PROLONGATION OF VECTOR FIELDS TO JET BUNDLES

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The main result of the present paper is that all natural operators transforming every projectable vector field on a fibred manifold Y into a vector field on its r-th jet prolongation J^rY are the constant multiples of the flow operator only. We also deduce a similar result for the natural operators transforming every vector field on a manifold M into a vector field on any bundle of contact elements over M.

All manifolds and maps are assumed to be infinitely differentiable.

1. Formulation of the result.

We shall need an analogy of natural bundles and natural operators defined on the category $\mathcal{FM}_{m,n}$ of all fibred manifolds with m-dimensional bases and n-dimensional fibres and their fibred local isomorphisms. Since in this case the general theory differs only slightly from the well known theory of natural bundles, [7], we will mention briefly the basic facts. A general setting of the bundle functors on categories over manifolds including a detailed exposition of the functors on $\mathcal{FM}_{m,n}$ will appear in [4]. We write $\mathcal{M}f_m$ for the category of m-dimensional manifolds and local diffeomorphisms. For any fibred manifold $Y \to M$ we denote by $C^\infty(Y)$ the space of all smooth global sections. Every fibred isomorphism $f\colon Y \to \bar{Y}$ extends into a map $f^*\colon C^\infty(Y) \to C^\infty(\bar{Y})$. Given two fibred manifolds $Y \to M$ and $\bar{Y} \to M$ over the same base, we denote by $C^\infty_M(Y,\bar{Y})$ the set of all base-preserving morphisms of Y into $\bar{Y}$.

Definition 1. Let $I : \mathcal{FM}_{m,n} \to \mathcal{M}f_{m+n}$ be the faithful functor forgetting the fibrations. A *bundle functor F on the category $\mathcal{FM}_{m,n}$* consists of a covariant functor $F : \mathcal{FM}_{m,n} \to \mathcal{M}f$ and of a natural transformation $p : F \to I$ satisfying the following localization condition. If $i : U \to Y$ is an inclusion of an open fibred submanifold, then $Fi : FU \to FY$ is an embedding onto $p_Y^{-1}(U)$.

Given a bundle functor $F\colon \mathcal{FM}_{m,n} \to \mathcal{M}f$, a system of subsets $\mathcal{D}_Y \subset C^\infty(FY)$, $(Y \to M) \in \mathrm{Ob}\mathcal{FM}_{m,n}$, is said to be natural, if the following conditions hold,

(i) $f^*\mathcal{D}_Y = \mathcal{D}_{\bar{Y}}$ for every isomorphism $f\colon Y \to \bar{Y}$

(ii) the restriction of every $s \in \mathcal{D}_Y$ to every open fibred submanifold $U \subset Y$ belongs to $\mathcal{D}_U$

(iii) the subset $\mathcal{D}^r_Y = \{j^r_x s;\ s \in \mathcal{D}_Y,\ x \in M\}$ is a fibred submanifold of $J^r(FY \to Y)$ for every positive integer r.

Definition 2. Let F, G_1, $G_2 : \mathcal{FM}_{m,n} \to \mathcal{M}f$ be bundle functors and $\mathcal{D} = \{\mathcal{D}_Y \subset C^\infty(FY);\ Y \in \mathrm{Ob}\mathcal{FM}_{m,n}\}$ be a natural system. A system of maps $A_Y : \mathcal{D}_Y \to C^\infty_Y(G_1Y, G_2Y)$, $Y \in \mathrm{Ob}\mathcal{FM}_{m,n}$, is said to be a natural operator $A : F \rightharpoonup$

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(G_1, G_2) with domain $\mathcal{D}$ if for every $f \in \mathcal{FM}_{m,n}(Y, \bar{Y})$, $s_1 \in C^\infty(FY)$, $s_2 \in C^\infty(F\bar{Y})$ the commutativity of the left-hand diagram implies the commutativity of the right-hand one

$$\begin{array}{ccc} FY & \xleftarrow{s_1} & Y \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle f} \\ F\bar{Y} & \xleftarrow{s_2} & \bar{Y} \end{array} \qquad\qquad \begin{array}{ccc} G_1Y & \xrightarrow{A_Y s_1} & G_2Y \\ \big\downarrow{\scriptstyle G_1 f} & & \big\downarrow{\scriptstyle G_2 f} \\ G_1\bar{Y} & \xrightarrow{A_{\bar{Y}} s_2} & G_2\bar{Y} \end{array}$$

and if smoothly parametrized families of sections are transformed into smoothly parametrized ones.

