NATURAL OPERATORS ON CONFORMAL MANIFOLDS

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Abstract. In this note, we discuss the problem of the classification of all linear local operators naturally defined on all conformal manifolds with fixed dimension $m > 2$. In particular, we present a general definition of the natural operators, we analyze the possible orders of the operators and we try to present a concise summary of some results, both in the real and complex settings, including also a full discussion on the so called singular cases. We omit a detailed treatment of the curved conformal manifolds where no complete classification has been obtained yet, and we focus on the special case of conformally flat manifolds. The results are heavily based on well known facts from the representation theory of parabolic subalgebras in the Lie algebras of the orthogonal groups, but we try to make our exposition as elementary as possible.

1. Natural operators

Our approach follows the general definition of bundle functors and natural operators on 'geometric categories' as developed in the monograph [Kolář, Michor, Slovák, 92]. The general theory can be applied directly to bundles corresponding to representations of the orthogonal groups (with conformal weights), but we extend it to the spin bundles as well. There are several other approaches available in the literature, some authors list directly the objects of their interest, some other ones present definitions which apply to the conformally flat manifolds and provide us with concrete constructions for the general case, some of them deal with the complex manifolds, another ones work in the real setting, cf. [Baston, 90], [Baston, Eastwood, 90], [Branson, 85], [Jakobsen, 86]. The advantage of our approach is that we get a precise definition for the whole category which even fits a more general framework. We believe that this setting, supported by some general ideas from [Kolář, Michor, Slovák, 92], will also lead to some progress in the classification of the operators on the curved conformal manifolds. Moreover, it follows that all the above mentioned approaches are equivalent.

In the sequel we shall deal with Riemannian manifolds with an arbitrary fixed signature (m, n). All mappings are smooth.

On each Riemannian manifold M, there is the canonical orthogonal frame bundle $P_{O(m,n)}M$, the reduction of the linear frame bundle P^1M to the (pseudo) orthogonal group. This means, there is a natural principal bundle $P_{O(m,n)}$ on Riemannian manifolds with signature (m, n). Each representation $\lambda\colon O(m,n) \to \mathrm{Diff}(V)$ gives rise to a natural

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bundle F_λ on the category of Riemannian manifolds with locally invertible isometries defined by the construction of the associated bundles.

The same construction works for every category $\mathcal{C}$ over m-dimensional manifolds with a distinguished natural principal bundle $P\colon \mathcal{C} \to \mathcal{PB}_m(G)$ (i.e. the values of P are in the category of principal fiber bundles with m-dimensional bases and structure group G). The natural operators are defined as the local operators acting on sections of natural bundles which intertwine the actions of the $\mathcal{C}$-morphisms on the sections. The Lie derivative of sections of natural bundles is defined for all vector fields with flows formed by $\mathcal{C}$-morphisms, the so called $\mathcal{C}$-fields, but the values are in the vertical bundles. If the bundles themselves are vector bundles, we recover the usual Lie derivative and it is easy to see that the linear natural operators commute with the Lie derivative and vice versa. For the proofs see [Kolář, Michor, Slovák, 92] or [Cap, Slovák, 92] where the result is proved in the non-linear setting. Each natural bundle F admits the so called flow operator $\mathcal{F}$, a natural operator which transforms $\mathcal{C}$-fields on M into vector fields on FM. The flow of its value $\mathcal{F}X$ is defined by the application of the functor F to the flow of the $\mathcal{C}$-field X. If $P\colon \mathcal{C} \to \mathcal{PB}_m(G)$ is a natural principal bundle, then $\mathcal{P}X$ is right invariant for all $\mathcal{C}$-fields X.

The next lemma is well known. Let us consider a linear representation λ of a Lie group G in a vector space V and the associated bundle $F_\lambda M$ to the principal bundle $p\colon PM \to M$. Let us write $\{u, v\}$ for the class in $F_\lambda M$ determined by $(u, v) \in PM \times V$. The Lie derivative of the V-valued functions on PM is defined as usual.

1.1. Lemma. *Let $\lambda\colon G \to GL(V)$ be a representation. Then the set of all smooth section $C^\infty(F_\lambda M)$ is identified with the set of G-equivariant mappings in $C^\infty(PM, V)^G$, $s \mapsto \tilde{s}$, $s(p(u)) = \{u, \tilde{s}(u)\}$, and for all $\mathcal{C}$-fields $X \in \mathcal{X}(M)$ and sections $s \in C^\infty(F_\lambda M)$, the Lie derivative $\mathcal{L}_X s$ corresponds to $\mathcal{L}_{\mathcal{P}X}\tilde{s}$.*

Proof. We have only to write down explicitely the definition of the Lie derivative and to compare it with the identification from the lemma. □

1.2. In view of the above discussion, we can define the natural linear operators D as those systems of operators for which $D_M(\mathcal{L}_{\mathcal{P}X}\tilde{s}) = \mathcal{L}_{\mathcal{P}X}(D_M\tilde{s})$ for all sections and $\mathcal{C}$-fields. We get exactly the linear natural operators acting on the natural bundles on categories over manifolds (defined separately for each manifold), but with this formulation we are able to involve also some covering fenomena. Let us consider P and G as in 1.1, a covering $\bar{G}$ of G and two representations λ_1, λ_2 of $\bar{G}$ in V and W. Then some of the natural bundles PM can be covered by principal $\bar{G}$-bundles $\bar{P}M$. Let us consider the manifolds M together with such coverings $\bar{P}M$ as distinguished objects. Now, each $\bar{P}M$ yields the bundles $F_{\lambda_i}M$ and each $\mathcal{C}$-field X determines a unique right invariant lift, denoted by the same symbol $\mathcal{P}X$, on $\bar{P}M$. Hence in this setting we can define natural operators between bundles corresponding to representations of the finite dimensional coverings of G. Of course, such operators need not to be defined on all $\mathcal{C}$-objects M, they are well defined only on those ones where the coverings $\bar{P}M$ do exist.

Definition. Let $\lambda_1\colon \bar{G} \to GL(V)$, $\lambda_2\colon \bar{G} \to GL(W)$ be finite dimensional linear representations. A local operator $D\colon C^\infty(F_{\lambda_1}M) \to C^\infty(F_{\lambda_2}M)$ is called *natural* if and only if $D(\mathcal{L}_{\mathcal{P}X}\tilde{s}) = \mathcal{L}_{\mathcal{P}X}(D\tilde{s})$ for all $\mathcal{C}$-fields X on M.

1.3. The conformal case. In particular, on conformal Riemannian manifolds with the signature (m, n) there is the natural principal bundle whose values are reductions of the second frame bundles P^2M to a structure group $B \subset O(m+1, n+1)$. The structure group is the first prolongation of the conformal linear group $CO(m,n)$ (also called the Poincaré conformal group), see [Kobayashi, 72], [Baston, 90]. Thus, each representation of B (or of the subgroup $CO(m,n)$ trivially extended to B) yields a natural bundle on conformal manifolds. The linear representations of the double covering of $CO(m,n)$ give rise to vector bundles on the so called spin manifolds. Our aim is to describe the linear natural operators which act between bundles corresponding to the irreducible representations.

