# INFINITESIMALLY NATURAL OPERATORS ARE NATURAL

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Abstract. It is well known that linear geometric operations (like the exterior differential) commute with the Lie derivative. A detailed analysis of both the concepts of geometric operations and of Lie differentiation leads to the proof of a converse implication even in the nonlinear case. So naturality is equivalent to commuting with Lie differentiation. We also generalize this result to the case of gauge natural operators.

## 1. Preliminaries

**1.1. Natural bundles.** The notion of a (local) geometric operation has got an explicit and well defined meaning in the concept of the so called natural operators between natural bundles, cf. [Nijenhuis, 72], [Kolář, 90].

Let us write $\mathcal{M}f$ for the category of manifolds and smooth mappings, $\mathcal{M}f_m$ for the category of $m$-dimensional manifolds and local diffeomorphisms (i.e. globally defined maps of maximal rank at each point) and $\mathcal{M}f_m^+$ for the category of oriented $m$–dimensional manifolds and orientation preserving local diffeomorphisms. Further let $\mathcal{FM}$ denote the category of fibered manifolds and fibered morphisms and $B\colon \mathcal{FM} \to \mathcal{M}f$ be the base functor.

A *bundle functor* (natural bundle) $F$ on $\mathcal{M}f_m$ is a functor $F\colon \mathcal{M}f_m \to \mathcal{FM}$ such that

(i) $B \circ F = \mathrm{id}_{\mathcal{M}f_m}$
(ii) for every inclusion $i_U\colon U \to M$ of an open submanifold, $FU$ is the restriction $p_M^{-1}(U)$ of the value $FM = (p_M\colon FM \to M)$ to $U$ and $Fi_U$ is the inclusion $p_M^{-1}(U) \to FM$.

A natural bundle $F$ is said to be of order $r$ if for every $x \in M$ and every local diffeomorphism $f\colon M \to M'$ the restriction of $Ff$ to the fiber $F_xM$ over $x$ depends only on the jet $j^r_x f$. Every natural bundle is of finite order, see [Palais, Terng, 77] and [Kolář–Michor–Slovák, Theorem 22.1].

**1.2. Natural operators.** Let $N \to M$ and $N' \to M$ be fibered manifolds. A mapping $D\colon C^\infty(N) \to C^\infty(N')$ is called a *local regular operator* if for each point $x \in M$ and every section $s \in C^\infty(N)$, $Ds(x)$ depends only on the germ of $s$ at $x$ and

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smoothly parameterized families of sections are mapped to smoothly parameterized families.

Let $F$, $G\colon \mathcal{M}f_m \to \mathcal{FM}$ be bundle functors. A *natural operator* $D\colon F \to G$ is a system $D_M$, $M \in \mathrm{Ob}\mathcal{M}f_m$, of local regular operators such that for each section $s$ and each diffeomorphisms $f: M \to N$ we have $Gf \circ D_M s \circ f^{-1} = D_N(Ff \circ s \circ f^{-1})$, and for each open submanifold $U \hookrightarrow M$ we have $(D_M s)|_U = D_U(s|_U)$.

A natural operator $D$ is said to be of order $r \le \infty$ if the values of all operators $D_M$ depend on $r$-jets only.

**1.3. The jet groups.** The Lie group $G^k_m := \mathrm{inv}J^k_0(\mathbb{R}^m, \mathbb{R}^m)_0$, $1 \le k < \infty$, with multiplication defined by composition of jets is called the *$k$-th jet group* in dimension $m$. The Lie algebra $\mathfrak{g}^k_m$ of the Lie group $G^k_m$ is the vector space $\{j^k_0 X\ ;\ X \in \mathcal{X}(\mathbb{R}^m),\ X(0) = 0\}$ of $k$–jets of vector fields on $\mathbb{R}^m$ at 0 with the bracket

$$[j^k_0 X, j^k_0 Y] = -j^k_0[X, Y] \tag{1}$$

and exponential mapping

$$\exp(j^k_0 X) = j^k_0\ \mathrm{Fl}^X_1, \qquad j^k_0 X \in \mathfrak{g}^k_m, \tag{2}$$

where $\mathrm{Fl}^X$ denotes the flow of the vector field $X$ (see [Terng, 78]). The direct limit of the groups $G^k_m$ is the infinite jet group $G^\infty_m := \mathrm{inv}J^\infty_0(\mathbb{R}^m, \mathbb{R}^m)_0$.

The first order jet group $G^1_m$ is identified with $GL(m, \mathbb{R})$ and $G^k_m$ is the semidirect product $G^1_m \circledS B^k_1$, where $B^k_1$ is the kernel of the jet projection $\pi^k_1 : G^k_m \to G^1_m$.