If functor G_2 is a composition $G_2 = H \circ G_1$, where H is a bundle functor defined on a suitable category of fibred manifolds, and if the values of operators in question are sections of the canonical projections $p^H_{G_1 M}\colon H(G_1 M) \to G_1 M$, then we write $A\colon F_2 \rightharpoonup HG_1$.

Proposition 1. *For every natural operator $F \rightharpoonup (G_1, G_2)$ and every $s \in C^\infty(FY)$, $z \in G_1 Y$, the value $A_Y s(z)$ depends on the germ of s at $p^{G_1}_Y(z)$.*

Proof. This follows directly from the locality of bundle functors and naturality of the domain.

By the r-th order distinguished frame bundle of a fibred manifold $Y \in \mathrm{Ob}\mathcal{FM}_{m,n}$ we mean the space of all r-jets of the local fibred manifold isomorphisms from $\mathbf{R}^{m+n} \to \mathbf{R}^m$ into Y with source $0 \in \mathbf{R}^{m+n}$. This is a principal fibre bundle with structure group $G^r_{m,n}$ of all r-jets of the local isomorphisms of $\mathbf{R}^{m+n} \to \mathbf{R}^m$ into itself with source and target $0 \in \mathbf{R}^{m+n}$. By the general theory, [4], the fibre S of $F(\mathbf{R}^{m+n} \to \mathbf{R}^m)$ over $0 \in \mathbf{R}^{m+n}$ is a manifold endowed with a canonical action of $G^r_{m,n}$ and every FY is the fibre bundle associated with the r-th order distinguished frame bundle of Y with standard fibre S. Moreover, there is a bijective correspondence between all natural transformations between two r-th order bundle functors on $\mathcal{FM}_{m,n}$ and the set of all $G^r_{m,n}$-equivariant maps between their standard fibres.

Next we restrict our attention to natural operators with domains defined on the projectable vector fields on fibred manifolds, which form a natural domain. Let G_1, $G_2 : \mathcal{FM}_{m,n} \to \mathcal{M}f$ be any bundle functors and T be the tangent functor. Consider a natural operator A transforming the projectable vector fields on Y into the elements in $C^\infty_Y(G_1 Y, G_2 Y)$. We shall write briefly $A : T_{\mathrm{proj}} \rightharpoonup (G_1, G_2)$. In particular, we are interested in the case when the values of A_Y are vector fields on GY for a bundle functor $G\colon \mathcal{FM}_{m,n} \to \mathcal{M}f$. Then we write $A : T_{\mathrm{proj}} \rightharpoonup TG$.

There is a canonical natural operator $\mathcal{G}\colon T_{\mathrm{proj}} \rightharpoonup TG$, called the flow operator, defined as follows. For every projectable vector field X on $Y \in \mathrm{Ob}\mathcal{FM}_{m,n}$ its flow $\exp tX$ is formed by local $\mathcal{FM}_{m,n}$-morphisms. Hence we can apply functor G to get a one parameter family $\varphi_t = G(\exp tX)$, which is smooth by regularity of G. The value $\mathcal{G}X$ is the vector field on GY corresponding to the flow φ_t. In particular, for the bundle functor $J^r : \mathcal{FM}_{m,n} \to \mathcal{M}f$ of the r-th jet prolongation of fibred manifolds we have the flow operator $\mathcal{J}^r : T_{\mathrm{proj}} \rightharpoonup TJ^r$. The main part of our paper is devoted to the proof of the following assertion.

Theorem 1. *Every natural operator $A : T_{\mathrm{proj}} \rightharpoonup TJ^r$ is a constant multiple of the flow operator $\mathcal{J}^r$.*

2. Finite order natural operators.

DEFINITION 3. A natural operator $A : F \rightharpoonup (G_1, G_2)$ with domain $\mathcal{D}$ is said to be of order r if for every s_1, $s_2 \in \mathcal{D}_Y$ and $z \in G_1 Y$ the condition $j^r s_1(p_Y^{G_1}(z)) = j^r s_2(p_Y^{G_1}(z))$ implies $A_Y s_1(z) = A_Y s_2(z)$.