For technical reasons, we shall consider only the connected component of the unit $CO_0(m,n) \subset CO(m,n)$. In the definite case this means that we shall deal with oriented manifolds, while for the general signature (m,n) there are four connected components in $CO(m,n)$. Each representation of $CO_0(m,n) = SO_0(m,n)\times\mathbb{R}$ consists of an irreducible representation of $SO_0(m,n)$ and an element from the dual of the center $-a \in \mathbb{R}^*$. The real number a is called the *conformal weight* of the representation. Similarly, the irreducible representations of the double covering correspond to irreducible representations of $Spin_0(m,n)$ with conformal weights. The sign convention of the conformal weights follows the generally adopted requirement that the Riemannian metrics themselves have weight two. Then the sections of the bundles corresponding to the above mentioned representations can be identified by means of a fixed metric from the conformal class with sections of the bundles on the underlying Riemannian manifolds, which 'rescale' by multiplication by the function f^a if we replace the fixed metric g by $f^2 g$.

Unfortunately, the linear representations of the whole Poincaré conformal group B need not to be completely reducible, so that we do not cover implicitly all natural vector bundles on conformal manifolds when dealing with the irreducible representations. On the other hand, we may consider only the irreducible representations of $CO_0(m,n)$ since every irreducible representation of B is a trivial extension of an irreducible representation of $CO_0(m,n)$.

1.4. Remark. If $f\colon M \to M$ is an isometry with respect to an arbitrary metric from the conformal class on M, then f is a conformal mapping with respect to all other ones from the class (this is easily seen from the definition of the conformal morphisms as morphisms of $CO_0(m,n)$-structures on M). Thus, our natural operators intertwine the actions of the isometries. This means that they must be natural also in the category of Riemannian manifolds (and we may add some homogeneity requirement connected with the conformal weights of the bundles). But the Riemannian case is very well understood and all such operators are expressed by universal formulas through the covariant derivatives of the sections, the curvature, its covariant derivatives and algebraic $O(m,n)$-invariant tensor operations, see e.g. [Slovák, 92a,b] or [Kolář, Michor, Slovák, 92]. Let us underline, that only the locality and naturality requirements are involved in the cited proofs. In particular, the natural operators on conformal manifolds as defined in 1.2 can be expressed by such universal formulas which cannot depend on our choice of the metric within the conformal class. So we see that our definition is equivalent to the alternative definition of 'invariant operators' which is used by several authors, see e.g. [Branson, 85, 89a, 89b]. In [Eastwood, Rice, 87] and [Baston, 90], the authors use an equivalent definition for the subcategory of the conformally flat manifolds and they discuss possible (highly non-trivial) constructions of natural operators in the general curved case, without specifying explicitely the definition.

1.5. The main idea how to describe all linear natural operators acting between the above mentioned bundles is rather simple. Each linear local operator has (locally) a finite order and so it factors (locally) through a mapping $D\colon J^k(F_{\lambda_1}M) \to F_{\lambda_2}M$ defined on the k-jets of sections. If we fix a point and switch to the dual mapping, we get a linear mapping $(F_{\lambda_2}M)^*_x \to J^k_x(F_{\lambda_1}M)^*$. Now, the domain is identified with the representation space W and, since the representations are irreducible, the latter mapping must be either zero or an inclusion. Since each finite dimensional representation of $CO_0(m,n)$ is determined by the weight of a highest weight vector in W and the action of the kernel of the jet projection $B \to CO(m,n)$ acts trivially on W, we have only to find highest weight vectors in $J^k_x(F_{\lambda_1}M)^*$ with the trivial action of the kernel, the so called *singular highest weight vectors.* Indeed, each such vector generates an irreducible $CO_0(m,n)$-subspace and hence an intertwining mapping $(F_{\lambda_2}M)^*_x \to J^k_x(F_{\lambda_1}M)^*$.

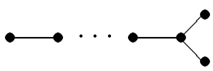
Fortunately, the (complexified) inverse limits of the finite dimensional duals, i.e. the spaces $J^\infty_x(F_\lambda M)^*$, are well known in the representation theory under the name generalized Verma modules and there is a very well developed theory of their homomorphisms. We shall explain this identification in 2.4.

This solves our problem on the level of individual fibers and so for subcategories homogeneous in the geometric sense, e.g. for the conformally flat Riemannian manifolds. (By homogeneous we mean that there is a fixed object U in the category such that for each point of any other object there is a neighborhood isomorphic with U, consult [Kolář, Michor, Slovák, 92] for more precise explanation.)

Let us remark that the same approach is applicable for many other 'geometric categories' like the whole category of m-dimensional manifolds, the manifolds with fixed volume forms and unimodular smooth mappings or the symplectic manifolds with symplectomorphisms. The duals to the jet spaces correspond to the induced representations of the Lie algebra of formal vector fields in the category in question. In the latter categories, these algebras are infinite dimensional, but the singular highest weight vectors were classified in [Rudakov, 74]. Since these categories are homogeneous in the above sense, we obtain a complete classification of the natural linear operators between first order natural vector bundles in this way. For more details see also [Kolář, Michor, Slovák, 92].

2. Natural vector bundles on conformal manifolds

Since we restrict ourselves to the connected components of the unit and since we admit the coverings of the orthogonal groups, we may discuss the whole problem on the Lie algebra level. We shall need some basic concepts and results from the representation theory which are available in several standard textbooks.

The semisimple complex algebras are classified by means of the so called Dynkin diagrams. The diagram corresponding to $\mathfrak{o}(2m,\mathbb{C})$ is  with m nodes, $m > 2$, while  with m nodes describes $\mathfrak{o}(2m+1)$, $m > 1$. Each node corresponds to one simple coroot. Since the weights are linear combinations of the dual elements to the simple coroots, we can express each weight by inscribing its values on the simple coroots over the nodes. For technical reasons, we shall increase all these values by one. In other words, the numbers over the nodes are the coefficients in the expression of the weight in question as a linear combination of the so called fundamental weights, increased by one (in fact we add the sum of the fundamental weights to each weight, i.e. half the sum of all positive roots).

2.1. Example. If e^i, $i = 1,\dots,m$, is the dual basis of the (real) Cartan algebra in $\mathfrak{o}(2m,\mathbb{C})$, then the fundamental weights are $\lambda_i = e^1 + \dots + e^i$, $1 \le i \le m-2$,

$\lambda_{m-1} = \frac{1}{2}(e^1 + \cdots + e^{m-1} - e^m)$, $\lambda_m = \frac{1}{2}(e^1 + \cdots + e^{m-1} + e^m)$. They correspond to the spaces of (complex) exterior forms of degrees $1 \leq i \leq m-2$, the remaining two representations are the so called spin representations. In our notation for the representations, the one-forms, i.e. $\mathbb{C}^{2m*}$, are denoted by $\overset{2}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!<\!\begin{smallmatrix}\overset{1}{\bullet}\\ \underset{1}{\bullet}\end{smallmatrix}$, while the last spin representation corresponds to $\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!<\!\begin{smallmatrix}\overset{1}{\bullet}\\ \underset{2}{\bullet}\end{smallmatrix}$.

