INVARIANT OPERATORS ON MANIFOLDS WITH ALMOST HERMITIAN SYMMETRIC STRUCTURES, I. INVARIANT DIFFERENTIATION

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Abstract. This is the first part of a series of three papers. The whole series aims to develop the tools for the study of all almost Hermitian symmetric structures in a unified way. In particular, methods for the construction of invariant operators, their classification and the study of their properties will be worked out.

In this paper we present the invariant differentiation with respect to a Cartan connection and we expand the differentials in the terms of the underlying linear connections belonging to the structures in question. Then we discuss the holonomic and non-holonomic jet extensions and we suggest methods for the construction of invariant operators.

1. Introduction

It is well known that the theories of conformal Riemannian structures and projective structures admit a unified exposition in terms of the so called $|1|$-graded Lie algebras $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, see e.g. [Kobayashi, 72], and there has been a wide discussion on geometries fitting into a similar scheme, see e.g. [Kobayashi, Nagano, 64, 65], [Ochiai, 70], [Tanaka, 79]. Following the original Cartan's ideas, the Cartan connections appeared there as the absolute parallelisms obtained on the last nontrivial prolongations of the original G-structure in question, and it turned out that they should play a role similar to that of the Levi-Civita connections in Riemannian geometry.

In this series of papers, we shall deal with any Lie group G with $|1|$-graded Lie algebra $\mathfrak{g}$. We denote B the subgroup with the Lie algebra $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Further there is the normal subgroup $B_1 \subset B$ with the Lie algebra $\mathfrak{g}_1$ and the Lie group $B_0 = B/B_1$ with the Lie algebra $\mathfrak{g}_0$. The corresponding geometric structures are then reductions of the linear frame bundles P^1M on $\dim \mathfrak{g}_{-1}$–dimensional manifolds M to the structure group B_0, as a rule, and it turns out that the flat (homogeneous) models for such structures are the Hermitian symmetric spaces G/B. Thus, following [Baston, 91], we call them the almost Hermitian symmetric structures, briefly the AHS structures. Similar structures were studied earlier by [Goncharov, 87].

Even much more general structures have been investigated thoroughly from the point of view of the equivalence problem, see e.g. [Cartan, 1908], [Tanaka, 93],

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however our present goal is different. We aim to develop a calculus for the Cartan connections similar to the Ricci calculus for the linear (Riemannian) connections. Thus, we first discuss the AHS structures in a more abstract form, as principal B-bundles P equipped with an analogy to the soldering form on the linear frame bundles. This corresponds to thinking about conformal and projective structures as being second order structures (i.e. reductions of the second order frame bundles P^2M). In particular, there is always a class of distinguished linear connections, parameterized by one-forms, and we work out tools for building invariant operators as expressions in terms of these linear connections. The kth order jet extensions of the bundles associated to the defining principal B-bundles P fail to be associated bundles to P in a natural way, except for the locally flat structures, however the semi-holonomic ones always are. We construct a universal invariant differential operator with values in the semi-holonomic jets, the iterated invariant derivative with respect to a Cartan connection. Our approach generalizes vastly most of the classical constructions of invariant operators in the conformal Riemannian geometries. We shall comment more explicitly on the direct links in the text.

In the next part of the series, we show that the AHS structures, defined as first order structures, give rise to canonical principal B-bundles equipped with the canonical soldering forms. Moreover we construct explicitly the canonical Cartan connections there. Thus, the calculus developed here suggests direct methods for the study of invariant operators on all AHS structures.

In the third part, we shall rewrite the recurrence procedure for the expansion of the invariant differentials in terms of the finite dimensional representation theory of the invidual Lie algebras. This will help us to achieve an explicit construction of large classes of invariant operators, even for the 'curved cases'. In particular, we shall show that all the operators on locally flat AHS manifolds, well known from the theory of the generalized Verma modules as the standard operators, have a canonical extension to all AHS manifolds. Moreover, we shall even present universal formulae for those operators in a closed form .