Let $A : F \rightharpoonup (G_1, G_2)$ be a natural operator of order r with domain $\mathcal{D}$. Then we have the so called *associated maps*

$$\mathcal{A}_Y : \mathcal{D}_Y^r \times_Y G_1 Y \to G_2 Y$$
$$\mathcal{A}_Y\big(j^r s(p_Y^{G_1}(z)), z\big) = A_Y s(z)$$

which are smooth by the regularity of A, [4]. Conversely, having the associated map $\mathcal{A}_Y$, the value of the operator A_Y on a section $s \in \mathcal{D}_Y$ is given by the left-hand side of the latter formula.

Proposition 2. *The maps $\mathcal{A}_Y$ are determined by the restriction*

$$\mathcal{A} = A_{(\mathbf{R}^{m+n} \to \mathbf{R}^m)} | S^r \times Z : S^r \times Z \to Q$$

where S^r or Z or Q are the fibres of $\mathcal{D}^r_{(\mathbf{R}^{m+n} \to \mathbf{R}^m)}$ or $G_1(\mathbf{R}^{m+n} \to \mathbf{R}^m)$ or $G_2(\mathbf{R}^{m+n} \to \mathbf{R}^m)$ over $0 \in \mathbf{R}^{m+n}$, respectively.

Proof. The Proposition follows immediately from Proposition 1 and the locality of bundle functors.

Let us denote

$$k := \max\{r + \text{order of } F, \text{order of } G_1, \text{order of } G_2\}.$$

Then the group $G^k_{m,n}$ acts on both $S^r \times Z$ and Q and, by the definition of naturality, the map $\mathcal{A} : S^r \times Z \to Q$ is $G^k_{m,n}$-equivariant. On the other hand, given a $G^k_{m,n}$-equivariant map $\mathcal{A} : S^r \times Z \to Q$, there is a unique natural operator A with $\mathcal{A}$ being the restriction of $\mathcal{A}_{(\mathbf{R}^{m+n} \to \mathbf{R}^m)}$ to the standard fibres, [2], [4]. Hence we have

Proposition 3. *There is a bijective correspondence between $G^k_{m,n}$-equivariant maps $\mathcal{A} : S^r \times Z \to Q$ and natural operators $A : F \rightharpoonup (G_1, G_2)$ with domain $\mathcal{D}$.*

If there is a natural transformation $\pi : G_2 \to G_1$ over the identical transformation on $\mathcal{FM}_{m,n}$ and if we require that the natural operators we are looking for transform the elements of $\mathcal{D}_Y$ into sections of $\pi_Y : G_2 Y \to G_1 Y$, then in the above correspondence we have to add the condition $\pi_0 \circ \mathcal{A} = \mathrm{pr}_2 : S^r \times Z \to Z$, where pr_2 is the projection onto the second factor and π_0 is the restriction of $\pi_{\mathbf{R}^{m+n} \to \mathbf{R}^m}$ to Q.

3. The finiteness of the order.

Proposition 4. *Let G_1, $G_2 : \mathcal{FM}_{m,n} \to \mathcal{M}f$ be bundle functors of orders less then or equal to $r+1$, $r \geq 0$. Then every natural operator $A : T_{\mathrm{proj}} \rightharpoonup (G_1, G_2)$ is of order less then or equal to r.*

The proof is based on a lemma.

Lemma 1. *Let $p : Y \to M$ be a fibred manifold and ξ, η be projectable vector fields on Y. Let $y \in Y$ satisfy $Tp \circ \xi(y) \neq 0$, $Tp \circ \eta \neq 0$. Then there is a locally defined fibred isomorphism f of Y on a neighbourhood of y transforming locally η into ξ. If moreover $j^r_y \xi = j^r_y \eta$, then there exists an isomorphism f with the property $j^{r+1}_y f = j^{r+1}_y \operatorname{id}_Y$.*

Proof. Let us first assume that in suitable fibred coordinates on Y it holds $\xi^1(0) \neq 0$ and $\eta = \frac{\partial}{\partial x^1}$. We are looking for a fibred map $f \colon \mathbf{R}^{m+n} \to \mathbf{R}^{m+n}$ satisfying

$$\begin{aligned} \xi^i(f^1(x), \dots, f^n(x)) &= \frac{\partial f^i}{\partial x^1}(x) \qquad 1 \leq i \leq n \\ \xi^p(f^1(x), \dots, f^{n+m}(x)) &= \frac{\partial f^p}{\partial x^1}(x) \qquad n < p \leq m+n \end{aligned}$$