Similarly, the (real) Cartan algebra of $\mathfrak{o}(2m+1,\mathbb{C})$ has a dual basis e^i, $1 \leq i \leq m$, and the fundamental weights are $\lambda_i = e^1 + \cdots + e^i$, $1 \leq i \leq m-1$, $\lambda_m = \frac{1}{2}(e^1 + \cdots + e^m)$. The last one is the weight of the spin representation, the first $m-1$ weights describe the exterior forms of degrees less then m.

In dimension six, the Dynkin diagram coincides with that for $\mathfrak{sl}(4,\mathbb{C})$: $\bullet\!-\!\bullet\!-\!\bullet$. If we want to use the above description with this shape of the diagram, we have to renumber the nodes in such a way that the first one is that in the middle. In dimension four we have $\mathfrak{o}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ and so its Dynkin diagram $\bullet\ \bullet$ is not connected. There are just two fundamental weights in dimension four, the spin representations.

2.2. We can use slightly modified Dynkin diagrams to describe all irreducible representations of the conformal algebras. The complexified algebra $\mathfrak{g} = \mathfrak{o}(m+1, n+1, \mathbb{C}) = (m+n+2,\mathbb{C})$ has a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a parabolic subalgebra in $\mathfrak{g}$, the complexified Lie algebra of the Poincaré conformal group B. The component of degree zero is just the Lie subalgebra $\mathfrak{g}_0 = \mathfrak{co}(m+n,\mathbb{C})$.

The above decomposition (of the complexified algebra) $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is the Levi decomposition of the parabolic subalgebra corresponding to a choice of one of the simple roots (the first one) and the nilpotent part $\mathfrak{g}_1$ acts trivially in each irreducible representation of $\mathfrak{b}$. Further, $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{z}$ where $\mathfrak{z}$ is the center and we can arrange the things so that the Cartan algebra $\mathfrak{h}$ decomposes as $\mathfrak{h} = (\mathfrak{h} \cap [\mathfrak{g}_0, \mathfrak{g}_0]) \oplus \mathfrak{z}$. So each irreducible representation can be denoted by a diagram with all coefficients positive integral except the chosen node where the value can be quite arbitrary. The (real or complex) representations are then viewed as representations of the orthogonal algebras in dimensions decreased by two (we simply forget the chosen node) and the remaining information is encoded in the conformal weight. We shall denote this omitted node by a cross. The conformal weight of the indicated representation is given by the coefficient of e^1 in the expression of the weight and it can be calculated easily from the coefficients inscribed over the nodes, see 2.1. The weights corresponding to such representations will be called dominant for $\mathfrak{b}$.

For example, $\overset{1}{\times}\!-\!\overset{2}{\bullet}\cdots\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!<\!\begin{smallmatrix}\overset{1}{\bullet}\\ \underset{1}{\bullet}\end{smallmatrix}$ with $m+1$ nodes is the highest weight of $\mathbb{C}^{2m*}$ with conformal weight one while $\overset{-1}{\times}\!-\!\overset{2}{\bullet}\cdots\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!<\!\begin{smallmatrix}\overset{1}{\bullet}\\ \underset{1}{\bullet}\end{smallmatrix}$ corresponds to $\mathbb{C}^{2m*}$ with conformal weight -1, i.e. $\mathbb{C}^{2m}$, $\overset{1}{\times}\!-\!\overset{p+1}{\bullet}\cdots\overset{1}{\bullet}\!-\!\overset{1}{\bullet}\!<\!\begin{smallmatrix}\overset{1}{\bullet}\\ \underset{1}{\bullet}\end{smallmatrix}$ has conformal weight p and describes the trace-free part of $S^p\mathbb{C}^{2m*}$. We have to be careful in dimension four, e.g. $\mathbb{C}^4 \simeq \overset{2}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{2}{\bullet}$.

Let us notice, that the real representations exponentiate to representations of the (real) Lie groups for arbitrary conformal weights, while in the complex case we have to be more careful.

2.3. The vector bundles. Let us write δ for the sum of all fundamental weights. We adopt the convention that a *vector bundle corresponding to an irreducible representation which is dual to that one with highest weight λ will be denoted by the diagram corresponding to the values of $\lambda + \delta$ on the simple coroots.* This seems to be a very strange notation, but the passing to the duals reflects the fact that we are describing the dual mappings to the operators and the shift by δ simplifies heavily our formulas. In fact, the dual representations are distinguished only by their opposite conformal weights (which is, of course, not the same as the inverting of the sign over the crossed node in general).

2.4. Generalized Verma modules. In general, given a representation of a subalgebra $\mathfrak{b} \subset \mathfrak{g}$ in $GL(V)$, we define the so called induced representation $\operatorname{Ind}(\mathfrak{g}, V) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V$. If $\mathfrak{g}$ is complex semisimple and $\mathfrak{b}$ is a Borel subalgebra, then we get the classical Verma modules. Then the irreducible representations of $\mathfrak{b}$ are the one-dimensional characters λ and the induced representation $M(\lambda)$ is generated (as a $\mathfrak{U}(\mathfrak{g})$-module) by the highest weight vector $1 \otimes 1$. More generally, if $\mathfrak{b}$ is a parabolic subalgebra and λ is a weight dominant for $\mathfrak{b}$ corresponding to a finite dimensional representation in V with highest weight vector $v \in V$, then the $\mathfrak{U}(\mathfrak{g})$-module $\operatorname{Ind}(\mathfrak{g}, V)$ is generated by $1 \otimes v$. The letter module is called the generalized Verma module $M_{\mathfrak{b}}(\lambda)$. We shall also use the notation $M_{\mathfrak{b}}(V)$ if we emphasize the representation space. In our case $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{b}$ and by virtue of the Birkhoff-Witt theorem we have $M_{\mathfrak{b}}(\lambda) = \mathfrak{U}(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} V = S(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} V$ as vector spaces. The elements from the real enveloping algebra can be viewed as conformal fields on the (pseudo) sphere and the action on the duals to the jet spaces is exactly that one determined by the Lie derivative. In this way we get the identification of the Verma modules with the inverse limit of the duals to the jet spaces mentioned in 1.5. For more details on this procedure see [Kolář, Michor, Slovák, 92], Section 34, where the same problem is discussed for some other categories over manifolds.

The grading of the symmetric algebra induces the grading on $M_{\mathfrak{b}}(\lambda)$ which coincides with the natural grading of the duals of the jet spaces.

If we start with complex representations and complex Riemannian manifolds, we get an analogous identification, cf. [Baston, 90].

