x^\beta \frac{\partial}{\partial x_i}\right] = (\beta_i - \alpha_i) x^{\alpha+\beta-1_i} \frac{\partial}{\partial x_i}$. The vector fields $x^\gamma \frac{\partial}{\partial x_i}$, $1 \leq i \leq m+n$, $|\gamma| = r$, and $\gamma_j = 0$ if $i \leq m$ and $j > m$, form a base of the linear space $\mathfrak{g}_{r-1}$. If $p \neq q$ and $p+q = r-1$, then for every $x^\gamma \frac{\partial}{\partial x_i} \in g_{r-1}$ we are able to find some α and β with $|\alpha| = p+1$, $|\beta| = q+1$ and $\alpha+\beta = \gamma+1_i$, $\beta_i \neq \alpha_i$, $x^\alpha \frac{\partial}{\partial x_i} \in g_p$, $x^\beta \frac{\partial}{\partial x_i} \in \mathfrak{g}_q$. Therefore, if $p+q = r-1$ and $p \neq q$, then $[\mathfrak{g}_p, \mathfrak{g}_q] = \mathfrak{g}_{r-1}$. Let us consider an element $a \in \mathfrak{g}_{r-1} \setminus \mathfrak{h}_{r-1}$ and let us choose $b \in \mathfrak{g}_p$ and $c \in \mathfrak{g}_q$ such that $[b, c] = a$. Hence either $b \notin \mathfrak{h}_p$ or $c \notin \mathfrak{h}_q$ and it follows that $\operatorname{codim} \mathfrak{h} \geq \frac{1}{2}(r-1)$. According to (1) we get $s \geq \frac{1}{2}(r-1)$ and consequently $r \leq 2s+1$. $\square$

The proof of the better estimates for higher dimensions is based on the same ideas but supported by some considerations from linear algebra. We choose some non-zero linear form C on $\mathfrak{g}_{r-1}$ with $\ker C \supset \mathfrak{h}_{r-1}$. Then given p, q, $p+q = r-1$, we define a bilinear form $f : \mathfrak{g}_p \times \mathfrak{g}_q \to \mathbb{R}$ by $f(a, b) = C([a, b])$ and we study the dimensions of the anihilators. The following simple Lemma can be found in [3].

Lemma 1. *Let V, W, be finite dimensional real vector spaces and let $f : V \times W \to \mathbb{R}$ be a bilinear form. Denote by V^0 or W^0 the anihilators of V or W related to f, respectively. Let $M \subset V$, $N \subset W$ be subspaces satisfying $f|(M \times N) = 0$. Then* $\operatorname{codim} M + \operatorname{codim} N \geq \operatorname{codim} V^0$.

If we fix a base on the vector space $\mathbb{R}^m$ then there are the induced base on the vector space $\mathfrak{g}_{r-1}$ and the induced coordinate expression of every linear form C on $\mathfrak{g}_{r-1}$. By the naturality of Lie bracket, using arbitrary coordinates on $\mathbb{R}^m$ the coordinate formula for the Lie bracket does not change. Since fiber respecting linear transformations of $\mathbb{R}^{m+n} \to \mathbb{R}^m$ conserve the projectability of vector fields, we can use arbitrary affine coordinates on the fibration $\mathbb{R}^{m+n} \to \mathbb{R}^m$ in our discussion on possible codimensions of the subalgebras, which is based on formula (1). The coordinate expression of C will be written like $C = (C^\alpha_i)$, $i = 1, \dots, m+n$, $|\alpha| = r$, where we put $C^\alpha_i = 0$ whenever $i \leq m$ and $\alpha_j > 0$ for some $j > m$. Further we shall use the symbol (j) for a multiindex α with $\alpha_i = 0$ for all $i \neq j$ and its length will be clear from the context. If suitable, we also write $\alpha = i_1 \cdots i_r$, where $|\alpha| = r$, $1 \leq i_j \leq m+n$, and α_j is the number of indices that equal j.