If $F$ is a natural bundle of order $r$, then there is the induced action of $G^r_m$ on the so called *standard fiber* $S = F_0\mathbb{R}^m$ of the natural bundle $F$. The $k$-th jet prolongation $J^k \circ F$ is a natural bundle of order $k + r$ with standard fiber $J^k_0(F\mathbb{R}^m)$ identified with $T^k_m S = J^k_0(\mathbb{R}^m, S)$.

Consider two $r$-th order natural bundles $F$ and $G$. A $k$-th order natural operator $D\colon F \to G$ is completely determined by the so called *associated map* $\mathcal{D}\colon T^k_m S \to Q$, where $Q := G_0\mathbb{R}^m$ and $\mathcal{D}(j^k_0 s) := D_{\mathbb{R}^m} s(0)$ for all $s \in C^\infty(F\mathbb{R}^m)$. By naturality $\mathcal{D}$ commutes with the induced actions of $G^{k+r}_m$ on the standard fibers, i.e. $\mathcal{D}$ is $G^{k+r}_m$-equivariant.

On the other hand, every $G^{r+k}_m$-equivariant map $f\colon T^k_m S \to Q$ gives rise to a unique natural operator $F \to G$ with associated map $f$.

The definitions and the theory of natural bundles and operators apply to the category $\mathcal{M}f^+_m$ without any essential change. We only have to replace the jet groups $G^r_m$ by their connected components of the units $G^k_m{}^+$.

**1.4. The flow operator.** For every bundle functor $F$ on $\mathcal{M}f_m$ and every vector field $X$ on an $m$-dimensional manifold $M$ we can apply $F$ to the flow of $X$ (cf. the locality condition for bundle functors). In this way, we obtain a flow of a vector field $\mathcal{F}X$ on the manifold $FM$. This construction defines the so called *flow operator* $\mathcal{F}$ which is an example of a more general concept of natural operators which extend the bases, cf. [Kolář, 90].

**1.5. The Lie derivative.** For every smooth map $f\colon M \to N$ and vector fields $X \in \mathcal{X}(M)$, $Y \in \mathcal{X}(N)$ we define the *generalized Lie derivative* $\tilde{\mathcal{L}}_{(X,Y)} f\colon M \to TN$

by $\tilde{\mathcal{L}}_{(X,Y)} = Tf \circ X - Y \circ f$, cf. [Trautmann, 72] and [Kolář, 82]. One computes directly

$$\tilde{\mathcal{L}}_{(X,Y)} f = \tfrac{\partial}{\partial t}\big|_0 (\mathrm{Fl}^Y_{-t} \circ f \circ \mathrm{Fl}^X_t).$$

If $N \to M$ is a fibered manifold, $s \in C^\infty(N)$ is a section and $Y$ is a projectable vector field over $X$, then $\tilde{\mathcal{L}}_{(X,Y)} s$ is a section of the vertical bundle $VN \to M$. In particular, if $F$ is a bundle functor then for every section $s \in C^\infty(FM)$ and every vector field $X \in \mathcal{X}(M)$ we define the *Lie derivative* $\tilde{\mathcal{L}}_X s = \tilde{\mathcal{L}}_{(X,\mathcal{F}X)} s$, where $\mathcal{F}$ is the flow operator. So

$$\tilde{\mathcal{L}}_X s = \tfrac{\partial}{\partial t}\big|_0 (F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t).$$

More generally a short computation shows:

$$\begin{aligned}\tilde{\mathcal{L}}_X((\mathrm{Fl}^X_t)^* s) &= \tilde{\mathcal{L}}_X(F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t) = \tfrac{\partial}{\partial t}(F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t) = \\ &= T(F(\mathrm{Fl}^X_{-t})) \circ \tilde{\mathcal{L}}_X s \circ \mathrm{Fl}^X_t = (\mathrm{Fl}^X_t)^*(\tilde{\mathcal{L}}_X s)\end{aligned} \tag{3}$$

For every natural bundle $F$ the Lie derivative is a natural operator $\tilde{\mathcal{L}}\colon T \times F \to VF$ defined on the sections of the fibered products $TM \times_M FM \to M$.

If $F$ is a natural vector bundle, then $VF$ is naturally equivalent to $F \oplus F$ and the second component of our Lie derivative is just the classical Lie derivative $\mathcal{L}$.

**1.6.** Consider $r$-th order natural bundles $F$ and $G$ with standard fibers $S$ and $Q$. Since there are the induced actions of the jet group $G^{r+k}_m$ on $T^k_m S$ and $Q$, we have the fundamental field mappings $\zeta^{(k)}\colon \mathfrak{g}^{r+k}_m \to \mathcal{X}(T^k_m S)$ and $\zeta^Q : \mathfrak{g}^{r+k}_m \to \mathcal{X}(Q)$.

