BUNDLE FUNCTORS ON FIBRED MANIFOLDS

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ABSTRACT. We give a proof of the regularity of bundle functors on certain class of categories over manifolds and a description of all bundle functors on fibred manifolds with fixed dimensions of bases and fibers. Further we describe in the terms of Weil algebras all bundle functors on fibred manifolds with fixed dimension of bases which preserve fibred products. Finally we discuss certain natural operations with vector fields.

In this paper, all manifolds are smooth and paracompact. We denote by $\mathbb{N}_0$ the set of all non-negative integers.

1. Preliminaries

The classical theory of natural bundles and operators originated by A. Nijenhuis has been developed and extended by several authors, see e.g. [Epstein,Thurston, 79], [Palais, Terng, 77], [Janyška, 85]. The foundations of a general theory are outlined in the survey paper [Kolář, 89], a collection of the most of basic results on both the bundles and operators is prepared in [Kolář, Michor, Slovák]. In this paper, we present some specific features of the theory of bundle functors on category $\mathcal{FM}_m$ of fibred manifolds with m-dimensional bases and fibred morphisms over local diffeomorphisms on the bases and we also describe all bundle functors on the category $\mathcal{FM}_{m,n}$ of fibred manifolds with m-dimensional bases, n-dimensional fibers and local fibred isomorphisms. Some of our results are proved for a wider class of categories and as a special case we also reprove well known results for categories $\mathcal{M}f$ or $\mathcal{M}f_m$ of all smooth manifolds and smooth maps or m-dimensional manifolds and local diffeomorphisms.

A category $\mathcal{C}$ endowed with a faithful functor $m: \mathcal{C} \to \mathcal{M}f$ is called a *category over manifolds.* A $\mathcal{C}$-object Y is said to be over the underlying manifold mY. Since there is the inclusion $\mathcal{C}(Y, \bar{Y}) \subset C^\infty(mY, m\bar{Y})$, we identify the $\mathcal{C}$-morphisms with their underlying smooth maps. In the sequel, we shall adopt the definition of an admissible category (over manifolds) which reflects the properties usually needed for differential geometric constructions, see [Kolář, 89]. (We should remark that we do not need all the requirements from the definition in our proofs.)

A category over manifolds $m: \mathcal{C} \to \mathcal{M}f$ is said to be *local*, if every $\mathcal{C}$-object Y and every open subset $U \subset mY$ determine a $\mathcal{C}$-subobject $L(Y, U)$ of Y over U

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such that for every $Y \in \mathrm{Ob}\mathcal{C}$ and every open subsets $V \subset U \subset mY$, we have $L(Y, mY) = Y$, $L(L(Y,U),V) = L(Y,V)$ and the aggregation of both the objects and morphisms is assumed.

A *locally defined morphism* of Y into $\bar{Y}$ in a local category is a $\mathcal{C}$-morphism $f\colon L(Y,U) \to L(\bar{Y},V)$ for some open subsets $U \subset mY$, $V \subset m\bar{Y}$.

A $\mathcal{C}$-object Y is said to be locally homogeneous if for every x, $y \in mY$ there exists a locally defined $\mathcal{C}$-isomorphism f of Y into Y such that $f(x) = y$. Local category $\mathcal{C}$ is called locally homogeneous, if each $\mathcal{C}$-object is locally homogeneous. A local skeleton of a locally homogeneous category $\mathcal{C}$ is a system (C_α), $\alpha \in I$, of $\mathcal{C}$-objects such that every $\mathcal{C}$-object Y is locally isomorphic to a unique C_α. In this case Y is said to be an object of type α. A pointed local skeleton of a locally homogeneous category $\mathcal{C}$ is a local skeleton (C_α), $\alpha \in I$, with a distinguished point $0_\alpha \in mC_\alpha$ for each $\alpha \in I$. So a local pointed skeleton of category $\mathcal{FM}_m$ is the sequence $(\mathbb{R}^{m+n} \to \mathbb{R}^m, 0)$, $0 \in \mathbb{R}^{m+n}$, $n = 0, 1, 2 \dots$.

The space $J^r(Y,\bar{Y})$ of all r-jets of a $\mathcal{C}$-object Y into a $\mathcal{C}$-object $\bar{Y}$ is defined as the subset of the manifold $J^r(mY, m\bar{Y})$ formed by all r-jets of the locally defined $\mathcal{C}$-morphisms of Y into $\bar{Y}$. If we add the following three conditions on $J^r(Y,\bar{Y})$, Y, $\bar{Y} \in \mathrm{Ob}\mathcal{C}$, we get the *infinitesimally admissible* categories.

(a) $J^r(Y,\bar{Y})$ is a submanifold of $J^r(mY, m\bar{Y})$,

(b) the jet projections $J^r(Y,\bar{Y}) \to J^k(Y,\bar{Y})$, $0 \le k < r$, are surjective submersions,

(c) if $X \in J^r(Y,\bar{Y})$ is an invertible r-jet of mY into $m\bar{Y}$, then X is generated by a locally defined $\mathcal{C}$-isomorphism.

Taken a fixed local pointed skeleton $(C_\alpha, 0_\alpha)$, $\alpha \in I$, of $\mathcal{C}$ we write $\mathcal{C}^r(\alpha,\beta) = J^r_{0_\alpha}(C_\alpha, C_\beta)_{0_\beta}$. If $\mathcal{C}$ is infinitesimally admissible, the restrictions of the jet compositions are smooth maps between smooth manifolds and we heave obtained a category $\mathcal{C}^r$ over $I \times I$, the *r-th order skeleton of* $\mathcal{C}$.

The last requirement on an admissible category $\mathcal{C}$ over manifolds is the *smooth splitting property.* This property reads that for a local pointed skeleton $(C_\alpha, 0_\alpha)$, $\alpha \in I$, and for every smooth curve $\gamma\colon \mathbb{R} \to J^r(C_\alpha, C_\beta) \subset J^r(mC_\alpha, mC_\beta)$, α, $\beta \in I$, there exists a smooth map $\Gamma\colon \mathbb{R} \times mC_\alpha \to mC_\beta$ such that $\gamma(t) = j^r_{c(t)}\Gamma(t,-)$, where $c(t)$ is the source of r-jet $\gamma(t)$. (Note that all $\Gamma(t,-)$ must be $\mathcal{C}$-morphisms by definition.)

Since there are the canonical polynomial representatives of jets of fibred morphisms, both the categories $\mathcal{FM}_m$ and $\mathcal{FM}_{m,n}$ are admissible. In the sequel, we shall denote $B\colon \mathcal{FM} \to \mathcal{M}f$ the base functor.

1.1. Definition. A bundle functor on a local category $\mathcal{C}$ over manifolds is a functor $F\colon \mathcal{C} \to \mathcal{FM}$ satisfying $B \circ F = m$ and the localization condition:

(i) For every inclusion of an open subset $i_U : U \hookrightarrow mY$, $F(L(Y,U))$ is the restriction $p_Y^{-1}(U)$ of $p_Y : FY \to mY$ over U and Fi_U is the inclusion $p_Y^{-1}(U) \hookrightarrow FY$.