There are also further applications of our development already. Let us mention at least the generalization of the Eastwood's 'curved translation principle' in the conformal Riemannian geometry, [Eastwood, 95], worked out for all AHS structures in [Čap, 96], and the algebraic implications of our non-holonomic jet considerations discussed in [Eastwood, Slovák]. Several links to earlier paper and some further discussion on applications is also available in [Slovák, 96].

Our motivation comes mostly from the wide range of results on the invariant operators on conformal Riemannian manifolds, in particular the series of papers by T. N. Bailey, R. J. Baston, T. P. Branson, M. G. Eastwood, C. R. Graham, H. P. Jakobsen, V. Wünsch and others, cf. the references at the end of this paper, but our development is probably most influenced by [Baston, 90, 91].

The work on this series of papers has also benefited from discussions with several mathematicians, the authors like to mention especially the fruitful communication with J. Bureš and M. Eastwood.

2. The invariant differentiation

We shall discuss the obvious operation on the frame forms of sections of associated bundles defined by means of the horizontal vector fields with respect to the Cartan connections. However, we exploit the very special properties of the structures and connections in question, and we can iterate our derivatives. The result of such an iterated differentiation of a section is not the frame form of a section in general (i.e. the required equivariance properties fail), but we shall see later that the invariant iterated differential defines a universal differential operator with values in semi-holonomic jet extensions of the bundles in question.

2.1. Cartan connections. Let G be a Lie group, $B \subset G$ a closed subgroup, and let $\mathfrak{g}$, $\mathfrak{b}$ be the Lie algebras of G and B. Further, let $P \to M$ be a principal fiber bundle with structure group B and let us denote by ζ_X the fundamental vector field corresponding to $X \in \mathfrak{b}$. A $\mathfrak{g}$-valued one form $\omega \in \Omega^1(P, \mathfrak{g})$ with the properties

(1) $\omega(\zeta_X) = X$ for all $X \in \mathfrak{b}$
(2) $(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in B$
(3) $\omega|_{T_uP} \colon T_uP \to \mathfrak{g}$ is a bijection for all $u \in P$

is called a *Cartan connection*. Clearly, $\dim M = \dim G - \dim B = \dim(G/B)$ if a Cartan connection exists.

The curvature $K \in \Omega^2(P, \mathfrak{g})$ of a Cartan connection ω is defined by the structure equation

$$d\omega = -\frac{1}{2}[\omega, \omega] + K.$$

The Cartan connection ω defines for each element $Y \in \mathfrak{g}$ the vector field $\omega^{-1}(Y)$ given by the equality $\omega(\omega^{-1}(Y)(u)) = Y$ for all $u \in P$. This defines an absolute parallelism on P.

From now on we assume that there is an abelian subalgebra $\mathfrak{g}_{-1}$ in $\mathfrak{g}$ which is complementary to $\mathfrak{b}$, so that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{b}$. Then $\omega^{-1}(\mathfrak{g}_{-1}) \subset TP$ is a smooth distribution which is complementary to the vertical subbundle, so we can consider ω as a generalized connection on P. Moreover ω splits as $\omega = \omega_{-1} + \omega_{\mathfrak{b}}$ according to the above decomposition and similarly for the curvature.

A direct computation using property (2) of Cartan connections shows that the curvature is always a horizontal 2-form, i.e. it vanishes if one of the vectors is vertical. Thus it is fully described by the function $\kappa \in C^\infty(P, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g})$,

$$\kappa(u)(X, Y) = K(\omega^{-1}(X), \omega^{-1}(Y))(u).$$

If we evaluate the structure equation on $\omega^{-1}(X)$ and $\omega^{-1}(Y)$ we obtain

$$\begin{aligned} -[X, Y] + K(\omega^{-1}(X), \omega^{-1}(Y)) &= \omega^{-1}(X)(\omega(\omega^{-1}(Y))) - \omega^{-1}(Y)(\omega(\omega^{-1}(X))) \\ &\quad - \omega([\omega^{-1}(X), \omega^{-1}(Y)]) \\ &= \ - \omega([\omega^{-1}(X), \omega^{-1}(Y)]). \end{aligned}$$

In particular for $X, Y \in \mathfrak{g}_{-1}$, we see that $\kappa(u)(X, Y) = -\omega(u)([\omega^{-1}(X), \omega^{-1}(Y)])$ so the $\mathfrak{b}$-part of κ is the obstruction against integrability of the horizontal distribution defined by ω.