**Lemma.** *For all $j^{r+k}_0 X \in \mathfrak{g}^{r+k}_m$ and $j^k_0 s \in T^k_m S$ we have*

$$\zeta^{(k)}_{j^{r+k}_0 X}(j^k_0 s) = \kappa(j^k_0(\tilde{\mathcal{L}}_{-X} s))$$

*where $\kappa$ is the map induced by the canonical natural equivalence $J^r V \to V J^r$.*

*Proof.* Writing $\lambda$ for the action of the jet group on $T^k_m S$ we have:

$$\begin{aligned}\zeta^{(k)}_{j^{r+k}_0 X}(j^k_0 s) &= \tfrac{\partial}{\partial t}\big|_0 \lambda(\exp t j^{r+k}_0 X)(j^k_0 s) = \tfrac{\partial}{\partial t}\big|_0 j^k_0(F(\mathrm{Fl}^X_t) \circ s \circ \mathrm{Fl}^X_{-t}) = \\ &= \kappa(j^k_0(\tfrac{\partial}{\partial t}\big|_0 (F(\mathrm{Fl}^X_t) \circ s \circ \mathrm{Fl}^X_{-t}))) = \kappa(j^k_0(\tilde{\mathcal{L}}_{-X} s)). \quad \square\end{aligned}$$

**1.7. Peetre theorem.** Given two natural vector bundles $F$ and $G$ and a linear mapping $D_M\colon C^\infty(FM) \to C^\infty(GM)$ we can compare the maps $\mathcal{L}_X \circ D_M$ and $D_M \circ \mathcal{L}_X$ for each vector field $X$ on $M$. By the classical Peetre theorem, if $D_M$ is a local regular operator and $M$ is compact, then $D_M$ is of finite order. Consequently, if $D_M$ comes from a natural operator $D$, then one easily shows that $\mathcal{L}_X \circ D_M = D_M \circ \mathcal{L}_X$ even for non compact $M$.

For the nonlinear case we need a generalization of the Peetre theorem due to [Slovák, 88]. The general result is rather technical and so we formulate a special case which we shall need.

**Proposition.** *Let $N \to M$ and $N' \to M$ be fibered manifolds and let $D\colon C^\infty(N) \to C^\infty(N')$ be a regular local operator. Then for every fixed section $s \in C^\infty(N)$ and for every compact set $K \subset M$, there is an order $r \in \mathbb{N}$ and a neighborhood $V$ of $s$ in the compact open $C^\infty$-topology such that for every $x \in K$ and $s_1,\ s_2 \in V$ the condition $j^r_x s_1 = j^r_x s_2$ implies $Ds_1(x) = Ds_2(x)$.*

**1.8. The vertical prolongation.** For two fibered manifolds $N \to M$ and $N' \to M$ and for a local regular operator $D\colon C^\infty(N) \to C^\infty(N')$ we define the *vertical prolongation* $VD\colon C^\infty(VN \to M) \to C^\infty(VN' \to M)$ of $D$ as follows: Every section $q \in C^\infty(VN \to M)$ is of the form $\frac{\partial}{\partial t}\big|_0 s_t$ for a family $s_t \in C^\infty(N)$ and we set

$$VD(q) = VD(\tfrac{\partial}{\partial t}\big|_0 s_t) = \tfrac{\partial}{\partial t}\big|_0 (Ds_t) \in C^\infty(VN' \to M).$$

We have to verify that this is a correct definition. So let us fix $q = \frac{\partial}{\partial t}\big|_0 s_t$ and $x \in M$.

By 1.7 the operator $D$ is of order $\le \infty$ and thus is induced by a map $\tilde D : J^\infty N \to N'$. Moreover each infinite jet has a neighborhood in the inverse limit topology on $J^\infty N$ on which $\tilde D$ only depends on $r$–jets for some finite $r$. Thus there is neighborhood $U$ of $x$ in $M$ and a locally defined smooth map $\tilde D^r : J^r N \to N'$ such that $Ds_t(y) = \tilde D^r(j^r_y s_t)$ for $y \in U$ and for $t$ sufficiently small. So we get

$$(VD)q(x) = \tfrac{\partial}{\partial t}\big|_0 (\tilde D^r(j^r_x s_t)) = T\tilde D^r(\tfrac{\partial}{\partial t}\big|_0 j^r_x s_t) = (T\tilde D^r \circ \kappa)(j^r_x q)$$

and thus the definition does not depend on the choice of the family $s_t$.

## 2. Infinitesimally natural operators

**2.1. Definition.** A local regular operator $D\colon C^\infty(FM) \to C^\infty(GM)$ is called *infinitesimally natural* if $\tilde{\mathcal{L}}_X(Ds) = VD(\tilde{\mathcal{L}}_X s)$ for all $X \in \mathcal{X}(M)$ and all $s \in C^\infty(FM)$.