Let us notice that the fiber projections $p_Y : FY \to Y$ form a natural transformation $p^F : F \to m$. In particular, a bundle functor on $\mathcal{M}f_m$ is a natural bundle over m-dimensional manifolds in the sense of A. Nijenhuis, cf. [Epstein, Thurston, 79], [Palais, Terng, 77].

We call a bundle functor F on $\mathcal{C}$ *regular* if F transforms smoothly parameterized families of $\mathcal{C}$-morphisms into smoothly parameterized families of smooth maps.

A bundle functor $F\colon \mathcal{C} \to \mathcal{FM}$ is said to be *of order* r if for every point $x \in mY$, $Y \in \mathrm{Ob}\mathcal{C}$, and every locally defined $\mathcal{C}$-morphisms f, g, the equality $j^r_x f = j^r_x g$ implies that the restrictions of Ff and Fg to the fiber $F_x Y = p_Y^{-1}(x)$ coincide.

An important role in the general theory is played by the so called *jet groups*. By definition, given an admissible category $\mathcal{C}$ with a local pointed skeleton $(C_\alpha, 0_\alpha)$, $\alpha \in I$, the r-th jet group of type α is the Lie subgroup

$$G^r_\alpha := \mathrm{inv} J^r_{0_\alpha}(C_\alpha, C_\alpha)_{0_\alpha} \subset \mathrm{inv} J^r_{0_\alpha}(mC_\alpha, mC_\alpha)_{0_\alpha}.$$

Similarly to the classical natural bundles, every r-th order bundle functor F on an admissible category $\mathcal{C}$ determines the *associated maps*

$$F_{Y,\bar Y}\colon J^r(Y, \bar Y) \times_Y FY \to F\bar Y, \quad (j^r_x f, y) \mapsto Ff(y).$$

An application of the well known Boman's theorem, [Boman, 67], yields that the associated maps to an r-th order bundle functor F are smooth if and only if F is regular. On the other hand, the restrictions of the associated maps to a regular r-th order functor to $\mathcal{C}^r(\alpha, \beta) \times F_{0_\alpha} C_\alpha$, α, $\beta \in I$, define the *induced smooth action* of the r-th skeleton $\mathcal{C}^r$ on the system of manifolds $S_\alpha := F_{0_\alpha} C_\alpha$ and from these data one can reconstruct the original functor up to a natural equivalence. See [Kolář, 89] or [Kolář, Michor, Slovák] for the details.

The main advantage of a formulation of the concept of geometric objects in the categorical language is that we also get a simple explicit meaning for "geometric operations" in the concept of natural operators. In general, an operator can be viewed as a mapping transforming smooth maps into smooth maps. An operator will be called regular if smoothly parameterized families of maps are transformed into smoothly parameterized ones.

1.2. Definition. Let F_1, F_2 and G_1 be bundle functors on an admissible category $\mathcal{C}$, G_2 be an arbitrary functor $\mathcal{C} \to \mathcal{M}f$. Let $E_Y \subset \mathcal{C}(F_1 Y, F_2 Y)$, $Y \in \mathrm{Ob}\mathcal{C}$, be subsets of morphisms over identities id_Y and let us consider the system $E = (E_Y)$, $Y \in \mathrm{Ob}\mathcal{C}$. A *natural operator* $D\colon (F_1, F_2) \rightharpoonup (G_1, G_2)$ *with domain* E is a system of regular operators $D_Y\colon E_Y \to C^\infty(G_1 Y, G_2 Y)$, $Y \in \mathrm{Ob}\mathcal{C}$, such that for every $s_1 \in E_Y$, $s_2 \in E_{\bar Y}$ and $f \in \mathcal{C}(Y, \bar Y)$ the right-hand diagram commutes whenever the left-hand one does.

$$\begin{array}{ccc} F_2 Y & \xleftarrow{s_1} & F_1 Y \\ \big\downarrow{\scriptstyle F_2 f} & & \big\downarrow{\scriptstyle F_1 f} \\ F_2 \bar Y & \xleftarrow{s_2} & F_1 \bar Y \end{array} \qquad\qquad \begin{array}{ccc} G_1 Y & \xrightarrow{D_Y s_1} & G_2 Y \\ \big\downarrow{\scriptstyle G_1 f} & & \big\downarrow{\scriptstyle G_2 f} \\ G_1 \bar Y & \xrightarrow{D_{\bar Y} s_2} & G_2 \bar Y \end{array}$$

The domain E is called natural if for every embedding $i\colon U \to Y$ of an open submanifold the pullbacks of maps in E_Y lie in E_U, i.e. $i^* E_Y \subset E_U$. (In words, we can restrict the maps to open submanifolds and pull them back by isomorphisms.)

If $F_1 = \mathrm{Id}_{\mathcal{C}}$ and $G_2 = H \circ G_1$ where H is a bundle functor on a suitable category, domain E is natural and if all values $D_Y s$, $s \in E_Y$, are sections of the canonical projections $p^H_Y\colon H \circ G_1 Y \to G_1 Y$, then D is called a natural operator between F_2 and HG_1 (with domain E) and we write $D\colon F_2 \rightharpoonup HG_1$.

It follows immediately from the definition that every natural operator $D\colon (\mathrm{Id}, F) \rightharpoonup (G_1, G_2)$ with natural domain E is formed by a system of local regular operators $D_Y : E_Y \subset C^\infty(FY) \to C^\infty(G_1Y, G_2Y)$ with respect to the canonical projections $p_Y^{G_1} : G_1Y \to Y$, i.e. their values depend on germs of the sections in the underlying points only. Further the locality of bundle functors implies that the whole operator D is determined by the values of D_{C_α} on germs of sections in E_{C_α} at the distinguished points 0_α, where $(C_\alpha, 0_\alpha)$ is a local pointed skeleton of $\mathcal{C}$.

If we assume that the r-jets of sections in E_Y form fibred submanifolds in J^rFY, $Y \in \mathrm{Ob}\mathcal{C}$, and if all morphisms in $\mathcal{C}$ are locally invertible, we can completely recover the classical theory of r-th order natural operators between natural bundles to our case $D\colon F \rightharpoonup HG$, i.e. we get a bijection between the natural operators and certain equivariant maps, see [Kolář, 89].

2. Regularity of bundle functors

In this section we will prove the regularity of bundle functors for a class of categories over manifolds. In the original method developed in [Epstein, Thurston, 79] for the bundle functors on category $\mathcal{M}f_m$, one first uses the Whitney extension theorem, [Tougeron, 72], in a rather involved proof of the continuous regularity. Then, after manipulation with infinite dimensional Lie groups leading to certain rough estimates on possible orders of bundle functors, one gets the smooth regularity. We modify this approach using a nonlinear version of Peetre theorem, [Slovák, 88], and so we get the regularity independently on the finiteness of the order in a more general context.