In particular, on the principal fiber bundle $G \to G/B$ over the homogeneous space G/B there is the (left) Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ which is a Cartan connection and Maurer-Cartan structure equation shows that the corresponding curvature is vanishing.

2.2. Let $\lambda\colon B\to GL(V_\lambda)$ be a linear representation and let $P\to M$ be a principal fiber bundle as above. Then the sections of the associated vector bundle $E_\lambda\to M$ correspond bijectively to B-equivariant smooth functions in $C^\infty(P,V_\lambda)$. We shall systematically use this identification without further comments. For a classical principal connection γ on P, the covariant derivative on E_λ along a vector field X on M can be defined as the B-equivariant function $\nabla^\gamma_X f$ with the value at $u\in P$ given by the usual derivative in the direction of the horizontal lift of X to P. For a general connection on a bundle P without any structure group, we can apply the same definition to vector fields on P. The idea is to view the Cartan connections as general connections, but to exploit their special properties.

Given a Cartan connection ω on P, there is the horizontal projection $\chi_\omega\colon TP\to TP$ defined for each $\xi\in T_uP$ by $\chi_\omega(\xi)=\xi-\zeta_{\omega_{\mathfrak b}(\xi)}(u)$, where ζ_Y means the fundamental vector field corresponding to $Y\in\mathfrak b$. The covariant exterior differential with respect to ω on vector-valued functions $s\in C^\infty(P,V)$, evaluated on a vector field ξ on P, is $d^\omega s(u)(\xi)=(\chi_\omega\circ\xi)\cdot s$. By definition, the value of the covariant exterior differential $d^\omega s(u)(\xi)$ with respect to ξ depends only on the horizontal projection $\chi_\omega(\xi(u))$, hence on $\omega(\chi_\omega(\xi(u)))=\omega_{-1}(\xi(u))$. This leads to the following

2.3. Definition. The mapping $\nabla^\omega\colon C^\infty(P,V)\to C^\infty(P,\mathrm{Hom}(\mathfrak g_{-1},V))$ defined by $\nabla^\omega s(u)(X)=\mathcal L_{\omega^{-1}(X)}s(u)=(\omega(u))^{-1}(X)\cdot s$ is called the *invariant differential* corresponding to the Cartan connection ω.

We shall often use the brief notation $\nabla^\omega_X s(u)$, $X\in\mathfrak g_{-1}$ for $(\nabla^\omega s)(u)(X)$.

Note that the invariant differential of a B–equivariant function is not B–equivariant in general, so the invariant differential of a section is not a section in general. Also, our brief notation suggests that ∇^ω_X should behave like the usual covariant derivative along a vector field, but the analogy fails in general because of the nontrivial interaction between $\mathfrak g_{-1}$ and $\mathfrak b$. There is however a possibility to form a section of a bundle out of a given section and its invariant differential. This point of view will be worked out in detail in section 5.

2.4. Proposition (Bianchi identity). *Let $\omega\in\Omega^1(P,\mathfrak g)$ be a Cartan connection. Then the curvature κ satisfies*

$$\sum_{\mathrm{cycl}}\big([\kappa(X,Y),Z]-\kappa(\kappa_{-1}(X,Y),Z)-\nabla^\omega_Z\kappa(X,Y)\big)=0$$

for all X, Y, $Z\in\mathfrak g_{-1}$, where $\sum_{\mathrm{cycl}}$ denotes the sum over all cyclic permutations of the arguments.