**2.2. Theorem.** *Every natural operator $D : F \to G$ between two bundle functors on $\mathcal{M}f^+_m$ consists of infinitesimally natural operators $D_M$.*

*Proof.*

$$\begin{aligned}(VD_M)(\tilde{\mathcal{L}}_X s) &= (VD_M)\left(\tfrac{\partial}{\partial t}\big|_0 (F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t)\right) = \\ &= \tfrac{\partial}{\partial t}\big|_0 D_M\left(F(\mathrm{Fl}^X_{-t}) \circ s \circ \mathrm{Fl}^X_t\right) \\ &= \tfrac{\partial}{\partial t}\big|_0 \left(G(\mathrm{Fl}^X_{-t}) \circ D_M s \circ \mathrm{Fl}^X_t\right) = \tilde{\mathcal{L}}_X(D_M s) \quad \square\end{aligned}$$

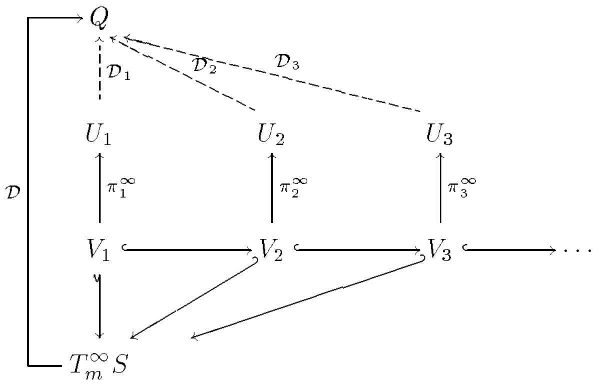
Since every natural operator $D$ between bundle functors defined on $\mathcal{M}f_m$ is uniquely determined by $D_{\mathbb{R}^m}$ the corresponding theorem for the category $\mathcal{M}f_m$ is a trivial consequence.

**2.3. Theorem.** *Let $F$ and $G$ be two bundle functors on $\mathcal{M}f^+_m$, $M$ be an $m$-dimensional manifold and let $D_M\colon C^\infty(FM) \to C^\infty(GM)$ be an infinitesimally natural operator. Then $D_M$ extends to a unique natural operator $D\colon F \to G$.*

The proof will require several steps. Let us fix an infinitesimally natural operator $D\colon F\mathbb{R}^m \to G\mathbb{R}^m$ and let us write $S$ and $Q$ for the standard fibers $F_0\mathbb{R}^m$ and $G_0\mathbb{R}^m$.

As it was noticed in 1.1 we may assume that both natural bundles $F$ and $G$ are of some finite order, say $k$. Thus we have actions of $G^{r+k}_m$ on $T^r_m S$ and, via the jet projections, also on $Q$ for all $r$. Since each local operator is locally of finite order by 1.7, there is the induced map $\mathcal{D}\colon T^\infty_m S \to Q$. Moreover, at each $j^\infty_0 s \in T^\infty_m S$ the application of 1.7 (with $K = \{0\}$) yields a smallest possible order $r = \chi(j^\infty_0 s)$ such that for every section $q$ with $j^r_0 q = j^r_0 s$ we have $Ds(0) = Dq(0)$. Let us define $\tilde V_r \subset T^\infty_m S$ as the subset of all jets with $\chi(j^\infty_0 s) \le r$. Let $V_r$ be the interior of $\tilde V_r$ in the inverse limit topology and put $U_r := \pi^\infty_r(V_r) \subset T^r_m S$.

By virtue of the nonlinear Peetre theorem $T^\infty_m S = \cup_r V_r$ and so the sets $V_r$ form an open filtration of $T^\infty_m S$. On each $V_r$, the map $\mathcal{D}$ factors to a map $\mathcal{D}_r : U_r \to Q$.




**2.4. Lemma.** *For all $r \in \mathbb{N}$ and $X \in \mathfrak{g}^{r+k}_m$ we have $T\mathcal{D}^r \circ \zeta^{(r)}_X = \zeta^Q_X \circ \mathcal{D}^r$ on $U_r$.*

*Proof.* Recall from 1.8 that $(VD)q(0) = (T\mathcal{D}^r \circ \kappa)(j^r_0 q)$ for all $j^r_0 q \in \kappa^{-1}(TU_r)$. Using 1.6 and the infinitesimal naturality of $D$ we compute (identifying $X$ with a polynomial vector field on $\mathbb{R}^m$):

$$\begin{aligned}(T\mathcal{D}_r \circ \zeta^{(r)}_X)(j^r_0 s) &= T\mathcal{D}_r(\kappa(j^r_0(\tilde{\mathcal{L}}_{-X} s))) = VD(\tilde{\mathcal{L}}_{-X} s)(0) = \\ &= \tilde{\mathcal{L}}_{-X}(Ds)(0) = \zeta^Q_X(Ds(0)) = \zeta^Q_X(\mathcal{D}_r(j^r_0 s)). \quad \square\end{aligned}$$