We shall need a corollary of the nonlinear Peetre theorem in a special case only. Let us call a subset $E \subset C^\infty(M, N)$ *Whitney-extendible* if for every $f \in C^\infty(M, N)$, every sequence $f_k \in E$, $f_0 \in E$ and every convergent sequence $x_k \to x$ in M satisfying for all $k \in \mathbb{N}$ $\mathrm{germ} f(x_k) = \mathrm{germ} f_k(x_k)$ and $j^\infty_x f_0 = j^\infty_x f$, there are $g \in E$ and $k_0 \in \mathbb{N}$ such that $\mathrm{germ} g(x_k) = \mathrm{germ} f_k(x_k)$ for all $k \geq k_0$.

2.1. Proposition, [Slovák, 88]. *Let X, Y, Z, W be manifolds, $p\colon Z \to X$ be a fibration and let $D\colon E \subset C^\infty(X, Y) \to C^\infty(Z, W)$ be a local operator with respect to p. If E is Whitney-extendible, then for every fixed mapping $f \in E$ and every compact set $K \subset Z$ there exists $r \in \mathbb{N}$ such that for all $x \in p(K)$, $g \in E$, the condition $j^r_x f = j^r_x g$ implies $Df|(\pi^{-1}(x) \cap K) = Dg|(\pi^{-1}(x) \cap K)$.*

Let us remark that the full version of the nonlinear Peetre theorem yields the finiteness of the order of D on certain neighborhood of the fixed mapping f. It is easy to find examples of Whitney-extendible domains. Beside the whole $C^\infty(X, Y)$ and local diffeomorphisms which are Whitney-extendible directly by the definition, the Whitney-extendibility of the set of all fibred morphisms between two fibrations or of the set of all sections of a fibration is verified by a simple application of Whitney extension theorem. In particular, both the categories $\mathcal{FM}_{m,n}$ and $\mathcal{FM}_m$ have Whitney-extendible sets of morphisms.

2.2. Definition. A category $\mathcal{C}$ over manifolds is called *locally flat* if $\mathcal{C}$ admits a local pointed skeleton $(C_\alpha, 0_\alpha)$ where each $\mathcal{C}$-object C_α is over some $\mathbb{R}^{m(\alpha)}$ and if all translations t_x on $\mathbb{R}^{m(\alpha)}$ belong to the $\mathcal{C}$-morphisms.

2.3. Theorem. *Let $\mathcal{C}$ be an admissible locally flat category over manifolds with Whitney-extendible sets of morphisms, $m\colon \mathcal{C} \to \mathcal{M}f$ be the faithful functor. Let $F\colon \mathcal{C} \to \mathcal{M}f$ be a functor endowed with a natural transformation $p\colon F \to m$ such that the locality condition 1.1.(i) holds. Then there are canonical diffeomorphisms*

$$mC_\alpha \times p_{C_\alpha}^{-1}(0_\alpha) \cong FC_\alpha, \quad (x,z) \mapsto Ft_x(z) \tag{1}$$

and for every $Y \in \mathrm{Ob}\mathcal{C}$ of type α the map $p_Y : FY \to Y$ is a locally trivial fiber bundle with standard fiber $p_{C_\alpha}^{-1}(0_\alpha)$, in particular F is a bundle functor on $\mathcal{C}$.

Proof. Let us fix a type α and let us denote $\tau_x := Ft_x$ the action of the abelian group $\mathbb{R}^m = mC_\alpha$ on FC_α. The first (and the most technical) step in the proof of the Theorem is to prove that this action is continuous. Analyzing the original proof in [Epstein,Thurston, 79] or its modification [Mikulski, 85] we find that this proof also applies in our more general situation without any essential change. This is done in detail in [Kolář, Michor, Slovák]. But then a general theorem in [Montgomery, Zippin, 55] implies that this action is smooth (we deal with a continuous action τ of a Lie group such that each τ_x is a diffeomorphism). It follows that for every $z \in p_{C_\alpha}^{-1}(0_\alpha)$ the map $s\colon \mathbb{R}^m \to FC_\alpha$, $s(x) \mapsto \tau_x(z)$ is smooth and $p_{C_\alpha} \circ s = \mathrm{id}_{\mathbb{R}^m}$. Therefore p_{C_α} is a submersion and $p_{C_\alpha}^{-1}(0_\alpha)$ is a manifold. Since both the maps $(x,z) \mapsto \tau(x,z)$ and $y \mapsto \tau(-p_{C_\alpha}(y), y)$ are smooth, (1) is a diffeomorphism. The rest of the Theorem follows now from the locality of functor F. □

In our next step towards the regularity we show that the bundle functors in question have finite order at least "locally".

2.4. Lemma. *Let F be a bundle functor on an admissible category $\mathcal{C}$ and let $\mathcal{C}(Y,Y) \subset C^\infty(mY, mY)$ be Whitney-extendible. Consider a point $x \in Y$ and a compact set $K \subset p_Y^{-1}(x) \subset FY$. We define $Q_K := \cup_{f \in \mathrm{inv}\mathcal{C}(Y,Y)} Ff(K)$. Then there is $r \in \mathbb{N}$ such that for all invertible $\mathcal{C}$-morphisms f, g and for every point $y \in Y$ the equality $j^r_y f = j^r_y g$ implies*

$$Ff|(Q_K \cap p_Y^{-1}(y)) = Fg|(Q_K \cap p_Y^{-1}(y)).$$

Proof. Let us fix the map $\mathrm{id}_Y \in \mathcal{C}(Y,Y)$ and let us apply Proposition 2.1 to $F\colon \mathcal{C}(Y,Y) \to C^\infty(FY,FY)$, $p = p_Y$ and K. We denote r the resulting order. For every $z \in Q_K$ there are $y \in K$ and $g \in \mathrm{inv}\mathcal{C}(Y,Y)$ with $Fg(y) = z$. Consider f_1, $f_2 \in \mathrm{inv}\mathcal{C}(Y,Y)$ such that $j^r f_1(p(z)) = j^r f_2(p(z))$. Then $j^r(f_1 \circ g)(p(y)) = j^r(f_2 \circ g)(p(y))$ and therefore $j^r(g^{-1} \circ f_1^{-1} \circ f_2 \circ g)(p(y)) = j^r \mathrm{id}_Y(p(y))$. Hence $Ff_1(z) = Ff_1 \circ Fg(y) = Ff_2 \circ Fg(y) = Ff_2(z)$. □

2.5. Theorem. *Let $\mathcal{C}$ be an admissible locally flat category over manifolds with Whitney-extendible sets of morphisms. If all $\mathcal{C}$-morphisms are locally invertible, then every bundle functor F on $\mathcal{C}$ is regular.*

Proof. Since all morphisms are locally invertible and the functors are local, we may restrict ourselves to objects of one fixed type, say α. We shall write $(C, 0)$ for $(C_\alpha, 0_\alpha)$, $mC = \mathbb{R}^m$, $p = p_C$. Let us consider a smoothly parameterized family

$g_s \in \mathcal{C}(C,C)$ with parameters in a manifold P. For any $z \in FC$, $x = p(z)$, $f \in \mathcal{C}(C,C)$ we have

$$Ff(z) = \tau_{f(x)} \circ F(t_{-f(x)} \circ f \circ t_x) \circ \tau_{-x} \tag{2}$$

and the mapping in the brackets transforms 0 into 0. Since τ is a smooth action according to Theorem 2.3, the regularity will follow from (2) if we show that for families with $g_s(0) = 0$ the restrictions of Fg_s to the standard fiber $S = p^{-1}(0)$ are smoothly parameterized. Since the case $m = 0$ is trivial, we may assume $m > 0$. By virtue of Lemma 2.4 F is of order ∞. We first show that the induced action of the group of infinite jets $G^\infty_\alpha = \operatorname{inv} J^\infty_0(C,C)_0$ on S is continuous with respect to the inverse limit topology.