Proof. Let X, Y, $Z\in\mathfrak g_{-1}$ and let us write $\tilde X$, $\tilde Y$, $\tilde Z$ for the vector fields $\omega^{-1}(X)$, $\omega^{-1}(Y)$, $\omega^{-1}(Z)$. Now we evaluate the structure equation $d\omega+\frac12[\omega,\omega]=K$ on the fields $[\tilde X,\tilde Y]$ and $\tilde Z$:

$$-\mathcal L_{\tilde Z}\omega([\tilde X,\tilde Y])-\omega([[\tilde X,\tilde Y],\tilde Z])-[\kappa(X,Y),Z]=-\kappa(\kappa_{-1}(X,Y),Z).$$

Using the definition of the invariant differential, we obtain

$$\omega([[\tilde X,\tilde Y],\tilde Z])=-[\kappa(X,Y),Z]+\kappa(\kappa_{-1}(X,Y),Z)+\nabla^\omega_Z\kappa(X,Y).$$

Forming the cyclic sum, the left hand side vanishes by the Jacobi identity for vector fields. □

2.5. The iterated invariant differential. The invariant differential with respect to any Cartan connection ω can be iterated, after k applications on $s \in C^\infty(P, V)$ we get $(\nabla)^k s = \nabla \dots \nabla s \in C^\infty(P, \otimes^k \mathfrak{g}_{-1}^* \otimes V)$.

Lemma. *For all $u \in P$ and $X, Y, \dots, Z \in \mathfrak{g}_{-1}$, $s \in C^\infty(P, V)$, we have*

$$(\nabla)^k s(u)(X, Y, \dots, Z) = (\mathcal{L}_{\omega^{-1}(Z)} \circ \dots \circ \mathcal{L}_{\omega^{-1}(Y)} \circ \mathcal{L}_{\omega^{-1}(X)}) s(u).$$

In particular, we obtain $(\nabla)^2 s(u)(X, Y) - (\nabla)^2 s(u)(Y, X) = \mathcal{L}_{\omega^{-1}(\kappa(X,Y))} s(u)$, the Ricci identity.

Proof. This is just the definition for $k = 1$. So let us assume that the statement holds for $k - 1$. If we replace s by its $(k-2)$-nd invariant differential, we shall deal with the case $k = 2$. By the definition, $\nabla(\nabla s)(u)(X, Y) = \mathcal{L}_{\omega^{-1}(Y)}(\nabla s)(u)(X) = \mathcal{L}_{\omega^{-1}(Y)}(\nabla s(\_)(X))(u)$ since the invariant differential is linear in X. But the expression in the last bracket is just $\mathcal{L}_{\omega^{-1}(X)} s$. Now,

$$(\mathcal{L}_{\omega^{-1}(Y)} \circ \mathcal{L}_{\omega^{-1}(X)} - \mathcal{L}_{\omega^{-1}(X)} \circ \mathcal{L}_{\omega^{-1}(Y)}) s = \mathcal{L}_{[\omega^{-1}(Y), \omega^{-1}(X)]} s = \mathcal{L}_{\omega^{-1}(\kappa(X,Y))} s$$

since $\mathfrak{g}_{-1}$ is abelian. □

2.6. Let us compare our approach with the classical covariant derivative with respect to a linear connection $\gamma \in \Omega^1(P^1M, \mathfrak{gl}(m))$ on the linear frame bundle P^1M. With the help of the soldering form $\theta \in \Omega^1(P^1M, \mathbb{R}^m)$, we can build the form $\omega = \theta \oplus \gamma \in \Omega^1(P^1M, \mathbb{R}^m \oplus \mathfrak{gl}(m))$, a so called affine connection on M. It is simple to check that ω is a Cartan connection in the above sense and that the horizontal lift of a vector $\xi \in T_xM$ which is determined by $X \in \mathbb{R}^m$ and $u \in P^1M$ is exactly $\omega^{-1}(X) \in T_u(P^1M)$. Thus the covariant differential of a section s of an associated bundle to P^1M is given by $\mathcal{L}_{\omega^{-1}(X)} \tilde{s}(u)$ where $\tilde{s}$ is the frame form of s. Therefore the iterated differential $(\nabla^\omega)^k$ coincides with the classical concept in this special case. The reason why this case is much simpler than the general one is that $\mathbb{R}^m$ is an abelian ideal in $\mathbb{R}^m \oplus \mathfrak{gl}(m)$ and not only a subalgebra.