**2.5. Lemma.** *The map $\mathcal{D}\colon T^\infty_m S \to Q$ is $G^{\infty\,+}_m$-equivariant.*

*Proof.* Given $a = j^\infty_0 f \in G^{\infty\,+}_m$ and $y = j^\infty_0 s \in T^\infty_m S$ we have to show $\mathcal{D}(a \cdot y) = a \cdot \mathcal{D}(y)$. Each $a$ is a composition of a jet of a linear map and of a jet from the kernel $B^\infty_1$ of the jet projection $\pi^\infty_1 : G^{\infty\,+}_m \to GL^+(m, \mathbb{R})$. If $f$ is linear, then there are linear maps $g_i$, $i = 1, 2, \dots, l$, lying in the image of the exponential map of $G^1_m$ such that $f = g_1 \circ \dots \circ g_l$. Since $T^\infty_m S = \cup_r V_r$ there must be an $r \in \mathbb{N}$ such that $y$ and all elements $(j^\infty_0 g_p \circ \dots \circ j^\infty_0 g_l) \cdot y$ are in $V_r$ for all $p \le l$. Thus $\mathcal{D}(a \cdot y) = \mathcal{D}_r(j^r_0 f \cdot j^r_0 s) = j^r_0 f \cdot \mathcal{D}_r(j^r_0 s) = a \cdot \mathcal{D}(y)$, since from 2.4 and the fact that the flows of $f$–related vector fields are $f$–related one easily concludes that $\mathcal{D}^r$ commutes with the actions of elements of $G^r_m$ which are in the image of the exponential map.

Since the kernel $B^r_1$ is nilpotent it lies in the image of the exponential map for each $r < \infty$ and thus an analogous consideration for $j^\infty_0 f \in B^\infty_1$ concludes the proof of the lemma. $\square$

**2.6. Lemma.** *The natural operator $\tilde{D}$ on $\mathcal{M}f_m^+$ which is determined by the $G_m^{\infty+}$-equivariant map $\mathcal{D}$ coincides on $\mathbb{R}^m$ with the operator $D$.*

*Proof.* There is the associated map $\tilde{\mathcal{D}}\colon J^\infty F\mathbb{R}^m \to G\mathbb{R}^m$ to the operator $\tilde{D}_{\mathbb{R}^m}$. Let us write $\tilde{\mathcal{D}}_0$ for its restriction $(J^\infty F)_0\mathbb{R}^m \to G_0\mathbb{R}^m$ to the standard fibers and similarly for the map $\mathcal{D}$ corresponding to the original operator $D$. Now let $t_x : \mathbb{R}^m \to \mathbb{R}^m$ be the translation by $x$. Then the map $\tilde{\mathcal{D}}$ (and thus the operator $\tilde{D}$) is uniquely determined by $\tilde{\mathcal{D}}_0$ and the fact that by naturality of $\tilde{D}$ we have $(t_{-x})^* \circ \tilde{D}_{\mathbb{R}^m} \circ (t_x)^* = \tilde{D}_{\mathbb{R}^m}$, since then

$$\begin{aligned}\tilde{D}_{\mathbb{R}^m}(s)(x) &= (G(t_x) \circ G(t_{-x}) \circ \tilde{D}_{\mathbb{R}^m}(s) \circ t_x)(0) = \\ &= G(t_x)((t_x)^*(\tilde{D}_{\mathbb{R}^m}(s))(0)) = G(t_x)(\tilde{D}_{\mathbb{R}^m}((t_x)^*s)(0)) = \\ &= G(t_x)(\tilde{\mathcal{D}}_0(j_0^\infty(t_x)^*s))\end{aligned}$$

But $t_x$ is the flow at time 1 of the constant vector field $x$ and for any complete vector field $X$ we compute using (3) and infinitesimal naturality:

$$\begin{aligned}&\tfrac{\partial}{\partial t}\big((\mathrm{Fl}^X_{-t})^*(D(\mathrm{Fl}^X_t)^*s)\big) = \\ &= -(\mathrm{Fl}^X_{-t})^*\tilde{\mathcal{L}}_X(D(\mathrm{Fl}^X_t)^*s) + (\mathrm{Fl}^X_{-t})^*\big((VD)((\mathrm{Fl}^X_t)^*\tilde{\mathcal{L}}_X s)\big) = \\ &= (\mathrm{Fl}^X_{-t})^*\big(-\tilde{\mathcal{L}}_X(D(\mathrm{Fl}^X_t)^*s) + (VD)(\tilde{\mathcal{L}}_X((\mathrm{Fl}^X_t)^*s))\big) = 0.\end{aligned}$$

Thus $(t_{-x})^* \circ D \circ (t_x)^* = D$ and since $\mathcal{D}_0 = \tilde{\mathcal{D}}_0$ this concludes the proof. □

Lemmas 2.5 and 2.6 imply the assertion of Theorem 2.3. Indeed, if $M = \mathbb{R}^m$ we get the result immediately and it follows for general $M$ by locality of the operators in question.