Consider converging sequences $z_n \to z$ in S and $j^\infty_0 f_n \to j^\infty_0 f_0$ in G^∞_α. We shall show that any subsequence of $Ff_n(z_n)$ contains a further subsequence converging to the point $Ff_0(z)$. On replacing f_n by $f_n \circ f_0^{-1}$, we may assume $f_0 = \mathrm{id}_C$. By passing to subsequences, we may assume that all absolute values of the derivatives of $(f_n - \mathrm{id}_C)$ at 0 up to order $2n$ are less then e^{-n}. Let us choose positive reals $\varepsilon_n < e^{-n}$ in such a way that on the open balls $B(0,\varepsilon_n)$ centered at 0 with diameters ε_n all the derivatives in question vary at most by e^{-n}. Let $x_n := (2^{-n}, 0, \dots, 0) \in \mathbb{R}^m$. By Whitney extension theorem there is a local diffeomorphism $f : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$f|B(x_{2n+1}, \varepsilon_{2n+1}) = \mathrm{id}_C \quad \text{and} \quad f|B(x_{2n}, \varepsilon_{2n}) = t_{x_{2n}} \circ f_{2n} \circ t_{-x_{2n}}$$

for all large n's. Since the sets of $\mathcal{C}$-morphisms are Whitney extendible, there is a $\mathcal{C}$-morphism h satisfying the same equalities for large n's. Now

$$\begin{aligned} \tau_{-x_n} \circ Fh \circ \tau_{x_n}(z_n) &= Ff_n(z_n) && \text{if } n \text{ is even} \\ \tau_{-x_n} \circ Fh \circ \tau_{x_n}(z_n) &= z_n && \text{if } n \text{ is odd.} \end{aligned}$$

Hence, by virtue of Theorem 2.3, $Ff_{2n}(z_{2n})$ converges to z and we have proved the continuity of the action of G^∞_α on S as required. Now, let us choose a relatively compact open neighborhood V of z and define $Q_V := (\cup_{\operatorname{inv}\mathcal{C}(C,C)} f(V)) \cap S$. This is an open submanifold in S and functor F defines an action of the group G^∞_α on Q_V. According to Lemma 2.4 this action factorizes to an action of a jet group G^r_α on Q_V which is continuous by the above part of the proof. Hence this action has to be smooth for the reason discussed in the proof of Theorem 2.3 and since smoothness is a local property and all $\mathcal{C}$-morphisms are locally invertible this concludes the proof. □

2.6. Corollary. *Every bundle functor on $\mathcal{FM}_{m,n}$ is regular.*

We can also deduce the regularity for bundle functors on $\mathcal{FM}_m$ using Theorems 2.3 and 2.5.

2.7. Corollary. *Every bundle functor on $\mathcal{FM}_m$ is regular.*

Proof. The system $(\mathbb{R}^{m+n} \to \mathbb{R}^m, 0)$, $n \in \mathbb{N}_0$, is a local pointed skeleton of $\mathcal{FM}_m$. Every morphism $f\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+k}$ is locally of the form $f = h \circ g$ where $g =$

$g_0 \times \mathrm{id}_{\mathbb{R}^n}\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ and h is a morphism over identity on $\mathbb{R}^m$ ($g_0 = f_0$, $h_1(x,y) = f_1(f_0^{-1}(x), y)$). So we can deal separately with this two special types of morphisms.

The restriction F_n of functor F to subcategory $\mathcal{FM}_{m,n}$ is a regular bundle functor according to 2.9 and the morphisms of the type $g_0 \times \mathrm{id}_{\mathbb{R}^n}$ are $\mathcal{FM}_{m,n}$-morphisms.

Hence it remains to discuss the latter type of morphisms. We may restrict ourselves to families $h_p\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+k}$, $p \in \mathbb{R}^q$, for some $q \in \mathbb{N}$. Let us consider $i\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n} \times \mathbb{R}^q$, $(x,y) \mapsto (x,y,0)$, $h\colon \mathbb{R}^{m+n+q} \to \mathbb{R}^{m+n}$, $h(-,-,p) = h_p$. Since all the maps h_p are over identity, h is a fibred morphism. We have $h_p = h \circ t_{(0,0,p)} \circ i$, so that $Fh_p = Fh \circ Ft_{(0,0,p)} \circ Fi$. According to Theorem 2.3 Fh_p is smoothly parameterized. □

3. Bundle functors on $\mathcal{FM}_{m,n}$ and $\mathcal{FM}_m$

Let us denote $G^r_{m,n}$ the r-th order jet group (of the only type) in $\mathcal{FM}_{m,n}$. This is the Lie subgroup in G^r_{m+n} formed by jets of local fibred isomorphisms $f\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ with source and target zero. Analyzing the structure of the Lie algebra of $G^r_{m,n}$ by means of methods similar to those used in [Zajtz, 87] the next Proposition was derived in [Slovák, 89].

3.1. Proposition. *Let a jet group $G^r_{m,n}$, $m \ge 1$, $n \ge 0$, act smoothly on a manifold S, $\dim S = s$, $s > 0$, and assume that r is essential, i.e. the action does not factorize to an action of $G^k_{m,n}$, $k < r$. Then $r \le 2s+1$. Moreover, if m, $n > 1$, then*

$$r \le \max\{\frac{s}{m-1}, \ \frac{s}{m}+1, \ \frac{s}{n-1}, \ \frac{s}{n}+1\}$$

and if $m > 1$, $n = 0$, then

$$r \le \max\{\frac{s}{m-1}, \ \frac{s}{m}+1\}.$$

All these estimates are sharp for all $m \ge 1$, $n \ge 0$, $s \ge 1$.