3. The classification

3.1. Lemma. *There is a bijective correspondence between the natural linear operators acting on sections of complex natural vector bundles over complex conformal manifolds which are determined by the highest weights λ dominant for $\mathfrak{b}$, and the homomorphisms of the complex generalized Verma modules $M_{\mathfrak{b}}(\lambda)$.*

Proof. As discussed in 1.5, the operators correspond to highest weight vectors in the generalized Verma module $S(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} V$ with trivial action of $\mathfrak{g}_1$. Once we find such a singular highest weight vector $w \in M_{\mathfrak{b}}(\lambda)$ with weight ρ dominant for $\mathfrak{b}$, the corresponding homomorphism is induced by the inclusion of the generating highest weight vector in $M_{\mathfrak{b}}(\rho)$ onto w. On the other hand, each inclusion of $M_{\mathfrak{b}}(\rho)$ into $M_{\mathfrak{b}}(\lambda)$ yields such a singular highest weight vector $w \in M_{\mathfrak{b}}(\lambda)$. □

3.2. The infinitesimal character. For each (complex) $\mathfrak{U}(g)$-module A which is generated by a highest weight vector, the elements from the center $\mathfrak{Z}(\mathfrak{g})$ of the enveloping algebra act by scalars. This action can be viewed as an algebra homomorphism $\xi \colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$ which is called the infinitesimal character of A. In particular, for each weight λ dominant for $\mathfrak{b}$ there is the $\mathfrak{U}(\mathfrak{g})$-module $M_{\mathfrak{b}}(\lambda)$ and we shall denote its infinitesimal character by ξ_λ. If $D \colon M_{\mathfrak{b}}(\lambda) \to M_{\mathfrak{b}}(\rho)$ is a $\mathfrak{U}(\mathfrak{g})$-module homomorphism, then obviously $\xi_\lambda = \xi_\rho$. A classical theorem by Harish-Chandra states that $\xi_\lambda = \xi_\rho$ if and only if $\lambda + \delta$ and $\rho + \delta$ are conjugate under the action of the Weyl group W. This means that $s.\lambda = \rho$ for some element $s \in W$ where the dot denotes the so called affine action of W, $s.\lambda = s(\lambda + \delta) - \delta$. The elements from the Weyl group W are generated by the reflections with respect to the hyperplanes orthogonal to the simple roots. There is at most one weight dominant for the whole $\mathfrak{g}$ in each orbit of the affine action of W. On the other hand, the elements which map at least some of the weights dominant for $\mathfrak{b}$ into weights dominant for $\mathfrak{b}$ form the so called parabolic subgraph $W^{\mathfrak{b}}$ of W. In the even dimensions $m = 2n$, we can describe $W^{\mathfrak{b}}$ symbolically by

$$\mathrm{id} \longrightarrow s_1 \longrightarrow s_1 s_2 \longrightarrow \dots$$

$$\begin{array}{ccccc} & & s_1 s_2 \dots s_{n-1} s_n & & \\ & \nearrow & & \searrow & \\ \longrightarrow s_1 s_2 \dots s_{n-1} & & & & s_1 s_2 \dots s_{n-1} s_n s_{n+1} \longrightarrow \\ & \searrow & & \nearrow & \\ & & s_1 s_2 \dots s_{n-1} s_{n+1} & & \end{array}$$

$$\longrightarrow s_1 s_2 \dots s_{n-1} s_n s_{n+1} s_{n-1} \longrightarrow \dots \longrightarrow s_1 s_2 \dots s_2 s_1$$

where the symbols s_i denote the reflections corresponding to the simple roots indicated in the diagram $\overset{s_1}{\times}\!\!-\!\!\overset{s_2}{\bullet} \cdots \overset{s_{n-1}}{\bullet}\!\!<^{\bullet\, s_n}_{\bullet\, s_{n+1}}$. If $m = 2n + 1$ we order the simple roots as indicated in the diagram $\overset{s_1}{\times}\!\!-\!\!\overset{s_2}{\bullet} \cdots \overset{s_n}{\bullet}\!\!\Rightarrow\!\!\overset{s_{n+1}}{\bullet}$ and we get

$$\mathrm{id} \longrightarrow s_1 \longrightarrow \dots \longrightarrow s_1 \dots s_{n+1} \longrightarrow$$

$$\longrightarrow s_1 \dots s_{n+1} s_n \longrightarrow \dots \longrightarrow s_1 s_2 \dots s_2 s_1$$

The arrows describe the so called Bruhat order on $W^{\mathfrak{b}}$, for a more detailed description see e.g. [Boe, Collingwood, 85].

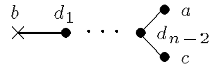
Definition. If λ is a weight dominant for $\mathfrak{b}$ such that $\lambda + \delta$ does not lie on a wall of a Weyl chamber, then the infinitesimal character ξ_λ is said to be *regular*. The infinitesimal characters of the weights λ with $\lambda + \delta$ lying on some wall are called *singular*. The infinitesimal characters of weights λ and ρ with the same cardinality of the stabilizer of $\lambda + \delta$ and $\rho + \delta$ in the Weyl group W are called *equisingular*.

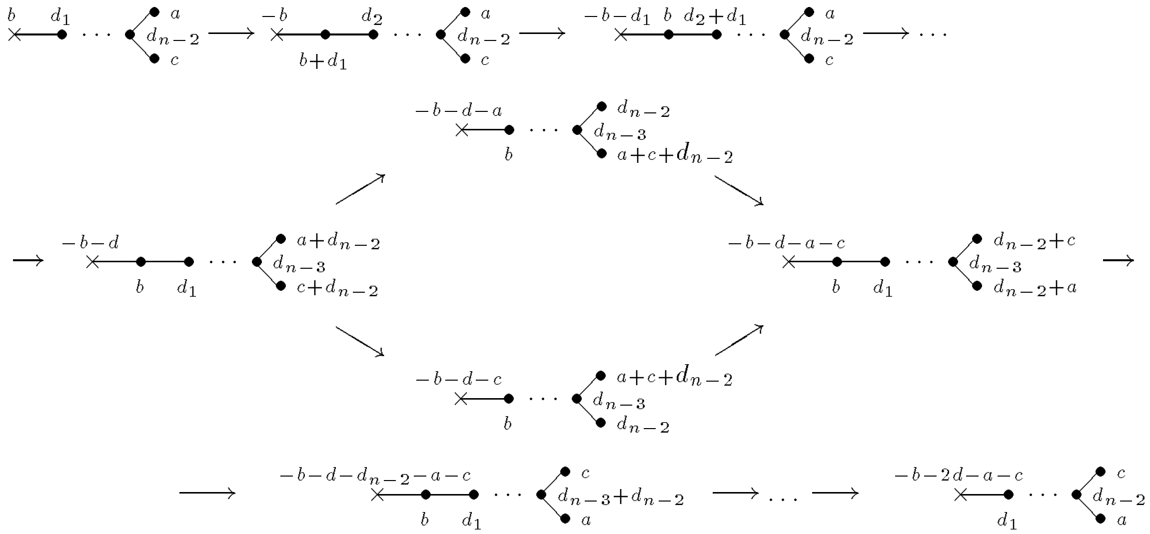
In particular, all regular infinitesimal characters are equisingular. If a weight has all coefficients over the nodes integral then its infinitesimal character is regular if and only if there is a weight ρ with the same infinitesimal character, which is dominant for the whole $\mathfrak{g}$. For such weights with regular infinitesimal characters, the meaning of the above patterns is easy to explain: We take the only weight λ dominant for $\mathfrak{g}$ with the infinitesimal character ξ_λ and we let the elements from $W^{\mathfrak{b}}$ act on $\lambda + \delta$ as indicated in the diagrams. In this way we get just all weights $\rho + \delta$ with ρ dominant for $\mathfrak{b}$ and with the same infinitesimal character ξ_λ.