Lemma 2. *Let C be a non-zero form on $\mathfrak{g}_{r-1}$, $m \geq 1$, $n \geq 0$. Then in suitable affine coordinates on the fibration $\mathbb{R}^{m+n} \to \mathbb{R}^m$, the induced coordinate expression of C satisfies one of the following conditions*

(i) $C^{(1)}_m \neq 0$ *and* $m > 1$

(ii) $C^{(1)}_1 \neq 0$; $C^\alpha_j = 0$ *whenever* $\alpha_j = 0$ *and* $1 \leq j \leq m$; $C^{\alpha+1_1}_1 = C^{\alpha+1_j}_j$ *(no summation) for all* $|\alpha| = r$, $1 \leq j \leq m$

(iii) $C^{(m+1)}_{m+n} \neq 0$, $n > 1$, *and* $C^\alpha_j = 0$ *whenever* $j \leq m$

(iv) $C^{(m+1)}_{m+1} \neq 0$; $C^\alpha_j = 0$ *if* $j \leq m$ *or* $\alpha_j = 0$; *and* $C^{\alpha+1_{m+1}}_{m+1} = C^{\alpha+1_j}_j$ *(no summation) for all* $|\alpha| = r$, $j \geq m+1$ *with* $\alpha_i = 0$, $1 \leq i \leq m$.

Proof. Let C be a non-zero form on $\mathfrak{g}_{r-1}$ with coordinates C_j^α in the canonical base on $\mathbb{R}^{m+n} \to \mathbb{R}^m$. Let us consider a matrix $A \in GL(m+n)$ whose first row consists of arbitrary real parameters $a_1^1 = t_1 \neq 0$, $a_2^1 = t_2, \dots, a_m^1 = t_m$, $a_j^1 = 0$ for $j > m$, and let all the other elements be like in the unit matrix. Let $\tilde{a}_j^i$ be the elements of the inverse matrix A^{-1}. If we perform this linear transformation, we get a new coordinate expression of C, in particular

$$\bar{C}_j^{(1)} = a_{i_1}^1 \cdots a_{i_r}^1 C_s^{i_1 \dots i_r} \tilde{a}_j^s \,. \tag{3}$$

Hence we get

$$\bar{C}_1^{(1)} = t_{i_1} \cdots t_{i_r} C_1^{i_1 \dots i_r} \frac{1}{t_1} \tag{4}$$

$$\bar{C}_j^{(1)} = t_{i_1} \cdots t_{i_r} \left(C_j^{i_1 \dots i_r} - \frac{t_j}{t_1} C_1^{i_1 \dots i_r} \right) \qquad \text{for } 1 < j \leq m. \tag{5}$$

Formula (4) implies that either we can obtain $\bar{C}_1^{(1)} \neq 0$ or $C_1^\alpha = 0$ for all multiindices α, $|\alpha| = r$. Let us assume $m > 1$ and let us try to get condition (i). According to (5), if (i) does not hold after performing any of our transformations, then the expression on the right hand side of (5) has to be identically zero for all values of the parameters and this implies $C_j^\alpha = 0$ whenever $\alpha_j = 0$, $|\alpha| = r$, and $C_1^{\alpha+1_1} = C_j^{\alpha+1_j}$ for all $|\alpha| = r-1$, $1 \leq j \leq m$. Hence we can summarize: either (i) can be obtained with $j \geq m > 1$, or (ii) holds, or $\bar{C}_j^\alpha = 0$ for all $1 \leq j \leq m$, $|\alpha| = r$, in suitable affine coordinates.

Analogously, let us take a matrix $A \in GL(m+n)$ whose $(m+1)$-st row consists of real parameters $t_1, \dots, t_{m+n}$, $t_{m+1} \neq 0$ and let the other elements be like in the unit matrix. The new coordinates of C are obtained as above

$$\bar{C}_{m+1}^{(m+1)} = t_{i_1} \cdots t_{i_r} C_{m+1}^{i_1 \dots i_r} \frac{1}{t_{m+1}} \tag{6}$$

$$\bar{C}_j^{(m+1)} = t_{i_1} \cdots t_{i_r} \left(C_j^{i_1 \dots i_r} - \frac{t_j}{t_{m+1}} C_{m+1}^{i_1 \dots i_r} \right) \,. \tag{7}$$