3. The second order structures

3.1. From now on we will assume that the group G is connected and semisimple, and that its Lie algebra is equipped with a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then the following facts are well known, see [Ochiai, 70]:

(1) $\mathfrak{g}_0$ is reductive with one–dimensional center
(2) the map $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ induced by the adjoint representation is the inclusion of a subalgebra
(3) the Killing form identifies $\mathfrak{g}_1$ as a $\mathfrak{g}_0$ module with the dual of $\mathfrak{g}_{-1}$
(4) the restrictions of the exponential map to $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are diffeomorphisms onto the corresponding closed subgroups of G.

By B we denote the closed (parabolic) subgroup of G corresponding to the Lie algebra $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then there is the normal subgroup B_1 in B with Lie algebra $\mathfrak{g}_1$. From (4) above we see that B_1 is a vector group. Finally, $B_0 := B/B_1$ is a reductive group with Lie algebra $\mathfrak{g}_0$, and the Lie group homomorphism induced by the inclusion of $\mathfrak{g}_0$ into $\mathfrak{b}$ splits the projection, so B is isomorphic to the semidirect product of B_0 and B_1.

3.2. In this setting any Cartan connection $\omega \in \Omega^1(P, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1)$ on a principal fiber bundle P with structure group B decomposes as $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$ and analogously does its curvature.

In order to involve certain covering phenomena, we shall slightly extend the classical definition of a structure on a manifold M. For principal fiber bundles P_1, P_2 over M with structure groups G_1, G_2, any morphism of principal fiber bundles $P_1 \to P_2$ over the identity on M, associated with a covering of a subgroup of G_2 by G_1, will be called a reduction of P_2 to the structure group G_1. For example, the spin structures on Riemannian manifolds will be incorporated into our general scheme in this way.

Now we show that in our setting, the canonical principal bundle $G \to G/B =: M$ can be viewed as a reduction of the second order frame bundle $P^2M \to M$ to the structure group B.

Lemma. *Let $O \in M$ be the coset of $e \in G$, $\varphi\colon \mathfrak{g}_{-1} \to M$, $\varphi(X) = \exp X \cdot O$. We define $i\colon G \to P^2M$, $i(g) = j^2_0(\ell_g \circ \varphi)$ and $i'\colon B \to G^2_m$, $i'(b) = j^2_0(\varphi^{-1} \circ \ell_b \circ \varphi)$. Then these two mappings define a reduction (in the above sense) of P^2M.*

Proof. We have $i(g.b) = j^2_0(\ell_{g.b} \circ \varphi) = j^2_0(\ell_g \circ \varphi).j^2_0(\varphi^{-1} \circ \ell_b \circ \varphi)$. Since the action of B on M is induced by conjugation, the conditions 3.1.(2) and (3) imply that the homomorphism i' induces an injection on the level of Lie algebras, so it is indeed a covering of a subgroup of G^2_m. □

Before we give the general definition of a B–structure on a manifold, we list some examples.

3.3. Examples. The semisimple Lie algebras $\mathfrak{g}$ which admit a grading of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ can be completely classified. In fact, the classification of these algebras in the complex case is equivalent to the classification of Hermitian symmetric spaces, see [Baston, 91] for the relation. The full classification can be found in [Kobayashi, Nagano, 64, 65] . Here we list some examples which are of interest in geometry:

(1) Let $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$, the algebra of matrices with trace zero, $\mathfrak{g}_0 = \mathfrak{sl}(p, \mathbb{R}) \oplus \mathfrak{sl}(q, \mathbb{R}) \oplus \mathbb{R}$ and $\mathfrak{g}_{\pm 1} = \mathbb{R}^{pq}$. The grading is easily visible in a block form with blocks of sizes p, q:

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \quad \mathfrak{g}_1 = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