But the solution $f = (f^i, f^p)$ of this system determined by the initial condition $f = \mathrm{id}$ on the hyperplane $x^1 = 0$ is a local fibred isomorphism at 0 with the required properties. Further, let ξ and η be arbitrary. According to our assumptions, we always can choose local fibred coordinates centred at y with both $\eta^1(0) \neq 0$ and $\xi^1(0) \neq 0$. By the first part of the proof we can find a local fibred isomorphism transforming $\frac{\partial}{\partial x^1}$ into η. Assume $j^r_0 \xi = j^r_0 \eta$. Then we have $\xi^a(x) = c^a + g^a(x)$ with $c^1 = 1$, all other c's equal to zero and $j^r_0 g = 0$. Consider the solution of the following system of equations

$$\begin{aligned} c^i + g^i(f^1(x), \dots, f^n(x)) &= \frac{\partial f^i}{\partial x^1}(x) \qquad 1 \leq i \leq n \\ g^p(f^1(x), \dots, f^{n+m}(x)) &= \frac{\partial f^p}{\partial x^1}(x) \qquad n < p \leq m+n \end{aligned}$$

determined by the initial condition $f = \mathrm{id}$ on the hyperplane $x^1 = 0$. We claim that the k-th order partial derivatives of the above solution f at the origin vanish for all $1 < k \leq r+1$. Indeed, if there is no derivative along the first axis, all the derivatives of order higher then one vanish according to the initial condition, and all the other cases follow directly from the equations. By the same argument we find that the first order partial derivatives of f at the origin coincide with the partial derivatives of the identity map.

Proof of Proposition 4. If $j^r_y \xi = j^r_y \eta$ and $\xi(y) \neq 0$, then there exists an f with $f^*\eta = \xi$ on a neighbourhood of y and $j^{r+1}_y f = j^{r+1}_y \mathrm{id}_Y$. Then Proposition 1 and the naturality imply for every $z \in G_1 Y$ with $p^{G_1}_Y(z) = y$

$$A_Y \xi(z) = A_Y(f^*\eta)(z) = G_2 f \circ A_Y \eta \circ G_1 f^{-1}(z) = A_Y \eta(z).$$

In the case $\xi(y) = 0$ we take a projectable vector field ζ on Y with $\zeta(y) \neq 0$ and consider the one-parameter families $\xi + t\zeta$, $\eta + t\zeta$, $t \in \mathbf{R}$. For every $t \neq 0$ we have $A_Y(\xi + t\zeta)(z) = A_Y(\eta + t\zeta)(z)$ by the first part of the proof. Since A is regular, this relation holds for $t = 0$ as well.

4. Proof of Theorem 1.

By Proposition 4 all natural operators $A : T_{\mathrm{proj}} \rightharpoonup TJ^r$ are of the order r, so that we can use the general procedure explained in section 2. First of all we should describe the action of $G^{r+1}_{m,n}$ on S^r. But according to Lemma 1, in every local fibred coordinates x^i, y^p on Y, all projectable vector fields with non-zero projections can be transformed into the vector field $\frac{\partial}{\partial x^1}$. Since the r-jets of these fields form a dense subset in the space of the r-jets of projectable vector fields, it suffices to show that the value of any natural operator A on $\frac{\partial}{\partial x^1}$ is a constant multilple of $\mathcal{J}^r \frac{\partial}{\partial x^1}$. That is why we shall deal with the restriction of $\mathcal{A} : S^r \times Z^r \to Q^r$ to the subsets $S^0 \times Z^r$ or $S_0 \times Z^r$, where Z^r or Q^r is the fibre of $J^r(\mathbf{R}^{m+n} \to \mathbf{R}^m)$ over $0 \in \mathbf{R}^{m+n}$ or the fibre $TJ^r(\mathbf{R}^{m+n} \to \mathbf{R}^m)$ over $0 \in \mathbf{R}^{m+n}$, respectively, S^0 is the subset of all constant vector fields and $S_0 \subset S^0$ is formed by vector fields with zero components in $\mathbf{R}^n$.