3.2. Theorem. *Let $F : \mathcal{FM}_{m,n} \to \mathcal{M}f$, $m \ge 1$, $n \ge 0$, be a functor endowed with a natural transformation $p\colon F \to m$ and satisfying the localization property 1.1.(i). Then $S := p^{-1}_{\mathbb{R}^{m+n}}(0)$ is a manifold of dimension $s \ge 0$ and for every $(Y \to M)$ in $\mathrm{Ob}\mathcal{FM}_{m,n}$ the mapping $p_Y : FY \to Y$ is a locally trivial fiber bundle with standard fiber S, i.e. $F\colon \mathcal{FM}_m \to \mathcal{FM}$. Functor F is a regular bundle functor of a finite order $r \le 2s+1$. If moreover $m > 1$, $n = 0$, then*

$$r \le \max\{\frac{s}{m-1}, \ \frac{s}{m}+1\},$$

and if $m > 1$, $n > 1$, then

$$r \le \max\{\frac{s}{m-1}, \ \frac{s}{m}+1, \ \frac{s}{n-1}, \ \frac{s}{n}+1\}.$$

All these estimates are sharp.

Let us notice that the action of $G^r_{m,n}$ on S induced by the functor F induces a bundle functor $\tilde{S}$ and according to the general theory, $\tilde{S}$ is canonically naturally equivalent to the functor F. Further there is a bijective correspondence between all natural transformations between two bundle functors on $\mathcal{FM}_{m,n}$ and the set of all $G^r_{m,n}$-equivariant maps between their standard fibers.

Proof of 3.2. We have only to prove the assertion concerning the order. The rest of the Theorem follows from Theorems 2.3 and 2.5. Since $\mathcal{FM}_{m,n}$ is locally flat, we have to prove that the action of the group G of germs of fibred morphisms $f\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, $f(0) = 0$, factorizes to an action of $G^r_{m,n}$ with the above bounds of r depending on s, m, n. As in the proof of Theorem 2.5, let $V \subset S$ be a relatively compact open set and $Q_V \subset S$ be the open submanifold invariant with respect to the action of G. By virtue of Lemma 2.4 the action of G on Q_V factorizes to an action of $G^k_{m,n}$ for some $k \in \mathbb{N}$. But then Proposition 3.1 yields the necessary estimates if $s > 0$. Moreover, if we consider the $G^r_{m,n}$-spaces with the extremal orders from Proposition 3.1, the above mentioned general construction gives bundle functors with the extremal orders.

The remaining case $s = 0$ follows immediately from the fact that given an action $\rho\colon G^r_{m,n} \to \mathrm{Diff}(S)$ on a zero-dimensional manifold S, then its kernel $\ker\rho$ contains the whole connected component of the unit. Since $G^r_{m,n}$ has two components and these can be distinguished by the first order jet projection, we see that the order can be at most one. Taken $S = \{1, -1\}$ we define the action of $j^1 f(0)$ to be the multiplication by the sign of the determinant of the linear map representing $j^1 f(0)$. The corresponding bundle functor $\tilde{S}$ is the bundle of elements of orientations and is of order 1. □

In the rest of this paper, we shall deal with functors on $\mathcal{FM}_m$. Consider a bundle functor $F\colon \mathcal{FM}_m \to \mathcal{FM}$ and let F_n be its restriction to $\mathcal{FM}_{m,n}$. We will write $S_n := p^{-1}_{\mathbb{R}^{m+n}}(0)$ for the standard fibers and $s_n := \mathrm{dim}S_n$. We have proved that functors F_n have finite orders $r(n)$ bounded by the estimates given in Theorem 3.2. Using ideas from [Kolář, Slovák, 89] and [Mikulski] we prove that all bundle functors on $\mathcal{FM}_m$ are of locally finite order.

3.3. Theorem. *Let $F\colon \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor. For all fibred manifolds Y with n-dimensional fibers and for all fibred maps $f, g : Y \to \bar{Y}$, the condition $j^{r(n+1)}_x f = j^{r(n+1)}_x g$ implies $Ff|F_xY = Fg|F_xY$. If $\mathrm{dim}\bar{Y} \le \mathrm{dim}Y$, then even the equality of $r(n)$-jets implies that the values on the corresponding fibers coincide.*

Proof. We may restrict ourselves to the case f, $g\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+k}$, $f(0) = g(0) = 0 \in \mathbb{R}^{m+k}$.

(a) First we discuss the case $n = k$. Let us assume $j^r_0 f = j^r_0 g$, $r = r(n)$ and consider families $f_t = f + t\mathrm{id}_{\mathbb{R}^{m+n}}$, $g_t = g + t\mathrm{id}_{\mathbb{R}^{m+n}}$, $t \in \mathbb{R}$. The Jacobians at zero are certain polynomials in t, so that the maps f_t and g_t are local diffeomorphisms at zero except a finite number of values of t. Since $j^r_0 f_t = j^r_0 g_t$ for all t, we have $Ff_t|S_n = Fg_t|S_n$ except a finite number of values of t. Hence the regularity of F implies $Ff|S_n = Fg|S_n$. As we deduced in the proof of Corollary 2.7, every fibred map $f \in \mathcal{FM}_m(\mathbb{R}^{m+n}, \mathbb{R}^{m+k})$ locally decomposes (in a canonical way) as $f = h \circ g$

where $g\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ and $h\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+k}$ is over the identity on $\mathbb{R}^m$. Hence in the rest of the proof we will restrict ourselves to morphisms over the identity.

(b) Next we assume $n = k + q$, $q > 0$, f, $g\colon \mathbb{R}^{m+k+q} \to \mathbb{R}^{m+k}$, and let $j^r_0 f = j^r_0 g$ with $r = r(n)$. Consider $\bar f = (f, \mathrm{pr}_2)$, $\bar g = (g, \mathrm{pr}_2)\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, where $\mathrm{pr}_2\colon \mathbb{R}^{m+k+q} \to \mathbb{R}^q$ is the projection onto the last factor. Since $j^r_0 \bar f = j^r_0 \bar g$, $f = \mathrm{pr}_1 \circ \bar f$ and $g = \mathrm{pr}_1 \circ \bar g$, the functoriality and (a) imply $Ff|S_n = Fg|S_n$.

(c) If $k = n + 1$ and $j^r_0 f = j^r_0 g$ with $r = r(n + 1)$, then we consider $\bar f$, $\bar g\colon \mathbb{R}^{m+n+1} \to \mathbb{R}^{m+n+1}$, $\bar f = f \circ \mathrm{pr}_1$, $\bar g = \mathrm{pr}_1$. Let us write $i\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n+1}$ for the inclusion $x \mapsto (x, 0)$. For every $y \in \mathbb{R}^{m+n+1}$ with $\mathrm{pr}_1(y) = 0$ we have $j^r_y \bar f = j^r_y \bar g$ and since $f = \bar f \circ i$, $g = \bar g \circ i$, we get $Ff|S_n = Fg|S_n$.

(d) Let $k = n + q$, $q > 0$, and $i\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n+q}$, $x \mapsto (x, 0)$. Analogously to (a) we may assume that f and g have maximal rank at 0. Hence according to the canonical local form of maps of maximal rank we may assume $g = i$.