Each weight with regular infinitesimal character appears just in one pattern and in a fixed position, cf. 3.4. It is surprisingly easy to write down the patterns for concrete weights since the action of the simple reflections s_i on the weights admits a very simple description, cf. [Baston, Eastwood, 89]:

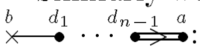
Let a be the coefficient at the i-th fundamental weight in the expression of the weight λ. In order to get the coefficients of the weight $s_i(\lambda)$, add a to the coefficients *corresponding to the adjacent nodes in the Dynkin diagram, with multiplicity if there is a multiple edge directed towards the adjacent node, and replace a by $-a$.*

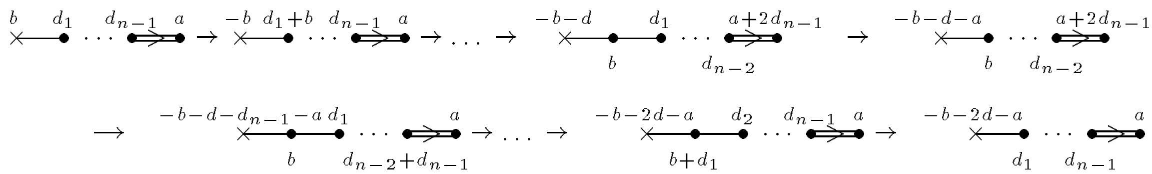
Since we inscribe the coefficients of $\lambda + \delta$ over the nodes, the affine action of the Weyl group on the weights is described exactly by the above procedure applied to the coefficients inscribed over the nodes of the Dynkin diagram.

3.3. Let us perform this explicitly for $\mathfrak{o}(m+n+2,\mathbb{C})$. We have to perform the action of the elements from W as indicated in 3.2. Let us fix first a weight  with all coefficients non-negative (but not necessarily integral).



where $d = d_1 + \cdots + d_{n-2}$.

Similarly we get the pattern for manifolds of dimension $2n+1$ starting with a weight :



where $d = d_1 + \cdots + d_{n-1}$.

In particular, the patterns in the small dimensions three and four are listed in 3.9 below.

3.4. Each position in the pattern corresponds to just one Weyl chamber and the weights λ which determine representations with regular infinitesimal character are those with $\lambda + \delta$ not lying on a wall of a Weyl chamber. Thus, the unique position of every representation with regular infinitesimal character can be read off the coefficients over the nodes. Let us call the non-negative coefficients $a, b, \dots$ over the left-most weight in the pattern the *coefficients of the pattern.*

If some of the coefficients of the pattern are not integral, then a lot of the listed weights are not dominant for $\mathfrak{b}$. If the stabilizer of a weight λ under the affine action of the Weyl group is not trivial, then its elements belong to $W^{\mathfrak{b}}$ and the pattern degenerates in such a way that some of the weights are not dominant for $\mathfrak{b}$ and the number of occurrences of the remaining weights appearing in the pattern equals to the cardinality of the stabilizer of each of them.

Lemma. *The number of occurences of the $\mathfrak{b}$-dominant weights in the pattern equals to the number of the zeros among its coefficients increased by one.*

Proof. This follows from the explicite description of the patterns in 3.3. □

3.5. The order. The conformal weights are easily computed by means of the coefficients in the Dynkin diagrams as described in 2.2. The conformal weight ω of the representation with the highest weight $\overset{b}{\times}\!\!-\!\!\overset{d_1}{\bullet}\cdots\overset{d_{n-2}}{\bullet}\!\!<\!{}^{\bullet\, a}_{\bullet\, c}$ is

$$\omega = b + d_1 + \cdots + d_{n-2} + \frac{1}{2}(a+c) - n$$

while that of $\overset{b}{\times}\!\!-\!\!\overset{d_1}{\bullet}\cdots\overset{d_{n-1}}{\bullet}\!\!\Rightarrow\!\!\overset{a}{\bullet}$ is

$$\omega = b + d_1 + \cdots + d_{n-1} + \frac{1}{2}a - \frac{1}{2}(2n+1).$$

If there is an operator $D\colon C^\infty((F_\lambda M)^*) \to C^\infty((F_\rho M)^*)$ between the complex bundles over complex manifolds, then its order is described easily be means of the conformal weights of λ and ρ. Let us remind that D corresponds to the inclusion of the representation space V_ρ into the Verma module $M_{\mathfrak{b}}(\lambda)$ (remember our convention with the duals). Since each homogeneous component in the grading of the Verma module is a $\mathfrak{g}_0$-submodule, the image of the inclusion must be contained in one homogeneous component. But the degree of this component is exactly the order of the operator D. If ω_1 is the conformal weight of λ, then the conformal weight of all irreducible representations in the i-th homogeneous component in $M_{\mathfrak{b}}(\lambda)$ is $\omega_1 - i$. Thus, the operator D has the order $r = \omega_1 - \omega_2$ where ω_2 is the conformal weight of ρ.

3.6. Translation functors. There is a general construction which allows to translate the results on homomorphisms of Verma modules from one pattern to another one, the so called Jantzen-Zuckerman functors, see e.g. [Zuckerman, 77]. As before, let us write V_μ for the finite dimensional irreducible representation with highest weight μ dominant for $\mathfrak{b}$. Further, write V_μ^* for the module contragradient to V_μ, i.e. V_μ^* has the lowest weight $-\mu$. Each $\mathfrak{U}(\mathfrak{g})$-module decomposes completely into submodules with different infinitesimal characters, see e.g. [Zuckerman, 77]. Let us write p_λ for the projections onto the modules with infinitesimal character ξ_λ. Hence given a weight λ dominant for $\mathfrak{b}$ and a weight μ dominant for $\mathfrak{g}$, we can define two functors

$$\varphi_{\lambda+\mu}^{\lambda} = p_{\lambda+\mu} \circ ((\ ) \otimes V_\mu) \circ p_\lambda$$
$$\psi_{\lambda}^{\lambda+\mu} = p_\lambda \circ ((\ ) \otimes V_\mu^*) \circ p_{\lambda+\mu}$$

where the action on the morphisms is defined by the tensor product with the identity.

These functors are defined on a large class of $\mathfrak{U}(\mathfrak{g})$-modules involving the generalized Verma modules. For technical reasons, we shall also allow λ to be an arbitrary weight with $s.\lambda$ dominant for $\mathfrak{b}$ for some $s \in W^{\mathfrak{b}}$ (then the projections p_λ and $p_{\lambda+\mu}$ are well defined), but we shall always assume that $\lambda+\delta$ belongs to the fundamental Weyl chamber which contains the weights corresponding to the representations appearing in the most left position in the patterns. In particular, this means that λ is dominant for $\mathfrak{g}$ if ξ_λ is regular and λ is integral.