**2.7. Remarks.** In general the situation changes if we consider the naturality over the whole category $\mathcal{M}f_m$. For example, the vector product is a natural operator on $\mathcal{M}f_3^+$ which transforms Riemannian metrics into sections of the values of $T \otimes T^* \otimes T^*$. Clearly the vector product is not natural on $\mathcal{M}f_3$.

Our general result covers an earlier result on natural transformations between natural bundles deduced in [Krupka, Janyška, 90]. The latter authors also discuss how to test whether a natural operator on $\mathcal{M}f_m^+$ is natural on $\mathcal{M}f_m$. The obvious necessary and sufficient condition is that the associated $G_m^{\infty+}$-equivariant map also commutes with the action of one element from the other connected component of $G_m^\infty$.

## 3. The multilinear case

As we already remarked for a natural vector bundle $F$ the vertical bundle $VF$ is naturally equivalent to $F \oplus F$ and the second component of our general Lie derivative is just the usual Lie derivative. Thus if $D\colon C^\infty(FM) \to C^\infty(GM)$ is linear we get the usual condition $D \circ \mathcal{L}_X = \mathcal{L}_X \circ D$ for infinitesimal naturality.

More generally, if $F$ is a sum of $k$ natural vector bundles, $G$ is a natural vector bundle and $D$ is $k$-linear, then we have:

$$\begin{aligned}(\mathrm{pr}_2 \circ VD)(\tilde{\mathcal{L}}_X(s_1, \dots, s_k)) &= \tfrac{\partial}{\partial t}\big|_0 D\big(F(\mathrm{Fl}^X_{-t}) \circ (s_1, \dots, s_k) \circ \mathrm{Fl}^X_t\big) = \\ &= \sum_{i=1}^k D(s_1, \dots, \mathcal{L}_X s_i, \dots, s_k).\end{aligned}$$

Hence for the $k$-linear operators theorem 2.2 implies

**3.1. Corollary.** *Let $E_1, \dots, E_k$ and $F$ be vector bundle functors on $\mathcal{M}f_m^+$. Every natural $k$-linear operator $D : E_1 \oplus \dots \oplus E_k \to F$ satisfies*

$$\mathcal{L}_X(D_M(s_1, \dots, s_k)) = \sum_{i=1}^k D_M(s_1, \dots, \mathcal{L}_X s_i, \dots, s_k) \tag{4}$$

*for all $s_1 \in C^\infty(E_1 M)$,…,$s_k \in C^\infty(E_k M)$ and all $X \in \mathcal{X}(M)$.*

Formula (4) covers many well known formulas for Lie derivatives of values of geometric operations. Let us mention e.g. the commutation with the exterior differential or the Jacobi identity for the Lie bracket.

**3.2.** For an important class of vector bundle functors we can prove a stronger version of theorem 2.3:
Let $E_1, \dots, E_k$ be $r$–th order natural vector bundles corresponding to actions $\lambda_i$ of the jet group $G_m^r$ on standard fibers $S_i$, and assume that with the restricted actions $\lambda_i | G_m^1$ the spaces $S_i$ are invariant subspaces in spaces of the form $\oplus_j(\otimes^{p_j}\mathbb{R}^m \otimes \otimes^{q_j}\mathbb{R}^{m*})$. In particular this applies to all natural vector bundles which are subbundles in tensor bundles. Given any natural vector bundle $F$ we have

**Theorem.** *Every local regular $k$-linear operator*

$$D_M : C^\infty(E_1 M) \oplus \dots \oplus C^\infty(E_k M) \to C^\infty(FM),$$

*over an $m$–dimensional manifold $M$ which satisfies (4) extends uniquely to a natural operator $\tilde{D}$ on $\mathcal{M}f_m$ with $\tilde{D}_M = D_M$.*

The theorem follows from the theorem 2.3 and the next lemma

**Lemma.** *Every $k$-linear natural operator $D : E_1 \oplus \dots \oplus E_k \to F$ on $\mathcal{M}f_m^+$ extends to a natural operator on $\mathcal{M}f_m$.*