Now we may assume $C_j^\alpha = 0$ whenever $1 \leq j \leq m$, for if not then (i) or (ii) could be obtained. As before, either there is a base relative to which $C_{m+1}^{(m+1)} \neq 0$ or $C_{m+1}^\alpha = 0$ for all $|\alpha| = r$. Further, according to (7) either we can get (iii) or $C_j^\alpha = 0$ whenever $\alpha_j = 0$, and $C_{m+1}^{\alpha+1_{m+1}} = C_j^{\alpha+1_j}$, for all $|\alpha| = r$, $j \geq m+1$. Therefore if both (iii) and (iv) do not hold after arbitrary transformations, then all C_j^α have to be zero, but this is contradictory to the fact that C is non-zero. $\square$

Lemma 3. *Let $p, q \in \mathbb{N}$, $p + q = r - 1 > 0$, $m > 1$, $n > 1$, $p \geq q \geq 0$ and let $\mathfrak{h}_p$ or $\mathfrak{h}_q$ be subspaces of $\mathfrak{g}_p$ or $\mathfrak{g}_q$, respectively. Let C be a non-zero linear form on $\mathfrak{g}_{r-1}$, let us suppose $[\mathfrak{h}_p, \mathfrak{h}_q] \subset \ker C$ and let C_i^α, $1 \leq i \leq m+n$, $|\alpha| = r$, be a coordinate expression of C satisfying one of the conditions (i)-(iv) in Lemma 2. Then*

$$\operatorname{codim} h_p + \operatorname{codim} \mathfrak{h}_q \geq \begin{cases} 2m-2, & \text{if (i) holds} \\ 2m, & \text{if (ii) holds and } q > 0 \\ m, & \text{if (ii) holds and } q = 0 \\ 2n-2, & \text{if (iii) holds} \\ 2n, & \text{if (iv) holds and } q > 0 \\ n, & \text{if (iv) holds and } q = 0. \end{cases}$$

Proof. We define a bilinear form

$$f : \mathfrak{g}_p \times \mathfrak{g}_q \to \mathbb{R} \qquad f(a,b) = C([a,b]) \ .$$

By our assumptions $f(\mathfrak{h}_p, \mathfrak{h}_q) = \{0\}$. Hence by Lemma 1 it suffices to prove that the codimension of the f-anihilator of $\mathfrak{g}_q$ in $\mathfrak{g}_p$ has the above lower bounds. Let $\mathfrak{h}_0$ be this anihilator and consider elements $a \in \mathfrak{h}_0$, $b \in \mathfrak{g}_q$. We get

$$C([a,b]) = \sum_{\substack{1 \leq i \leq m+n \\ |\alpha| = r}} C_i^\alpha [a,b]_\alpha^i = 0$$

Using formula for the bracket (1) we obtain

$$0 = \sum_{\substack{1 \leq i,j \leq m+n \\ |\mu| = q+1 \\ |\lambda| = p+1}} C_i^{\mu+\lambda-1_j} \left(\lambda_j b_\mu^j a_\lambda^i - \mu_j a_\lambda^j b_\mu^i \right) =$$

$$= \sum_{\substack{1 \leq i,j \leq m+n \\ |\mu| = q+1 \\ |\lambda| = p+1}} \left(C_i^{\mu+\lambda-1_j} \lambda_j - C_j^{\mu+\lambda-1_i} \mu_i \right) a_\lambda^i b_\mu^j \ .$$

Since $b \in \mathfrak{g}_q$ is arbitrary, we have got a system of linear equations for the anihilator $\mathfrak{h}_0$ containing one equation for each couple (j, μ), where $1 \leq j \leq m+n$, $|\mu| = q+1$ and $\mu_i = 0$ whenever $i > m$ and $j \leq m$. The (j,μ)-equation reads

$$\sum_{\substack{1 \leq i \leq m+n \\ |\lambda| = p+1}} \left(C_i^{\mu+\lambda-1_j} \lambda_j - C_j^{\mu+\lambda-1_i} \mu_i \right) a_\lambda^i = 0. \tag{8}$$

A lower bound of the codimension of $\mathfrak{h}_0$ is given by any number of linearly independent (j,μ)-equations and this has to be discussed separately for the cases (i)-(iv).