Lemma.

(1) *The functor* $\psi_\lambda^{\lambda+\mu}$ *is left adjoint to* $\varphi_{\lambda+\mu}^\lambda$.

(2) *If the weights* λ *and* $\lambda+\mu$ *are equisingular, then* $\psi_\lambda^{\lambda+\mu}(M_{\mathfrak b}(s.(\lambda+\mu))) = M_{\mathfrak b}(s.\lambda)$ *and* $\varphi_{\lambda+\mu}^\lambda(M_{\mathfrak b}(s.\lambda)) = M_{\mathfrak b}(s.(\lambda+\mu))$ *whenever* $s.\lambda$ *is dominant for* $\mathfrak b$.

Proof. Since V_μ is finite dimensional, the space of homomorphisms $\mathrm{Hom}(M_{\mathfrak b}(s.(\lambda+\mu))\otimes V_\mu^*, M_{\mathfrak b}(s'.\lambda))$ is naturally isomorphic to $\mathrm{Hom}(M_{\mathfrak b}(s.(\lambda+\mu)), M_{\mathfrak b}(s'.\lambda)\otimes V_\mu)$. In view of 3.2, only the summand $p_\lambda(M_{\mathfrak b}(s.(\lambda+\mu))\otimes V_\mu^*)$ can contribute to $\mathrm{Hom}(M_{\mathfrak b}(s.(\lambda+\mu))\otimes V_\mu^*, M_{\mathfrak b}(s'.\lambda))$ and similarly only $p_{\lambda+\mu}(M_{\mathfrak b}(s'.\lambda)\otimes V_\mu))$ contributes to the other homomorphisms. This shows the required natural equivalence

$$\mathrm{Hom}(\psi_\lambda^{\lambda+\mu}(M_{\mathfrak b}(s.(\lambda+\mu))), M_{\mathfrak b}(s'.\lambda)) \simeq \mathrm{Hom}(M_{\mathfrak b}(s.(\lambda+\mu)), \varphi_{\lambda+\mu}^\lambda(M_{\mathfrak b}(s'.\lambda))).$$

The other assertion is more difficult to prove. A general theorem reads that if the weights λ and $\lambda+\mu$ are equisingular, then the functors $\psi_\lambda^{\mu+\lambda}$ and $\varphi_{\lambda+\mu}^\lambda$ are the mutually inverse natural equivalences on their definition domains, see [Zuckerman, 77]. If we fix such weights λ and $\lambda+\mu$, then for each $s\in W^{\mathfrak b}$ the weights $s.\lambda$ and $s.(\lambda+\mu)$ determine representations appearing at the same position in the patterns starting with λ and $\mu+\lambda$. The infinitesimal characters are the same ones for the whole pattern and so the projection p_λ is the identity on $M_{\mathfrak b}(s.\lambda)$. Further

$$\begin{aligned} M_{\mathfrak b}(V_{s.\lambda})\otimes V_\mu &= \bigoplus_{i=0}^\infty (S^i(\mathfrak g_{-1})\otimes(V_{s.\lambda}\otimes V_\mu)) \\ &= \bigoplus_{i=0}^\infty (S^i(\mathfrak g_{-1})\otimes(\oplus_{j=1}^k V_{\nu_j})) = \bigoplus_{j=1}^k M_{\mathfrak b}(V_{\nu_j}) \end{aligned}$$

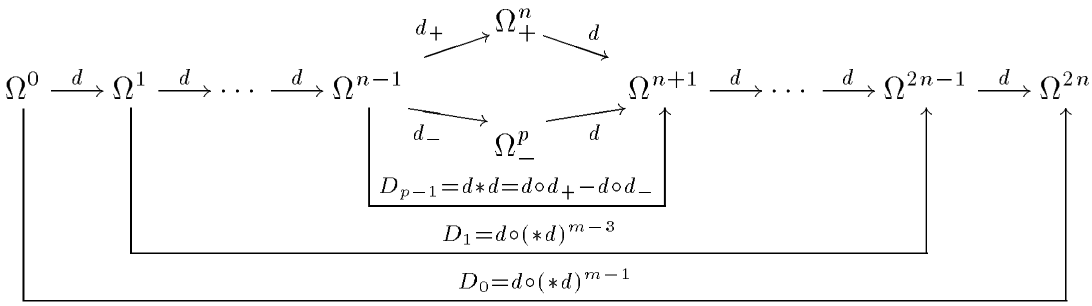
The weights ν_j appearing in the tensor product and their multiplicities can be determined using one of the consequences of the Weyl character formula, e.g. the well known Brower's formula or Klimyk's formula. Finally, the projection $p_{\lambda+\mu}$ selects just those ν_j which lead to the prescribed infinitesimal character $\xi_{\mu+\lambda}$.

So we see that the value of $\varphi_{\lambda+\mu}^\lambda$ on a generalized Verma module must be a sum of generalized Verma modules. If we replace V_μ and λ by V_μ^* and $\lambda+\mu$, we get the same result for the functor $\psi_\lambda^{\lambda+\mu}$. But since $\psi_\lambda^{\lambda+\mu}\circ\varphi_{\lambda+\mu}^\lambda$ is naturally equivalent to the identity, the values can always consist of only one generalized Verma module. But there is certainly the weight $\nu = s.(\lambda+\mu)$ involved among the weights ν_j and this appears with multiplicity one. Thus for all $s\in W^{\mathfrak b}$ we have $\varphi_{\lambda+\mu}^\lambda(M_{\mathfrak b}(s.\lambda)) = M_{\mathfrak b}(s.(\lambda+\mu))$ if $s.\lambda$ is dominant for $\mathfrak b$.

Similarly we can analyze the functor $\psi_\lambda^{\lambda+\mu}$ with μ and λ replaced by $-\mu$ and $\mu+\lambda$ and we get $\psi_\lambda^{\lambda+\mu}(M_{\mathfrak b}(s.(\lambda+\mu))) = M_{\mathfrak b}(s.\lambda)$. □

As a consequence of the lemma, we can pass from one pattern to another one by adding integral weights with regular infinitesimal character. In particular, once we describe all operators between the representations in one pattern, we can get all operators in many other patterns by applying the above translations.

3.7. The operators on exterior forms. All linear natural operators on (pseudo) Riemannian manifolds which do not disappear on flat manifolds and which behave well with respect to constant rescaling of the metric were described in [Slovák, 92b]. They are indicated in the following two diagrams. In the even dimension $m = 2n$ they are all composed from the exterior differential d and the Hodge star operator $*$.