*Proof.* By the multilinear version of the Peetre theorem (c.f. 1.7 and [Slovák, 88]) $D$ is of some finite order $\ell$. Thus $D$ is determined by the associated $k$-linear $(G_m^{r+\ell})^+$-equivariant map $\mathcal{D}\colon T_m^\ell S_1 \times \ldots \times T_m^\ell S_k \to Q$ (cf. 1.3). Recall that the jet group $G_m^{r+\ell}$ is the semidirect product of $GL(m, \mathbb{R})$ and the kernel $B_1^{r+\ell}$, while $(G_m^{r+\ell})^+$ is the semidirect product of the connected component $GL^+(m, \mathbb{R})$ of the unit and the same kernel $B_1^{r+\ell}$. Thus in particular the map $\mathcal{D}\colon T_m^\ell S_1 \times \ldots \times T_m^\ell S_k \to Q$ is $k$–linear and $GL^+(m, \mathbb{R})$ equivariant. By the descriptions of $(G_m^{r+\ell})^+$ and $G_m^{r+\ell}$ above we only have to show that any such map is $GL(m, \mathbb{R})$ equivariant, too. Using the standard polarization technique we can express the map $\mathcal{D}$ by means of a $GL^+(m, \mathbb{R})$ invariant tensor. But looking at the proof of the invariant tensor theorem (c.f. [Gurevich, 48] and [Kolář–Michor–Slovák, Theorem 24.4]) one concludes that the spaces of $GL^+(m, \mathbb{R})$ invariant and of $GL(m, \mathbb{R})$ invariant tensors coincide, so the map $\mathcal{D}$ is $GL(m, \mathbb{R})$ equivariant. □

## 4. Generalizations

**4.1. Categories over manifolds.** In geometry we often meet operations depending on some further structures of the underlying manifolds like a fibration, a symplectic structure, etc. The original theory of natural bundles and operators has been modified for these more general situations using the concept of a *category over manifolds*, i.e. a category $\mathcal{C}$ endowed with a forgetful functor into $\mathcal{M}f$, see [Kolář, 90] and [Kolář–Michor–Slovák, Chapter V]. In this case, the flow operator is defined on the so called $\mathcal{C}$-fields, the fields whose flows consist of local $\mathcal{C}$-morphisms. The concept of Lie differentiation of sections of the values of these more general bundle functors on $\mathcal{C}$ with respect to $\mathcal{C}$-fields is defined in the same way as above. The jet groups then consist of jets of local $\mathcal{C}$-morphisms and their Lie algebras are formed by jets of $\mathcal{C}$-fields. Analyzing the above proofs, we get results analogous to theorems 2.2 and 2.3, e.g. for categories of manifolds with fixed volume form and volume preserving local diffeomorphisms, symplectic manifolds and symplectic local diffeomorphisms, fibered manifolds with fixed dimensions of bases and fibers and local fibered isomorphisms.

**4.2. Gauge-naturality.** Another geometric situation was reflected in D. Eck's definition of gauge-natural bundles, see [Eck, 81].

Let us fix a Lie group $G$ and write $\mathcal{PB}_m(G)$ for the category whose objects are principal $G$-bundles over $m$–dimensional manifolds and whose morphisms are the morphisms of principal $G$-bundles $f\colon P \to \bar{P}$ with the base map $Bf\colon BP \to B\bar{P}$ lying in $\mathcal{M}f_m$.

A *gauge natural bundle* is a functor $F\colon \mathcal{PB}_m(G) \to \mathcal{FM}$ such that:

(i) every $\mathcal{PB}_m(G)$-object $\pi\colon P \to BP$ is transformed into a fibered manifold $q_P\colon FP \to BP$ over $BP$,
(ii) every $\mathcal{PB}_m(G)$-morphism $f\colon P \to \bar{P}$ is transformed into an $\mathcal{FM}$-morphism $Ff\colon FP \to F\bar{P}$ over $Bf$,
(iii) for every open subset $U \subset BP$, the inclusion $i\colon \pi^{-1}(U) \to P$ is transformed into the inclusion $Fi\colon q_P^{-1}(U) \to FP$.

Let $F$ and $E$ be two gauge-natural bundles on $\mathcal{PB}_m(G)$. A *gauge-natural operator* $D\colon F \to E$ is a system of regular operators $D_P\colon C^\infty(FP) \to C^\infty(EP)$ for all $\mathcal{PB}_m(G)$-objects $\pi\colon P \to BP$ such that

(a) $D_{\bar{P}}(Ff \circ s \circ Bf^{-1}) = Ff \circ D_P s \circ Bf^{-1}$ for every $s \in C^\infty(FP)$ and every $\mathcal{PB}_m(G)$-isomorphism $f\colon P \to \bar{P}$,
(b) $D_{\pi^{-1}(U)}(s|U) = (D_P s)|U$ for every $s \in C^\infty(FP)$ and every open subset $U \subset BP$.

Note that for the trivial structure group we recover the concepts of natural bundles and operators.

We define $\mathcal{PB}_m^+(G)$ as the category of principal $G$–bundles over oriented $m$–dimensional manifolds and morphisms over orientation preserving local diffeomorphisms.