Having the canonical coordinates x^i and y^p on $\mathbf{R}^{m+n}$, let X^i, Y^p be the induced coordinates on S^0, let y^p_α, $1 \le |\alpha| \le r$, be the induced coordinates on Z^r and $Q^i = dx^i$, $Q^p = dy^p$, $Q^p_\alpha = dy^p_\alpha$ be the additional coordinates on Q^r. The restriction $\mathcal{A} : S^0 \times Z^r \to Q^r$ is given by some functions

$$\begin{aligned} Q^i &= f^i(X^i, Y^q, y^s_\beta) \\ Q^p &= f^p(X^i, Y^q, y^s_\beta) \\ Q^p_\alpha &= f^p_\alpha(X^i, Y^q, y^s_\beta). \end{aligned}$$

Let us denote by g^i, g^p, g^p_α the restrictions of the corresponding f's to $S_0 \times Z^r$. The flows of constant vector fields are formed by translations, so that their r-jet prolongations are the induced translations of $J^r(\mathbf{R}^{m+n} \to \mathbf{R}^m)$ identical on all fibres. Therefore $\mathcal{J}^r \frac{\partial}{\partial x^1} = \frac{\partial}{\partial x^1}$ and it suffices to prove

$$g^i = kX^i, \qquad g^p = 0, \qquad g^p_\alpha = 0. \tag{1}$$

Let us remark that the coordinate formula for $\mathcal{J}^1\eta$ is

$$\mathcal{J}^1\eta = \eta^i \frac{\partial}{\partial x^i} + \eta^p \frac{\partial}{\partial y^p} + \left(\frac{\partial \eta^p}{\partial x^i} + \frac{\partial \eta^p}{\partial y^q} y^q_i - \frac{\partial \eta^j}{\partial x^i} y^p_j \right) \frac{\partial}{\partial y^p_i},$$

provided $\eta = \eta^i(x)\frac{\partial}{\partial x^i} + \eta^p(x,y)\frac{\partial}{\partial y^p}$, and to evaluate $\mathcal{J}^r\eta$, we have to iterate this formula and to use the canonical inclusion $J^r(Y \to M) \hookrightarrow J^1(J^{r-1}(Y \to M))$, cf. [5].

We shall prove (1) by induction on the order r. We have an inclusion $GL(m,\mathbf{R}) \times GL(n,\mathbf{R}) \hookrightarrow G^{r+1}_{m,n}$ determined by the products of linear isomorphisms of $\mathbf{R}^m$ and $\mathbf{R}^n$. It is easy to see that the action of $GL(m,\mathbf{R}) \times GL(n,\mathbf{R})$ on all quantities is tensorial. Using the equivariancy with respect to the homotheties in $GL(n,\mathbf{R})$, we obtain $f^i(X^j, Y^p, y^q_l) = f^i(X^j, kY^p, ky^q_l)$, $k \in \mathbf{R}$, $k \ne 0$, so that f^i depends on X^j only. Then the equivariancy of f^i with respect to $GL(m,\mathbf{R})$ implies $f^i = kX^i$, $k \in \mathbf{R}$, cf. [2]. The equivariancy of f^p with respect to the homotheties in $GL(n,\mathbf{R})$ gives $kf^p(X^i, Y^q, y^s_j) = f^p(X^i, kY^q, ky^s_j)$. This kind of homogeneity implies $f^p =$

$h^p_q(X^i)Y^q + h^{pj}_q(X^i)y^q_j$ with some smooth functions h^p_q, h^{pj}_q. Using the homotheties in $GL(m,\mathbf{R})$, we then obtain $h^p_q = \text{const}$ and $h^{pj}_q(X^i) = c^{pj}_{qi}X^i$. Then Lemma 3 of [2] yields $f^p = aY^p + by^p_i X^i$, a, $b \in \mathbf{R}$. Applying the same procedure to f^p_i, we find $f^p_i = cy^p_i$, $c \in \mathbf{R}$.

Let G^r_m denote the group of all invertible r-jets of $\mathbf{R}^m$ into $\mathbf{R}^m$ with source and target 0. Consider further the injection $G^2_n \hookrightarrow G^2_{m,n}$ determined by the products with the identities on $\mathbf{R}^m$. The action of an element (a^p_q, a^p_{qr}) of the latter subgroup is given by

$$\bar{y}^p_i = a^p_q y^q_i \tag{2}$$

$$\bar{Q}^p_i = a^p_{qt} y^q_i Q^t + a^p_q Q^q_i \tag{3}$$

and S_0 is an invariant subspace. In particular, (3) with $a^p_q = \delta^p_q$ gives an equivariancy condition

$$cy^p_i = ba^p_{qt} y^q_i y^t_j X^j + cy^p_i .$$

This yields $b = 0$, so that $g^p = 0$. Further, the subspace S^0 is invariant with respect to the inclusion of $G^1_{m,n}$ into $G^2_{m,n}$ determined by the 2-jets of linear transformations. The equivariancy of f^p_i with respect to an element $(\delta^i_j, \delta^p_q, a^p_i) \in G^1_{m,n}$ means $cy^p_i = c(y^p_i + a^p_i)$. Hence $c = 0$, which completes the proof for $r = 1$.