(e) Let us write $f = (\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^k)\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+k}$, $k > n$, and assume $j^r_0 f = j^r_0 i$ with $r = r(n + 1)$. We define $h\colon \mathbb{R}^{m+n+1} \to \mathbb{R}^{m+k}$

$$h(x, y) = (\mathrm{id}_{\mathbb{R}^m}, f^1(x), \dots , f^n(x), y, f^{n+2}(x), \dots , f^k(x)).$$

Then we have

$$h \circ (\mathrm{id}_{\mathbb{R}^m}, \mathrm{id}_{\mathbb{R}^n}, f^{n+1}) = f$$
$$h \circ i = (\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, 0, f^{n+2}, \dots , f^k).$$

Since $j^r_0(\mathrm{id}_{\mathbb{R}^m}, \mathrm{id}_{\mathbb{R}^n}, f^{n+1}) = j^r_0 i$, part (c) of this proof implies

$$F(\mathrm{id}_{\mathbb{R}^{m+n}}, f^{n+1})|S_n = Fi|S_n$$

and we get for every $z \in S_n$

$$Ff(z) = Fh \circ Fi(z) = F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, 0, f^{n+2}, \dots , f^k)(z).$$

Now, we shall proceed by induction. Let us assume

$$Ff(z) = F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, 0, \dots , 0, f^{n+s}, \dots , f^k)(z), \qquad s > 1,$$

for every $z \in S_n$ and $j^{r(n+1)}_0 f = j^{r(n+1)}_0 i$. Let $\sigma\colon \mathbb{R}^{m+n+k} \to \mathbb{R}^{m+n+k}$,

$$\begin{aligned}\sigma(x, x^1, \dots , x^n, x^{n+1}, \dots , x^{n+s}, \dots , x^k)&\\ = (x, x^1, \dots , x^n, x^{n+s}, \dots , x^{n+1}, \dots , x^k).&\end{aligned}$$

We get

$$\begin{aligned}&F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, 0, \dots , 0, f^{n+s}, \dots , f^k)(z) =\\ &\quad = F\big(\sigma \circ (\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, f^{n+s}, 0, \dots , 0, f^{n+s+1}, \dots , f^k)\big)(z) =\\ &\quad = F\sigma \circ F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, 0, \dots , 0, f^{n+s+1}, \dots , f^k)(z) =\\ &\quad = F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots , f^n, 0, \dots , 0, f^{n+s+1}, \dots , f^k)(z).\end{aligned}$$

So the induction yields $Ff(z) = F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots, f^n, 0, \dots, 0)$. Since we always have $r(n+1) \geq r(n)$, (a) implies

$$F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots, f^n)|S_n = F\,\mathrm{id}_{\mathbb{R}^{m+n}}|S_n.$$

Finally, we get

$$\begin{aligned} Ff|S_n &= F(\mathrm{id}_{\mathbb{R}^m}, f^1, \dots, f^n, 0, \dots, 0)|S_n \\ &= F(i \circ (\mathrm{id}_{\mathbb{R}^m}, f^1, \dots, f^n))|S_n = Fi|S_n \end{aligned}$$

and the Theorem is proved. □

4. Bundle functors on $\mathcal{FM}_m$ preserving fibred products

Let us notice that every m-dimensional manifold M can be viewed as the trivial fibration $\mathrm{id}_M\colon M \to M$. Analogously to [Kolář, Slovák, 89] we say that a bundle functor $F\colon \mathcal{FM}_m \to \mathcal{FM}$ has the *point property*, if $FM = M$ for all m-dimensional manifolds M.

4.1. Lemma. *Let $F\colon \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor with the point property. For every fibration $q_Y\colon Y \to M$ in $\mathcal{FM}_m$ we have the fibration $Fq_Y\colon FY \to M$ which equals to the composition $q_Y \circ p_Y$ and the assignment*

$$C^\infty(q_Y\colon Y \to M) \ni s \mapsto Fs \in C^\infty(Fq_Y\colon FY \to M)$$

defines a natural transformation $C^\infty(\ ) \to C^\infty(\ ) \circ F$ over fibred isomorphisms.

Proof. By definition, we have the commutative diagram

$$\begin{array}{ccc} FY & \xrightarrow{Fq_Y} & FM \\ {\scriptstyle p_Y}\big\downarrow & & \big\| \\ Y & \xrightarrow{q_Y} & M \end{array}$$

and for every $f \in \mathrm{inv}\mathcal{FM}_m(Y, \bar{Y})$, we have $F(f \circ s \circ (Bf)^{-1}) = Ff \circ Fs \circ (Bf)^{-1}$. □

4.2. Remark. The situation is somewhat special in the case $m = 0$. Indeed, for connected zero-dimensional base manifolds we have $C^\infty(\ ) = \mathrm{Id}_{\mathcal{M}f}$. Then Lemma 4.1 yields canonical natural sections $Y \to FY$, cf. [Kolář, Slovák, 89]. If $m > 0$, then such sections do not exist in general, e.g. taken $F = J^1$ there are no natural connections.

Let us denote $S_n := F_0\mathbb{R}^{m+n}$ and $k(n) := \dim S_n$.

4.3. Lemma. *Let $F\colon \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor with the point property. For every $Y \in \mathrm{Ob}\mathcal{FM}_{m,n}$ the fibers of $p_Y\colon FY \to Y$ are diffeomorphic to $\mathbb{R}^{k(n)}$.*

Proof. We have to prove that S_n is diffeomorphic to $\mathbb{R}^{k(n)}$ for all $n \in \mathbb{N}$. Let us fix the section $i\colon \mathbb{R}^m \to \mathbb{R}^{m+n}$ defined by $s(x) = (x, 0)$ and let $f_t\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$ be the fiber homotheties $f_t(x,y) = (x,ty)$. So $f_t(0) = 0$ for all t and f_0 coincides with the composition

$$\mathbb{R}^{m+n} \xrightarrow{\mathrm{pr}_1} \mathbb{R}^m \xrightarrow{i} \mathbb{R}^{m+n}.$$

Hence Lemma 4.1 implies that $Ff_0(S_n) = Fi(0) \in S_n$. Further Ff_t is smoothly parameterized and $Ff_t(S_n) \subset S_n$. In this situation, a lemma from differential topology, see [Kolář, Slovák, 89] implies our assertion. □

4.4. Proposition. *Let $F\colon \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor with the point property. We have $k(n+p) \geq k(n) + k(p)$ and for every $\mathcal{FM}_m$-objects $q_Y\colon Y \to M$, $q_{\bar Y}\colon \bar Y \to M$ the canonical map $\pi\colon F(Y \times_M \bar Y) \to FY \times_M F\bar Y$ is a surjective submersion. The equality holds if and only if F preserves fibred products in dimensions n and p of the fibers. So F preserves fibred products if and only if $k(n) = n.k(1)$ for all $n \in \mathbb{N}_0$.*

Proof. Let us consider the diagram

$$\begin{array}{ccc} F(Y \times_M \bar Y) & & \\ & & \\ FY \times_M F\bar Y & \xrightarrow{\mathrm{pr}_2} & F\bar Y \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle Fq_{\bar Y}} \\ FY & \xrightarrow{Fq_Y} & M \end{array}$$

where p and $\bar p$ are the projections on $Y \times_M \bar Y$.