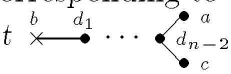


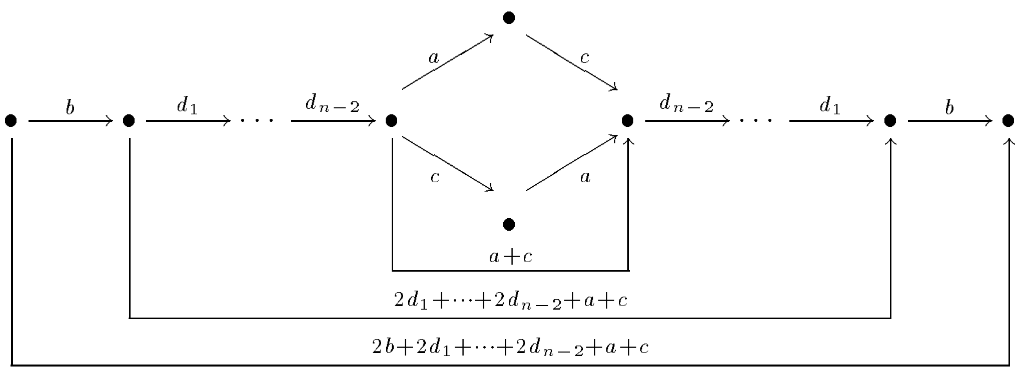
The odd-dimensional case ($m = 2n + 1$) coincides with the de Rham resolvent:

$$\Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Omega^{m-1} \xrightarrow{\ d\ } \Omega^m$$

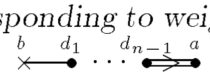
All of them are natural on flat conformal manifolds and there are no other natural linear operators there. In view of the translation procedure and the form of our patterns, this solves the existence problem for operators which act between bundles determined by integral weights with regular infinitesimal character. In particular, there is at most one operator between any two such bundles.

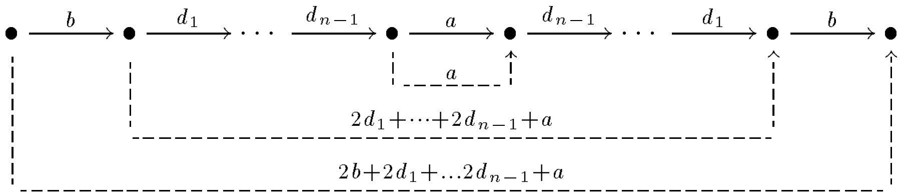
3.8. Theorem. *For every two weights ρ, λ dominant for $\mathfrak{b}$, the space of the natural linear operators $D\colon C^\infty(F_\lambda M) \to C^\infty(F_\rho M)$ acting on smooth sections of complex natural vector bundles over complex conformal Riemannian manifolds is at most one dimensional. All such non-trivial operators, i.e. those different from constant multiples of the identities, are indicated in the patterns below. The labels over the arrows indicate their orders.*

If $\dim M = 2n$, $n > 1$, then the non-trivial linear natural operators act between bundles corresponding to weights with integral coefficients. The pattern starting with the weight  *$b, d_1, \dots, d_{n-2}, a, c \geq 0$, is*



All arrows in the diagram which join weights dominant for $\mathfrak{b}$ describe a non-zero linear natural operator on conformally flat manifolds.

If the dimension of M is $2n+1$, $n > 0$, then the non-zero linear natural operators act between bundles corresponding to weights with integral and half-integral coefficients. If the pattern starts with  *and all the coefficients are non-negative integers,* *then the operators are exhausted exactly by those which are indicated by the solid arrows in the diagram*

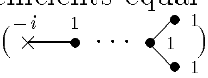


while if some of the coefficients are half-integral and the infinitesimal character is regular, then we get exactly those operators indicated by the dashed arrows which join weights dominant for $\mathfrak{b}$*. If the infinitesimal character of the pattern is singular, then there are no non-trivial operators in odd dimensions.*

Exactly the same classification applies to natural linear operators acting on smooth sections of real natural vector bundles over conformal Riemannian manifolds with an arbitrary signature (m', n')*,* $m' + n' = 2n \geq 4$ *or* $m' + n' = 2n + 1 \geq 3$*.*

Proof. The description of the general patterns and the computation of the conformal weights in 3.3 and 3.4 yield the possible orders of natural operators as indicated on the labels over the arrows in the diagrams above. Since the order must be a non-negative integer, the coefficients of the patterns must be half-integral. Moreover, if these coefficients are not integral and the dimension is even, then the only possibility to find a weight dominant for $\mathfrak{b}$ is either to choose b half-integral or to take two half-integral coefficients over the adjacent nodes in the left-most weight or one of the couples (d_{n-2}, a), (d_{n-2}, c), (a, c) must be half-integral, while all other coefficients must be integral. But then we can choose the half-integral coefficients $\frac{1}{2}$ while the integral can be set to one. All other choices are then covered by the translation procedure. In the case (a, c) is halfintegral, the only two weights dominant for $\mathfrak{b}$ are the two weights just in the middle, which are different but the order should be zero. Thus there is no non-zero operator available in this case. In all other cases listed above, the operator should transform complex functions with suitable conformal weights into complex functions with another conformal weight, but the orders should be odd. However, if we apply the methods leading to the description of the Riemannian invariants in 3.7, then we see that there is no such non-zero operator in the even dimensional case. The reason is that after applying an odd number of covariant derivatives we get into an odd tensor power of the covectors, but then there is no way how to come to functions using the orthogonal invariant tensor operations. Hence there are no non-zero linear natural operators acting between bundles with non-integral coefficients in the even dimensions.

In order to finish the description of the even dimensional case, we have now to discuss case by case the infinitesimal characters by means of the translations between the equisingular ones. If the infinitesimal character of the pattern is regular, then the assertion of the theorem follows from 3.7. We have seen in 3.4 that the two patterns have equisingular infinitesimal characters if and only if they posses the same number of zeros among their coefficients. On the other hand, if there should exist a weight dominant for $\mathfrak{b}$ in the pattern, then there can appear at most one zero, except the case $a = c = 0$, see 3.3.

Assume first $d_i = 0$ for some $0 < i \leq n-2$, or $b = 0$. Then there are only two weights dominant for $\mathfrak{b}$. Let us choose all other coefficients equal to one. Hence the operator should be defined on complex functions $C^\infty($$)$ with values in

$C^\infty(\overset{-2n+i+2}{\times}\!\!-\!\!\!\underset{1}{\bullet}\cdots\bullet\!\!<\!\!\begin{smallmatrix}\bullet\,1\\ 1\\ \bullet\,1\end{smallmatrix})$ (we set $i=0$ if we have chosen $b=0$). Such a natural operator on conformally flat manifolds exists and it is unique up to scalar multiples. This is the so called conformally invariant $n-i-1$-st power of the Laplacian which is defined by the complete contraction of the suitable iteration of the covariant derivative. Its uniqueness is clear from the considerations in the category of Riemannian manifolds. In particular, if $d_{n-2}=0$ we obtain the usual conformally invariant Laplacian on conformally flat manifolds.

If we choose $a=0$ and all other coefficients equal to one, we have also only two weights which are dominant for $\mathfrak{b}$. The corresponding operator $C^\infty(\overset{-n+1}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\bullet\!\!<\!\!\begin{smallmatrix}\bullet\,1\\ 1\\ \bullet\,2\end{smallmatrix})\to C^\infty(\overset{-n}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\bullet\!\!<\!\!\begin{smallmatrix}\bullet\,2\\ 1\\ \bullet\,1\end{smallmatrix})$ exists and is unique up to constant multiples. It is just the conformally invariant Dirac operator. The choice $c=0$ leads to the other Dirac operator on the basic spin representations. The last choice, $a=c=0$ yields four identical weights and operators of order zero. This finishes the discussion on the even dimensions.