For $r \geq 2$ it suffices to discuss the g's only. Using the homotheties in $GL(n,\mathbf{R})$ we find that $g^p_{i_1\cdots i_s}(X^j, y^q_\beta)$, $1 \leq |\beta| \leq r$, is linear in y^q_β. The homotheties in $GL(m,\mathbf{R})$ and Lemma 3 from [2] then yield

$$g^p_{i_1\cdots i_s} = W^p_{i_1\cdots i_s} + c_s y^p_{i_1\cdots i_s i_{s+1}\cdots i_r} X^{i_{s+1}} \dots X^{i_r} \tag{4}$$

where $W^p_{i_1\cdots i_s}$ do not depend on $y^p_{i_1\cdots i_r}$, $s = 1, \dots, r-1$, and

$$g^p_{i_1\cdots i_r} = c_r y^p_{i_1\cdots i_r} \tag{5}$$

$$g^p = b_1 y^p_i X^i + \cdots + b_r y^p_{i_1\cdots i_r} X^{i_1} \dots X^{i_r}. \tag{6}$$

Similarly to the first order case, we have an inclusion $G^{r+1}_n \hookrightarrow G^{r+1}_{m,n}$ determined by the products of diffeomorphisms on $\mathbf{R}^n$ with the identity of $\mathbf{R}^m$. One finds easily the following transformation law

$$\bar{y}^p_{i_1\cdots i_s} = a^p_q y^q_{i_1\cdots i_s} + F^p_{i_1\cdots i_s} + a^p_{q_1\cdots q_s} y^{q_1}_{i_1} \dots y^{q_s}_{i_s} \tag{7}$$

where $F^p_{i_1\cdots i_s}$ is a polynomial expression linear in a^p_α with $2 \leq |\alpha| \leq s-1$ and independent on $y^p_{i_1\cdots i_s}$. This implies

$$\bar{Q}^p_{i_1\cdots i_s} = a^p_q Q^q_{i_1\cdots i_s} + G^p_{i_1\cdots i_s} + a^p_{q_1\cdots q_s q_{s+1}} y^{q_1}_{i_1} \dots y^{q_s}_{i_s} Q^{q_{s+1}} \tag{8}$$

where $G^p_{i_1\cdots i_s}$ is a polynomial expression linear in a^p_α with $2 \leq |\alpha| \leq s$ and linear in Q^p_α, $0 \leq |\alpha| \leq s-1$.

We deduce that every $g^p_{i_1\cdots i_s}$, $0 \le s \le r-1$, is independent on $y^p_{i_1\cdots i_r}$. On the kernel of the jet projection $G^{r+1}_n \to G^r_n$, (8) for $r = s$ gives

$$0 = a^p_{q_1\cdots q_r q_{r+1}} y^{q_1}_{i_1} \dots y^{q_r}_{i_r} g^{q_{r+1}}$$

Hence $g^p = 0$. On the kernel of the jet projection $G^r_n \to G^{r-1}_n$, (8) with $s = 1, \dots, r-1$, implies

$$0 = c_s a^p_{q_1\cdots q_r} y^{q_1}_{i_1} \dots y^{q_r}_{i_r} X^{i_{s+1}} \dots X^{i_r}$$

i.e. $c_s = 0$. By projectability, g^i and g^p_α, $0 \le |\alpha| \le r-1$, correspond to a $G^r_{m,n}$-equivariant map $S_0 \times Z^{r-1} \to Q^{r-1}$. By the induction hypothesis, $g^p_\alpha = 0$ for all $0 \le |\alpha| \le r-1$. Then on the kernel of the jet projection $G^{r+1}_n \to G^{r-1}_n$ (8) gives

$$0 = c_r a^p_{q_1\cdots q_r} y^{q_1}_{i_1} \dots y^{q_r}_{i_r}$$

i.e. $g^p_{i_1\cdots i_r} = 0$.