Let us first assume that (i) holds, i.e. $C_m^{(1)} \neq 0$, $m > 1$. We denote by E_s the $(s,(1))$-equation, $1 \leq s < m$ and by F_k the $(m,(1)+1_k)$-equation, $1 \leq k < m$ (note if $q = 0$ then $(1) + 1_k = 1_k$). We shall show that this subsystem is of full rank. For this purpose, consider a linear combination $\sum_{s=1}^{m-1} a^s E_s + \sum_{k=1}^{m-1} b^k F_k = 0$ $a^s, b^k \in \mathbb{R}$. From (8) we get

$$\sum_{\substack{1 \leq i \leq m+n \\ |\lambda| = p+1}} \Bigg(\sum_{s=1}^{m-1} \big(C_i^{(1)+\lambda-1_s} \lambda_s - C_s^{(1)+\lambda-1_i} \delta_i^1 (q+1) \big) a^s + \tag{9}$$

$$+ \sum_{k=1}^{m-1} \big(C_i^{(1)+1_k+\lambda-1_m} \lambda_m - C_m^{(1)+1_k+\lambda-1_i} (\delta_i^1 q + \delta_i^k) \big) b^k \Bigg) a_\lambda^i = 0.$$

Hence all the coefficients at the variables a_λ^i with $1 \leq i \leq m+n$, $\lambda = p+1$, and $\lambda_j = 0$ whenever $j > m$ and $i \leq m$, have to vanish. Therefore, we get equations on reals a^s, b^k, whenever we choose i and λ. We have to show that all these reals are zero.

First, let us substitute $\lambda = (1)$ and $i = m$. Then (9) implies $C_m^{(1)}(p+1)a^1 = 0$ and consequently $a^1 = 0$. Now we choose $\lambda = (1) + 1_v$, $i = m$, with $1 < v < m$, and we get $C_m^{(1)} a^v = 0$ so that $a^s = 0$ for $1 \le s \le m-1$. Further, take $\lambda = (1)$ and $1 < i < m$ to obtain $-C_m^{(1)} b^i = 0$. Finally, the choice $i = 1$ and $\lambda = (1)$ leads to $-C_m^{(1)}(q+1)b^1 = 0$. In this way, we have proved that the chosen $2m-2$ equations E_s and F_k are independent and this implies the first lower bound in Lemma 3.

Now suppose (ii) takes place and let us denote E_s the $(s,(1))$-equation, $1 \le s \le m$, and if $q > 0$, then F_k will be the $(m, (1) + 1_m + 1_k)$-equation, $1 \le k \le m$. As before, we assume $\sum_{s=1}^m a^s E_s + \sum_{k=1}^m b^k F_k = 0$ for some reals a^s and b^k and we compair the coefficients at a^i_λ to show that all these reals are zeros. But before doing this, we can simplify all (j,μ)-equations with $1 \le j \le m$ using the relations from (ii). Indeed, (8) reduces to

$$\sum_{\substack{1\le i\le m\\ |\lambda|=p+1}} C_j^{\mu+\lambda-1_i}(\lambda_j - \mu_i) a^i_\lambda = 0.$$

Consequently E_s and F_k have the forms

$$\sum_{\substack{1\le i\le m\\ |\lambda|=p+1}} C_s^{(1)+\lambda-1_i}(\lambda_s - \delta^1_i(q+1)) a^i_\lambda = 0$$

$$\sum_{\substack{1\le i\le m\\ |\lambda|=p+1}} C_m^{(1)+1_k+1_m+\lambda-1_i}(\lambda_m - \delta^1_i(q-1) - \delta^k_i - \delta^m_i) a^i_\lambda = 0.$$