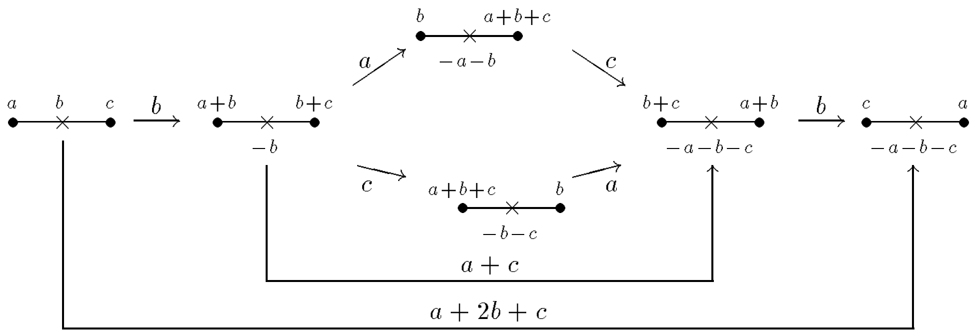
A quite different situation appears in the odd dimensions. There we must admit also the half-integral weights. If we combine our knowledge of the possible orders with the requirement that the arrows which could indicate a natural operator must join the nodes with weights dominant for $\mathfrak{b}$, we see that the only possibility is either to consider b half-integral or two adjacent coefficients d_i, d_{i+1} half-integral or d_{n-1} half-integral. But then either the orders indicated over the solid arrows are not integral or the weights are not dominant for $\mathfrak{b}$, so they are all excluded. Now we can discuss the individual positions of the diagram for functions with suitable half-integral conformal weights. The whole discussion is quite similar to the above description of the sigular patterns in even dimensions. Let us first show this procedure on the case of the longest arrow. We consider the weight $\overset{\frac12}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{1}{\bullet}$, i.e. the operator should act on the complex functions with conformal weight $\frac12$. The order $r=2n$ of the operator is now even and the complete contraction of the r-th iterated covariant derivative is just the n-th power of the Laplacian which is conformally invariant on flat manifolds as an operator acting on functions with conformal weight $\frac12$ with values in functions with conformal weight $\frac12+2n$. The uniqueness up to constant multiples is proved easily in the category of Riemnnian manifolds. Similarly we obtain $(n-i)$-th powers of the Laplacians $C^\infty(\overset{-i+\frac12}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{1}{\bullet})\to C^\infty(\overset{-2n+i+\frac12}{\times}\!\!-\!\!\underset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{1}{\bullet})$ in the remaining cases listed above. The last possibility is $d_{n-1}=\frac12$ which leads to the unique operator $C^\infty(\overset{-n+\frac12}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{2}{\bullet})\to C^\infty(\overset{-n-\frac12}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{2}{\bullet})$ which is the conformally invariant Dirac operator on the basic spin representation.

If the dimension is three, the whole patterns of weights starting with the functions with conformal weight $\frac12$ survives and the middle arrow corresponds to the conformally invariant Dirac operator acting on spinors with conformal weight one.

If the pattern has a singular infinitesimal character, then the weights must be integral. Indeed, with some half-integral coefficient, we need the summation to neglect it, but then we cannot get off the zero among the coeffiecients. Similarly, there can appear only one zero among the coefficients. If all non-zero coefficients equal one, then independent of our choice of the zero, we should find a non-trivial operator acting on complex functions with an odd order. This is not possible for the reason discussed above. Thus, there are no non-trivial operators acting between bundles with singular infinitesimal character in the even dimensions.

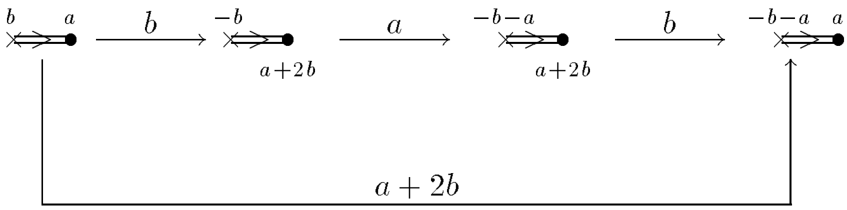
If we want to describe the natural operators in the real setting, then we also have to describe the singular highest weight vectors, but in the real generalized Verma modules, see 1.5. But if we complexify the duals to the jet spaces, then either we obtain the same set of highest weight vectors or some of them can be doubled. In any case no new singular highest weights appear. Since the spaces of the natural operators are always at most one-dimensional in the complex case, either the highest weight vector generating the whole Verma module is doubled, or no other one can be doubled. Thus we may look for the singular highest weight vectors in the complex $\mathfrak{U}(\mathfrak{g})$-module $M_{\mathfrak{b}}(\lambda)$. This also implies the pleasant fact that the existence of the operators and some of their characteristics do not depend on the signature (m', n'). □

3.9. Examples. Let us write down the complete patterns with the orders of the operators inscribed above the arrows, which exhaust all operators in dimensions three and four. If some weights are not dominant for $\mathfrak{b}$ they have to be ignored.



All coefficients are non-negative integers. All linear natural operators on 4-dimensional flat conformal manifolds are involved.

In dimension three we start with $\overset{b}{\times}\!\Longrightarrow\!\overset{a}{\bullet}$ with all coefficients integral or half-integral and non-zero. If the order is not integral we have to omit the corresponding arrow.



Using the general patterns, we can sometimes answer rather general questions. For example, if we want to find all linear natural operators, say of order two, on conformal manifolds of dimension $2n$ such that their source and target bundles coincide up to conformal weights, then they must correspond to the 'long' arrows in our pattern and $a = c$, cf. [Branson, 89, Theorem 3.14]. Now the exact formulas for the orders yield lists of possible sources. In particular, we find the operators $D_{2,k}$ discovered by Branson for $k < \frac{n}{2}$. The operators $D_{2,n}$ appear in the central diamond, e.g. $D_{2,2}\colon C^\infty(\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{3}{\bullet}) \to C^\infty(\overset{3}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet})$ in the pattern which should start with $\lambda = \overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{2}{\bullet}$, cf. [Branson, 89].

4. The curved case

We cannot extend the above results easily for the whole category of conformal manifolds. On the contrary, since the conformal manifolds are highly non-homogeneous in general, we cannot restrict our considerations to one point in the base manifold. A complete general classification has not been obtained yet, but several authors got nice partial results based on various approaches, see e.g. [Branson, 85], [Baston, 90], or the survey paper [Baston, Eastwood, 90]. We shall not go into any detail here, let us only remark, that nearly all of the operators discussed above admit a canonical extension to the whole category of conformal manifolds which is obtained by adding suitable lower order correction terms involving Ricci curvature. The only exception among the patterns with regular infinitesimal characters is the 'longest' arrow from the patterns. These operators might admit an extension, but they do not in general.

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