5. Prolongation of vector fields to the bundles of contact elements.

In this section we describe a class of the classical natural bundles with the property that the only natural operators (in the classical sense, [2]) transforming every vector field on the base manifold into a vector field on the total space of the bundle are the constant multiples of the flow operator only.

We recall that the bundle $T^r_n M \to M$ of all n-dimensional velocities of order r on a manifold M is the space $J^r_0(\mathbf{R}^n, M)$ of all r-jets of $\mathbf{R}^n$ into M with source $0 \in \mathbf{R}^n$. For $n \le m$, a velocity $A \in T^r_n M$ at $x \in M$ is called regular, if its underlying 1-jet corresponds to a linear map $T_0\mathbf{R}^n \to T_x M$ of the maximal rank. There is a canonical right action of the jet group G^r_n on $T^r_n M$ given by the jet composition. The equivalence class $A \circ G^r_n$ of a regular velocity $A \in T^r_n M$ is called a contact element of dimension n and order r on M, cf. [1]. The space $K^r_n M$ of all such elements has a canonical structure of a fibred manifold over M. (The elements of $K^r_n M$ can be viewed as the equivalence classes of n-dimensional submanifolds having r-th order contact at a common point, [1].) Using the jet composition, we extend every local diffeomorphism $f\colon M \to \bar M$ into a map $K^r_n f\colon K^r_n M \to K^r_n \bar M$. Thus, K^r_n is a bundle functor on the category $\mathcal{M}f_m$ of all m-dimensional manifolds and their local diffeomorphisms, or, which is the same, a classical natural bundle over m-manifolds, [7]. For $M = \mathbf{R}^m$, the elements of $K^r_n \mathbf{R}^m$ transversal to the canonical fibration $\mathbf{R}^m = \mathbf{R}^n \times \mathbf{R}^{m-n} \to \mathbf{R}^n$, which form an open dense subset in $K^r_n \mathbf{R}^m$, coincide with the elements of the r-th jet prolongation $J^r(\mathbf{R}^m \to \mathbf{R}^n)$ of this fibred manifold.

Denote by $\mathcal{K}^r_n$ the flow operator corresponding to the functor K^r_n in the classical sense, [3]. We remark that for a vector field $\xi = \xi^p(x)\frac{\partial}{\partial x^p} + \xi^u(x)\frac{\partial}{\partial x^u}$, $p, q = 1, \dots, n$, $u, v = n+1, \dots, m$ the coordinate expression of $\mathcal{K}^1_n \xi$ is

$$\xi^p \frac{\partial}{\partial x^p} + \xi^u \frac{\partial}{\partial x^u} + \left(\frac{\partial \xi^u}{\partial x^p} + \frac{\partial \xi^u}{\partial x^v} x^v_p - \frac{\partial \xi^q}{\partial x^p} x^u_q - \frac{\partial \xi^q}{\partial x^v} x^v_p x^u_q \right) \frac{\partial}{\partial x^u_p}$$

where x^u_p are the induced local coordinates on $K^1_n \mathbf{R}^m$, see [6].

Theorem 2. *Every natural operator $A\colon T \rightharpoonup TK_n^r$ is a constant multiple of the flow operator $\mathcal{K}_n^r$.*

Proof. It suffices to discuss the case $M = \mathbf{R}^m$. The above mentioned fibration $\mathbf{R}^m \to \mathbf{R}^n$ identifies an open dense subset in $K_n^r\mathbf{R}^m$ with $J^r(\mathbf{R}^m \to \mathbf{R}^n)$. By definition, on this open dense subset it holds $\mathcal{J}^r\xi = \mathcal{K}_n^r\xi$ for every projectable vector field ξ on $\mathbf{R}^m \to \mathbf{R}^n$. Since operator A commutes with the action of all diffeomorphisms preserving fibration $\mathbf{R}^m \to \mathbf{R}^n$, the restriction of A to $\frac{\partial}{\partial x^1}$ is a constant multiple of $\mathcal{K}_n^r\left(\frac{\partial}{\partial x^1}\right)$ by Theorem 1. But every vector field on $\mathbf{R}^m$ can be locally transformed into $\frac{\partial}{\partial x^1}$ in a neighbourhood of any non-zero point. This proves Theorem 2.

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