Assume first $q > 0$. If we choose $1 < i \le m$, $\lambda = (1)$, then the variables a^i_λ do not appear in the equations E_s at all. Hence the choice $i = m$, $\lambda = (1)$ gives (see (ii)) $0 = -2C_m^{(1)+1_m} b^m = -2C_1^{(1)} b^m$ and $1 < i < m$, $\lambda = (1)$ now imply $-C_m^{(1)+1_m} b^i = 0$. Hence $b^i = 0$ for all $1 < i \le m$. Further, we take $i = m$, $\lambda = (1) + 1_v + 1_m$, $v \ne m$ (note $p \ge q > 0$), so that all the coefficients in F_1 are zero. In particular, $v = 1$ implies $C_1^{(1)} a^1 = 0$ so that $a^1 = 0$. Now, if $1 < v < m$, then $C_v^{(1)+1_v} a^v = C_1^{(1)} a^v = 0$ and what remains are a^m and b^1, only. Taken $\lambda = (1)$, $i = 1$, we see $0 = -(q+1)C_m^{(1)} a^m - qC_m^{(1)+1_m} b^1 = C_1^{(1)} b^1$ and, finally, the choice $i = m$ and $\lambda = (1) + 1_m + 1_m$ gives $C_m^{(1)+1_m} 2a^m = 0$. This completes the proof of the second lower bound in Lemma 3. But if $q = 0$ and (ii) holds, we can perform the above procedure after forgetting all the equations F_k which are not defined. We have only to note $p + q = r - 1 > 0$, so that $|\lambda| = p + 1 = r \ge 2$.

The remaining three parts of the proof are complete recapitulations of the above ones. This becomes clear if we notice, that we have used neither any information on C_j^α, $j > m$, nor the fact that $C_j^\alpha = 0$ if $j \le m$ and $\alpha_i \ne 0$ for some $i > m$. That is why we can go step by step through the above proof on replacing 1 or m by $m+1$ or $m+n$, respectively. $\square$

Proposition 1. *Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}^r_{m,n}$, $m \ge 1$, $n \ge 0$, $r \ge 2$, which does not contain $\mathfrak{g}^r_{m,n}(r-1)$. Then*

$$\operatorname{codim} \mathfrak{h} \ge \frac{1}{2}(r-1). \tag{10}$$

Moreover, if $m > 1$, $n > 1$, then

$$\operatorname{codim} \mathfrak{h} \geq \min\{r(m-1),\ (r-1)m,\ r(n-1),\ (r-1)n\} \tag{11}$$

and if $m > 1$, $n = 0$, then

$$\operatorname{codim} h \geq \min\{r(m-1),\ (r-1)m\}. \tag{12}$$

Proof. We may suppose that $\mathfrak{h}$ is a subalgebra with grading $\mathfrak{h} = \mathfrak{h}_0 \oplus \cdots \oplus \mathfrak{h}_{r-1}$, $\mathfrak{h}_{r-1} \neq \mathfrak{g}^r_{m,n}(r-1)$, $\mathfrak{h}_i \subset \mathfrak{g}_i$, and we have proved the lower bound (10). Let us assume $m > 1$, $n > 1$, and choose a non-zero form C on $\mathfrak{g}^r_{m,n}(r-1)$ with $\ker C \supset \mathfrak{h}_{r-1}$. Then we know $[\mathfrak{h}_j, \mathfrak{h}_{r-j-1}] \subset \mathfrak{h}_{r-1} \subset \ker C$ and according to Lemma 2 there is a suitable coordinate expression of C satisfying one of the conditions (i)-(iv). Therefore we can apply Lemma 3.

Assume first C^α_i satisfies (i). Then for all j $\operatorname{codim} \mathfrak{h}_j + \operatorname{codim} \mathfrak{h}_{r-j-1} \geq 2m-2$ and consequently $\operatorname{codim} \mathfrak{h} = \sum_{j=0}^{r-1} \operatorname{codim} \mathfrak{h}_j \geq r(m-1)$.