**4.3.** If two $\mathcal{PB}_m(G)$-morphisms $f, g\colon P \to \bar{P}$ satisfy $j^r_y f = j^r_y g$ at a point $y \in P_x$ of the fiber of $P$ over $x \in BP$, then the fact that the right translations of principal bundles are diffeomorphisms implies $j^r_z f = j^r_z g$ for every $z \in P_x$. In this case we write $\mathbf{j}^r_x f = \mathbf{j}^r_x g$ and say that $f$ and $g$ have the same *fiber $r$–jet* at $x$, cf. [Kolář, 84]. The set of fiber $r$-jets between $P$ and $\bar{P}$ is denoted by $\mathbf{J}^r(P, \bar{P})$.

A gauge natural bundle $F$ is said to be of order $r$, if $\mathbf{j}^r_x f = \mathbf{j}^r_x g$ implies $Ff|F_x P =$

$Fg|F_xP$. Every gauge natural bundle is of finite order (c.f. [Eck, 81] and [Kolář–Michor–Slovák, Theorem 51.7]).

In the theory of gauge-natural bundles the role of the jet groups is played by the so called principal prolongations $W^r_mG$ of the Lie group $G$. We define $W^r_mG$ as the set of invertible fiber jets in $\mathbf{J}^r_0(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_0$. The group $W^r_mG$ is the semidirect product $G^r_m ⓢ T^r_mG$, cf. [Kolář, 71]. If $G$ is connected, then the theory of gauge natural bundles and operators on the category $\mathcal{PB}^+_m(G)$ is obtained by replacing the group $W^r_mG$ by its connected component of the unit ${G^r_m}^+ ⓢ T^r_mG$.

The Lie algebra $W^r_m\mathfrak{g}$ of $W^r_mG$ consists of the fiber jets of right invariant vector fields, and repeating the description of the Lie algebra $\mathfrak{g}^r_m$ with jets replaced by fiber jets, we immediately get the formulas for the bracket and the exponential mapping:

$$[\mathbf{j}^k_0X, \mathbf{j}^k_0Y] = -\mathbf{j}^k_0[X,Y] \tag{5}$$

$$\exp(\mathbf{j}^k_0X) = \mathbf{j}^k_0 \operatorname{Fl}^X_1 . \tag{6}$$

**4.4. Infinitesimally gauge-natural operators.** For a right invariant vector field $X \in \mathcal{X}(P)$ the definition of the Lie derivative $\tilde{\mathcal{L}}_Xs$ of a section $s$ of a gauge natural bundle is similar to the one given in 1.5: $\tilde{\mathcal{L}}_Xs := \tilde{\mathcal{L}}_{(\bar{X},\mathcal{F}X)}s$, where $\bar{X}$ is the vector field on the base manifold corresponding to $X$ and $\mathcal{F}$ is the obvious analog of the flow operator. Thus we have:

$$\begin{aligned} \tilde{\mathcal{L}}_X((\operatorname{Fl}^X_t)^*s) &= \tilde{\mathcal{L}}_X(F(\operatorname{Fl}^X_{-t}) \circ s \circ \operatorname{Fl}^{\bar{X}}_t) = \tfrac{\partial}{\partial t}(F(\operatorname{Fl}^X_{-t}) \circ s \circ \operatorname{Fl}^{\bar{X}}_t) = \\ &= T(F(\operatorname{Fl}^X_{-t})) \circ \tilde{\mathcal{L}}_Xs \circ \operatorname{Fl}^{\bar{X}}_t = (\operatorname{Fl}^X_t)^*(\tilde{\mathcal{L}}_Xs) \end{aligned} \tag{7}$$

Consider two gauge natural bundles $F$ and $E$, and a principal fiber bundle $P$ with structure group $G$.

A local regular operator $D\colon C^\infty(FP) \to C^\infty(EP)$ is called *infinitesimally gauge-natural* if we have $\tilde{\mathcal{L}}_X(Ds) = VD(\tilde{\mathcal{L}}_Xs)$ for all right invariant vector fields $X \in \mathcal{X}(P)$ and for all sections $s \in C^\infty(FM)$.

Using formulas (5), (6) and (7) instead of (1), (2) and (3), we can repeat the procedure leading to theorems 2.2 and 2.3 with some obvious modifications to get:

**4.5. Theorem.** Every gauge natural operator $D\colon F \to E$ between two gauge-natural bundles on $\mathcal{PB}_m(G)$ consists of infinitesimally gauge-natural operators $D_M$.

**4.6. Theorem.** *Let $F$ and $E$ be two gauge natural bundles on $\mathcal{PB}^+_m(G)$ where $G$ is connected, and let $P$ be an object of $\mathcal{PB}^+_m(G)$. If $D_P : C^\infty(FP) \to C^\infty(EP)$ is an infinitesimally gauge-natural operator then $D_P$ extends to a unique gauge-natural operator $D\colon F \to E$ on $\mathcal{PB}^+_m(G)$.*

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