According to locality of bundle functors, it suffices to restrict ourselves to objects from a local pointed skeleton. In particular, we shall deal with the values of F on trivial bundles $Y = M \times S$. In the special case $m = 0$, the Proposition was proved in [Kolář, Slovák, 89] and instead of modifying the original proof we shall deduce our result from this case.

For every point $x \in M$ we write $(FY)_x := (Fq_Y)^{-1}(x)$ and we define a functor $G = G_x\colon \mathcal{M}f \to \mathcal{FM}$ as follows. We set $G(Y_x) := (FY)_x$ and for every map $f = \mathrm{id}_M \times f_1\colon Y \to \bar Y$, $f_1\colon Y_x \to \bar Y_x$ we define $Gf_1 := Ff|(FY)_x\colon GY_x \to G\bar Y_x$. If we restrict all the maps in the diagram to the appropriate preimages, we get the product $(FY)_x \xleftarrow{\mathrm{pr}_1} (FY)_x \times (F\bar Y)_x \xrightarrow{\mathrm{pr}_2} (F\bar Y)_x$ and $\pi_x\colon G(Y_x \times \bar Y_x) \to GY_x \times G\bar Y_x$. Since G has the point property, π_x is a surjective submersion, see [Kolář, Slovák, 89, Proposition 5].

Hence π is a fibred morphism over the identity on M which is fiber-wise a surjective submersion and consequently π is a surjective submersion. Since $\mathbb{R}^{m+n} \times_{\mathbb{R}^m} \mathbb{R}^{m+p} = \mathbb{R}^{m+(n+p)}$, the inequality $k(n+p) \geq k(n) + k(p)$ follows.

Now, similarly to [Kolář, Slovák, 89, Proposition 8], if the equality holds, then π is a covering. Since $F\mathbb{R}^{m+n}$ is simply connected for all n according to Lemma 4.3, we see that π is a global isomorphism. □

4.5. Example. We shall define a class of bundle functors on $\mathcal{FM}_m$ which preserve fibred products, the so called *vertical Weil bundles.* Let A be any Weil algebra (i.e.

a real, commutative, unital, local algebra) and $T_A : \mathcal{M}f \to \mathcal{FM}$ be the corresponding Weil bundle (i.e. $T_A M = \mathrm{Hom}(C^\infty(M), A)$ and the action on morphisms is defined by the composition), see [Kainz, Michor, 87] or [Kolář, 88] for definitions. We define $V_A : \mathcal{FM}_m \to \mathcal{FM}$ as follows. For every $q_Y : Y \to M$, $V_A Y := \cup_{x\in M} T_A Y_x$ and for all $f \in \mathcal{FM}_m(Y, \bar{Y})$, $f_x = f|Y_x$, $x \in M$, we set $V_A f|(V_A Y)_x := T_A f_x$. Since $V_A(\mathbb{R}^{m+n} \to \mathbb{R}^m) = \mathbb{R}^m \times T_A \mathbb{R}^n$ carries a canonical smooth structure, every fibred atlas on $Y \to M$ induces a fibred atlas on $V_A Y \to Y$. It is easy to verify that V_A is a bundle functor which preserves fibred products. In the special case of the algebra D of dual numbers we get the vertical tangent bundle V.

4.6. Theorem. *Let $F \colon \mathcal{FM}_m \to \mathcal{FM}$ be a bundle functor with the point property. The following conditions are equivalent.*

(i) *F preserves fibred products*
(ii) *For all $n \in \mathbb{N}$ it holds* $\dim S_n = n(\dim S_1)$
(iii) *There is a Weil algebra A such that $FY = V_A Y$ for every trivial bundle $Y = M \times S$ and for every mapping $f_1 \colon S \to \bar{S}$ we have $F(id_M \times f_1) = V_A(id_M \times f_1) \colon F(M \times S) \to F(M \times \bar{S})$.*

Proof. We proved the equivalence of (i) and (ii) in Proposition 4.4. Obviously condition (iii) implies (ii). So we complete the proof if we show that (i) implies (iii).

Consider a trivial bundle $Y = M \times S$ and let us repeat the construction of the product preserving functors $G = G_x$, $x \in M$, from the proof of Proposition 4.4. Since according to the general result in [Kainz, Michor, 87] $G_x = T_A$ for certain Weil algebra $A = A_x$, the conclusion is that $F(\mathrm{id}_M \times f_1)|(FY)_x = G(f_1) = V_A(\mathrm{id}_M \times f_1)|(FY)_x$. At the same time the general theory of bundle functors implies (we take $A = A_0$) $F\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n \times S_n = \mathbb{R}^n \times A^n = V_A \mathbb{R}^{m+n}$ for all $n \in \mathbb{N}$ (including the actions of jets of maps of the form $\mathrm{id}_{\mathbb{R}^m} \times f_1$). So all the algebras A_x coincide and since the bundles in question are trivial, we can always find an atlas $(U_\alpha, \varphi_\alpha)$ on Y such that the chart changings are over the identity on M. But a cocycle defining the topological structure of FY is obtained if we apply F to these chart changings and therefore the resulting cocycle coincides with that obtained from the functor V_A. □

4.7. Remark. Two maps f, $g \colon Y \to Q$ defined on a fibred manifold Y are said to have the same *(r,s)-jet* $j^{r,s}_y f = j^{r,s}_y g$ at $y \in Y$ if $j^r_y f = j^r_y g$ and $j^s_y(f|Y_x) = j^s_y(g|Y_x)$. Bundle functor F on $\mathcal{FM}_m$ is said to be of order (r,s) if for every f, $g \colon Y \to \bar{Y}$, $y \in Y$, the condition $j^{r,s}_y f = j^{r,s}_y g$ implies $Ff|F_y Y = Fg|F_y Y$. Analyzing the above considerations we get immediately the following assertion.

4.8. Proposition. *A bundle functor $F \colon \mathcal{FM}_m \to \mathcal{FM}$ of order $(0,s)$ preserves fibred products if and only if F is naturally equivalent to a vertical Weil bundle.*

4.9. Remark. If we drop the point property in the formulation of Theorem 4.6, then the conditions (i) and (iii) are still equivalent, but they are not equivalent to condition (ii), in general. There can appear some covering fenomena. Take as an example the functor $F(Y \to M) = GM \times_M Y$ where $G \colon \mathcal{M}f_m \to \mathcal{FM}$ is the bundle functor of elements of orientations, $Ff = G(Bf) \times_M f$.

5. Natural operators $T \rightharpoonup TV_A$ on projectable vector fields

We deduce that beside possible invariant vector fields on the bundles V_AY the flow operator is the only natural one. We shall write briefly $D\colon T_{\rm proj} \rightharpoonup TV_A$ for natural operators with the natural domain of all projectable vector fields.