Let us suppose that (ii) holds. Then $\operatorname{codim} \mathfrak{h}_0 + \operatorname{codim} \mathfrak{h}_{r-1} \geq m$ and $\operatorname{codim} \mathfrak{h}_j + \operatorname{codim} \mathfrak{h}_{r-j-1} \geq 2m$ for $1 \leq j \leq r-2$, so that $\operatorname{codim} \mathfrak{h} \geq m + (r-2)m = (r-1)m$. This completes the proof of (12) and analogous considerations lead to the estimate (11) if our coordinate expression of C satisfies (iii) or (iv). $\square$

Example 1. Let $\mathfrak{h}_1 \subset \mathfrak{g}^r_m$, $m > 1$ be defined by

$$\mathfrak{h}_1 = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i}; a^j_{(1)} = 0 \text{ for } j = 2, \ldots, m,\ 1 \leq |(1)| \leq r\}.$$

The linear subspace $\mathfrak{h}_1$ consists just of polynomial vector fields of degree r tangent to the line $x_2 = x_3 = \cdots = x_m = 0$, so that $\mathfrak{h}_1$ clearly is a Lie subalgebra in $\mathfrak{g}^r_m$ of codimension $r(m-1)$. Consider now the subalgebra in $\mathfrak{h} \subset \mathfrak{g}^r_{m,n}$ consisting of projectable polynomial vector fields of degree r over polynomial vector fields from $\mathfrak{h}_1$. This is a subalgebra of codimension $r(m-1)$ in $\mathfrak{g}^r_{m,n}$.

Example 2. Let $\mathfrak{h}_2 \subset \mathfrak{g}^r_{m,n}$, $n > 1$, be the subalgebra in $\mathfrak{g}^r_{m+n}$ of polynomial vector fields tangent to the line $x_1 = \cdots = x_m = x_{m+2} = \cdots = x_{m+n} = 0$. and let us define $\mathfrak{h} = \mathfrak{h}_2 \cap \mathfrak{g}^r_{m,n}$. Since every vector field in $\mathfrak{g}^r_{m,n}$ is tangent to the fiber over zero, this clearly is a Lie subalgebra with coordinate description

$$\mathfrak{h} = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i}; a^j_{(m+1)} = 0 \text{ for } m+1 < j \leq m+n,\ 1 \leq |(m+1)| \leq r\}$$

and hence its codimension is $r(n-1)$.

Example 3. The divergence of a homogeneous polynomial vector field on $\mathbb{R}^m$ of degree k is a homogeneous polynomial of degree $k-1$. For every element $a = a^i_\lambda x^\lambda \frac{\partial}{\partial x_i}$ we have

$$\operatorname{div} a = \sum_{\substack{1 \leq i \leq m \\ 1 \leq |\lambda| \leq r}} \lambda_i a^i_\lambda x^{\lambda - 1_i}.$$

Let M be the line in $\mathbb{R}^m$, $m > 1$, defined by $x_2 = x_3 = \cdots = x_m = 0$ and let us denote by $\mathfrak{h}_3$ the linear subspace in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{r-1}$ (note $\mathfrak{g}_0$ is missing!) $\mathfrak{h}_3 = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i} \in$

$\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{r-1}\,;\ \left(\operatorname{div}(a^i_\lambda x^\lambda \frac{\partial}{\partial x_i})\right)\big|_M = 0\}$. Of course, $\mathfrak{h}_3$ is not a Lie subalgebra in $\mathfrak{g}^r_m$. Further, consider the Lie subalgebra $\mathfrak{h}_4 \subset \mathfrak{g}^{r-2}_m$ consisting of all polynomial vector fields of degree at most $r-1$ without absolute terms and tangent to M, cf. Example 1. Let $\mathfrak{h}_5 = \left(\pi^r_{r-1}\right)^{-1}\mathfrak{h}_4$ and let us define a linear subspace $\mathfrak{h}_6 = (\mathfrak{h}_5 \cap \mathfrak{g}_0) \oplus (\mathfrak{h}_3 \cap \mathfrak{h}_5)$. First we claim that $\mathfrak{h}_3 \cap \mathfrak{h}_5$ is a subalgebra. Indeed, if X, $Y \in \mathfrak{h}_3 \cap \mathfrak{h}_5$, then either the degree of both of them is less then r or their bracket is zero. But in the first case, X and Y are tangent to M and their divergences are zero on M, so that $\operatorname{div}([X,Y])|_M = 0$. Now, consider a polynomial vector field X from the subalgebra $\mathfrak{h}_5 \cap \mathfrak{g}_0$ and a field $Y \in \mathfrak{h}_3 \cap \mathfrak{h}_5$. Since every field from $\mathfrak{g}_0$ has constant divergence everywhere and X is tangent to M, we get $\operatorname{div}([X,Y])|_M = 0$. In coordinates, we have

$$\mathfrak{h}_6 = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i} \in \mathfrak{g}^r_m;$$
$$a^j_{(1)} = 0,\ \sum_{i=1}^m a^i_{(1)+1_i}(1 + \delta^1_i |(1)|) = 0$$
$$\text{for } j = 2, \dots, m,\ 1 \le |(1)| \le r-1\}$$