We obtain easily the formulae for the commutators. Let $X \in \mathfrak{g}_{-1}$, $Z \in \mathfrak{g}_1$, $A = (A_1, A_2) \in \mathfrak{g}_0$. Then

$$\begin{aligned} [\ ,\ ]\colon \mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, \quad & [A, X] = A_2.X - X.A_1 \\ [\ ,\ ]\colon \mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_1, \quad & [Z, A] = Z.A_2 - A_1.Z \\ [\ ,\ ]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0, \quad & [X, Z] = (-Z.X, X.Z). \end{aligned}$$

The corresponding homogeneous space is the real Grassmannian, the corresponding structures are called *almost Grassmannian.* In the special case $p = 1$, $q = m$, we obtain the classical projective structures on m-dimensional manifolds.

(2) Let $\mathfrak{g} = \mathfrak{so}(m+1, n+1, \mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{co}(m, n, \mathbb{R}) = \mathfrak{so}(m, n, \mathbb{R}) \oplus \mathbb{R}$, $\mathfrak{g}_{-1} = \mathbb{R}^{m+n}$, $\mathfrak{g}_1 = \mathbb{R}^{(m+n)*}$. For technical reasons we choose the defining bilinear form $\langle\ ,\ \rangle$ on $\mathbb{R}^{m+n+2}$ given by $2x_0x_{m+n+1} + g(x_1, \dots, x_{m+n})$, where g is the standard pseudometric with signature (m, n) given by the matrix $\mathbb{J}$. In block form with sizes $1, m+n, 1$, we get

$$\begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & -p^T\mathbb{J} & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \qquad \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{g}_0, \qquad \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & -\mathbb{J}q^T \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{g}_1,$$

where $A \in \mathfrak{so}(m, n, \mathbb{R})$ and $a\mathbb{I}_{m+n}$ is in the center of $\mathfrak{co}(m, n, \mathbb{R})$.

The commutators are

$$\begin{aligned} [\ ,\ ]\colon \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0, &\qquad [(A,a),(A',a')] = (AA' - A'A, 0) \\ [\ ,\ ]\colon \mathfrak{g}_0 \times \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}, &\qquad [(A,a),p] = Ap + ap \\ [\ ,\ ]\colon \mathfrak{g}_1 \times \mathfrak{g}_0 \to \mathfrak{g}_1, &\qquad [q,(A,a)] = qA + aq \\ [\ ,\ ]\colon \mathfrak{g}_{-1} \times \mathfrak{g}_1 \to \mathfrak{g}_0, &\qquad [p,q] = (pq - \mathbb{J}(pq)^T\mathbb{J}, qp) \end{aligned}$$

where (A,a), $(A',a') \in \mathfrak{so}(m,n) \oplus \mathbb{R} = \mathfrak{g}_0$, $p \in \mathbb{R}^m = \mathfrak{g}_{-1}$, $q \in \mathbb{R}^{m*} = \mathfrak{g}_1$.

The homogeneous spaces are the conformal pseudo–Riemannian spheres for metrics with signatures (m, n).

(3) The symplectic algebra $\mathfrak{sp}(2n, \mathbb{R})$ admits the grading with $\mathfrak{g}_{-1} = S^2\mathbb{R}^n$, $\mathfrak{g}_1 = S^2\mathbb{R}^{n*}$, $\mathfrak{g}_0 = \mathfrak{gl}(n, \mathbb{R})$. We can express this grading in the block form:

$$\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \in \mathfrak{g}_{-1}, \qquad \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \in \mathfrak{g}_0, \qquad \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix} \in \mathfrak{g}_1$$

The commutators are $[X, Z] = X.Z \in \mathfrak{gl}(n, \mathbb{R})$, $[A, X] = A.X + (A.X)^T \in \mathfrak{g}_{-1}$, $[A, Z] = -(Z.A + (Z.A)^T) \in \mathfrak{g}_1$. The corresponding homogenous spaces are the Lagrange Grassmann manifolds and the corresponding structures are called almost Lagrangian.