For every projectable vector field on Y its flow consists of local fibred isomorphisms. If we apply any bundle functor F on $\mathcal{FM}_m$ to the flow, we get a flow of a vector field on FY, the value of the flow operator $\mathcal{F}$. In particular, we get the flow operators $\mathcal{V}_A$. Let us further recall a construction of certain invariant vector fields, cf. [Kolář, 88]. According to the general theory of Weil bundles, every element d in the Lie group $Aut(A)$ of the algebra automorphisms determines a natural transformation $d\colon T_A \to T_A$. Applying this construction fiber-wise, we get a natural transformation $d\colon V_A \to V_A$. So every element $a \in \mathfrak{Aut}(A)$, the Lie algebra of $Aut(A)$, tangent to a one-parametric subgroup $d(t)$ in $Aut(A)$ determines vertical vector fields $X_Y : V_AY \to TV_AY$ tangent to the maps $(d(t))_Y$ at $t = 0$. The way of the construction implies that the constant maps $\mathrm{op}(a)_Y : X \mapsto X_Y$ form a natural operator $\mathrm{op}(a)\colon T_{\rm proj} \rightharpoonup TV_A$.

5.1. Theorem. *Let A be a Weil algebra, $V_A : \mathcal{FM}_m \to \mathcal{FM}$, $m > 0$, be the corresponding vertical Weil bundle. Then all natural operators $D\colon T_{\rm proj} \rightharpoonup TV_A$ are of the form*

$$D = k\mathcal{V}_A + op(a) \qquad k \in \mathbb{R},\, a \in \mathfrak{Aut}(A).$$

Proof. As we discussed in section 1, D is fully determined by the values of $D_{\mathbb{R}^{m+n}}$ on the germs of vector fields at $0 \in \mathbb{R}^{m+n}$. Every germ of a vector field on $\mathbb{R}^{m+n}$ with a non-zero projection to $\mathbb{R}^m$ can be transformed into the germ of $\frac{\partial}{\partial x^1}$ where $(x^1, \dots, x^m, y^1 \dots, y^n)$ are the canonical global coordinates on $\mathbb{R}^{m+n} \to \mathbb{R}^m$ and $\frac{\partial}{\partial x^1}$ is invariant under translations $t_{(x,y)}$ on $\mathbb{R}^{m+n}$. Since our operators are regular, the naturality implies that for every bundle functor $F\colon \mathcal{FM}_m \to \mathcal{FM}$ and for every functor $G\colon \mathcal{FM}_m \to \mathcal{M}f$, two natural operators D, $\bar D\colon T_{\rm proj} \rightharpoonup (F,G)$ on projectable vector fields are equal if and only if

$$D_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})|F_0\mathbb{R}^{m+n} = \bar D_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})|F_0\mathbb{R}^{m+n}.$$

Further $D_{\mathbb{R}^{m+n}}(\frac{\partial}{\partial x^1})$ must be invariant with respect to the isotropy group $G \subset \mathrm{inv}\mathcal{FM}_m(\mathbb{R}^{m+n}, \mathbb{R}^{m+n})$ of $\frac{\partial}{\partial x^1}$.

Let us consider a natural operator $D\colon T_{\rm proj} \rightharpoonup TV_A$. For every fibred manifold $q_Y : Y \to BY$ we denote $P_Y = Tq_Y \circ Tp_Y : TV_AY \to TBY$, where $p_Y : V_A \to Y$ is the canonical bundle functor projection. These maps form a natural transformation $P\colon TV_A \to TB$ and so the composition $\tilde D = P \circ D$ is a natural operator $\tilde D\colon T_{\rm proj} \rightharpoonup (V_A, TB)$.

Let us choose a point z in the fiber $S_n = (V_A)_0\mathbb{R}^{m+n}$ and let us discuss the possible values $\tilde D_{\mathbb{R}^{m+n}}(\frac{\partial}{\partial x^1})(z)$. For every $f \in G$ of the form $f = f_0 \times \mathrm{id}_{\mathbb{R}^n}$, $f_0(0) = 0$, the naturality yields

$$\begin{aligned}\tilde D_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})(z) &= \tilde D_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1}) \circ V_Af(z)\\ &= Tf_0(\tilde D_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})(z)).\end{aligned}$$

This implies that the value is an element in $T_0\mathbb{R}^m$ which is invariant under the action of all maps $f_0 = (x^1g^1, g^2, \dots, g^m)$ where g^k's are arbitrary functions in variables $x^2, \dots, x^m$. Hence

$$\tilde{D}_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})(z) = k(z)(\tfrac{\partial}{\partial x^1})(0)$$

for some smooth function $k\colon S_n \to \mathbb{R}$.

Consider now maps $f \in G$ of the form $f = \mathrm{id}_{\mathbb{R}^m} \times f_1$, $f(0) = 0$. Naturality with respect to these maps reads

$$\begin{aligned} k(z)\tfrac{\partial}{\partial x^1}(0) &= \mathrm{id}_{T\mathbb{R}^m} \circ \tilde{D}_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})(z) \\ &= \tilde{D}_{\mathbb{R}^{m+n}}(\tfrac{\partial}{\partial x^1})(V_A f(z)) \\ &= k(V_A f(z))\tfrac{\partial}{\partial x^1}(0) \end{aligned}$$

If f_1 is a homothety on $\mathbb{R}^n$, $y \mapsto ty$, $t \in \mathbb{R}$, then $V_A f(z) = tz$ and we get $k(z) = k(tz)$. Consequently k is a constant function.

Now let us assume that $\tilde{D}_{\mathbb{R}^{m+n}}(\frac{\partial}{\partial x^1})|S_n = 0$. Then the restrictions $X_{\mathbb{R}^n} := D_{\mathbb{R}^{m+n}}(\frac{\partial}{\partial x^1})|(V_A\mathbb{R}^{m+n})_0$, $0 \in \mathbb{R}^m$, are sections of the fibrations $TT_A\mathbb{R}^n \to T_A\mathbb{R}^n$ and the naturality with respect to the maps $f = \mathrm{id}_{\mathbb{R}^m} \times f_1\colon \mathbb{R}^{m+n} \to \mathbb{R}^{m+k}$, $f_1\colon \mathbb{R}^n \to \mathbb{R}^k$, $f(0) = 0$, implies that $X_{\mathbb{R}^n}$ are invariant vector fields on $T_A\mathbb{R}^n$, i.e. they form an absolute operator X in the sense of [Kolář, 88]. By virtue of [Kolář, 88, Proposition 1], X is of the form $\mathrm{op}(a)\colon T \rightharpoonup TT_A$, $a \in \mathfrak{Aut}(A)$. In view of our definitions this implies that D coincides with $\mathrm{op}(a)\colon T_{\mathrm{proj}} \rightharpoonup TV_A$.

Finally, let $\mathcal{V}_A$ be the flow operator on projectable vector fields and let k be the constant from the first part of the proof. Let us define a natural operator $D_1 = D - k\mathcal{V}_A$. Then $\tilde{D}_1(\frac{\partial}{\partial x^1})|S_n = 0$ so that $D_1 = \mathrm{op}(a)$ for some $a \in \mathfrak{Aut}(A)$ and this concludes the proof. □

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