Finally, we take the subalgebra $\mathfrak{h}$ in $\mathfrak{g}^r_{m,n}$ consisting of polynomial vector fields over the fields from $\mathfrak{h}_6$. The codimension of $\mathfrak{h}$ is $(r-1)m$.

Example 4. Analogously to Example 2, let us consider the subalgebra $\mathfrak{h}_7$ in $\mathfrak{g}^r_{m+n}$, $n > 1$,

$$\mathfrak{h}_7 = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i} \in \mathfrak{g}^r_{m+n};$$
$$a^j_{(m+1)} = 0,\ \sum_{i=1}^{m+1} a^i_{(m+1)+1_i}(1 + \delta^{m+1}_i |(m+1)|) = 0$$
$$\text{for } j = 1, \dots, m+n, j \ne m+1\ 1 \le |(m+1)| \le r-1\}$$

and let us define $\mathfrak{h} = \mathfrak{h}_7 \cap \mathfrak{g}^r_{m+n}$. We set

$$\mathfrak{h} = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i} \in \mathfrak{g}^r_{m,n};$$
$$a^j_{(m+1)} = 0,\ \sum_{i=m+1}^{m+n} a^i_{(m+1)+1_i}(1 + \delta^{m+1}_i |(m+1)|) = 0$$
$$\text{for } j = m+2, \dots, m+n,\ 1 \le |(m+1)| \le r-1\}$$

and we have found a Lie subalgebra in $\mathfrak{g}^r_{m,n}$ of codimension $(r-1)n$.

In the first and the second examples, the corresponding Lie groups consist of polynomial isomorphisms keeping invariant the given lines. These are closed subgroups. In the remaining two examples, we have to consider analogous subgroups in $G^{r-1}_{m,n}$, then to take their preimages in the group homomorphism π^r_{r-1}. Further we consider the subgroups of polynomial local isomorphisms at the origin identical in linear terms and without the absolute ones. Their subsets consisting just of maps keeping the volume form along the given lines also are subgroups. Finally, we take the intersections of these subgroups. All these subgroups are closed.

Proof of Theorem 1. The main idea of the proof was explained during the proof of the estimate $r \leq 2s+1$. But now, we can use Proposition 1 to get a better lower bound of the codimensions for every $m > 1$, $n > 1$. Consequently

$$r \leq \max\{\frac{s}{m-1}, \frac{s}{m}+1, \frac{s}{n-1}, \frac{s}{n}+1\}.$$

If $n = 0$ we get

$$s \geq \min\{r(m-1), (r-1)m\},$$

so that $r \leq \max\{\frac{s}{m-1}, \frac{s}{m}+1\}$. Since all the groups determined by the subalgebras we have constructed above are closed, the corresponding homogeneous spaces are examples of manifolds with actions of $G^r_{m,n}$ with the extreme values of r. If $m = 1$, let us consider $\mathfrak{h} \subset \mathfrak{g}^{2s+1}_{1,n}$ with the coordinate description $\mathfrak{h} = \{a^i_\lambda x^\lambda \frac{\partial}{\partial x_i};\ a^1_{(1)} = 0$ for $2 \leq |(1)| \leq s$ or $|(1)| = 2s+1\}$. The formula for the Lie bracket shows that $\mathfrak{h}$ is a subalgebra and one can see that the corresponding subgroup H in $G^{2s+1}_{1,n}$ is closed. Then $\dim(G^{2s+1}_{1,n}/H) = s$ and $G^{2s+1}_{1,n}$ acts non trivially.

Analogously we get the remaining example. □

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