(4) If we use the symmetric form $\begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}$ instead of the antisymmetric one in the previous example, then we obtain the grading $\mathfrak{so}(2n, \mathbb{R}) = \Lambda^2\mathbb{R}^n \oplus \mathfrak{gl}(n, \mathbb{R}) \oplus \Lambda^2\mathbb{R}^{n*}$ with the commutators given by $[X, Z] = X.Z \in \mathfrak{gl}(n, \mathbb{R})$, $[A, X] = A.X - (A.X)^T \in \mathfrak{g}_{-1}$, $[A, Z] = -(Z.A - (Z.A)^T) \in \mathfrak{g}_1$. The corresponding homogeneous spaces are the isotropic Grassmann manifolds. They can be identified with the spaces of pure spinors, so the structures are called almost spinorial.

Some of the above examples coincide in small dimensions. Further there are similar structures corresponding to the exceptional Lie groups and we could also work in the complex setting or choose different real forms. For more information on these structures, see e.g. [Baston, 91].

3.4. Definition. Let G, B be as in 3.1 and let M be a manifold of dimension $m = \dim(\mathfrak{g}_{-1})$. A *$B$-structure on M* is a principal fiber bundle $P \to M$ with structure group B which is equipped with a differential form $\theta = \theta_{-1} \oplus \theta_0 \in \Omega^1(P, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ such that

(1) $\theta_{-1}(\xi) = 0$ if and only if ξ is a vertical vector

(2) $\theta_0(\zeta_{Y+Z}) = Y$ for all $Y \in \mathfrak{g}_0$, $Z \in \mathfrak{g}_1$
(3) $(r_b)^*\theta = \mathrm{Ad}(b^{-1})\theta$ for all $b \in B$, where Ad means the action on the vector space $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \simeq \mathfrak{g}/\mathfrak{g}_1$ induced by the adjoint action.

The form θ is called the *soldering form* or *displacement form.* A homomorphism of B–structures is just a homomorphism of principal bundles, which preserves the soldering forms.

The *torsion* T of the B–structure is defined by the structure equation

$$d\theta_{-1} = [\theta_{-1}, \theta_0] + T.$$

In the next part of this series, we shall apply the classical theory of prolongations of G–structures to show that in all cases, except the projective structures, each classical first order B_0–structure on M gives rise to a distinguished B–structure on M in the above sense. The construction based on certain subtle normalizations extends essentially the results on reductions of the second order frame bundles P^2M due to [Ochiai, 70], and it leads also to an explicit construction of the canonical Cartan connection. An illustration of this procedure in the special case of the conformal structures is presented in the last section of this paper. The next lemma shows that in the Ochiai's approach we loose all structures with non-zero torsion T. However, the torsion is quite often the only obstruction against the local flatness, see [Baston, 91] or [Čap, Slovák, 95]. For example, Ochiai deals in fact only with spaces locally isomorphic to the homogeneous spaces for all higher dimensional Grassmannian structures.

3.5. Lemma. *Let M be an m–dimensional manifold and let $P \to M$ be a reduction of the second order frame bundle $P^2M \to M$ to the group B over the homomorphism i', as in 3.2. Then there is a canonical soldering form θ on P such that (P, θ) is a B–structure on M, and this B–structure has zero torsion.*

Proof. The second frame bundle P^2M, is equipped with a canonical soldering form a form $\theta^{(2)} \in \Omega^1(P^2M, \mathbb{R}^m \oplus \mathfrak{g}^1_m)$ defined as follows. Each element $u \in P^2_xM$, $u = j^2_0\varphi$, determines a linear isomorphism $\tilde{u}\colon \mathbb{R}^m \oplus \mathfrak{g}^1_m \to T_{\pi^2_1(u)}P^1M$ (in fact $T_0(P^1\varphi)\colon T_{(0,e)}(\mathbb{R}^m \times G^1_m) \to TP^1M$). Now if $X \in T_uP^2M$ then $\theta^{(2)}(X) = \tilde{u}^{-1}(T\pi^2_1(X))$, i.e. $\theta^{(2)}(X) = j^1_0(P^1\varphi^{-1} \circ \pi^2_1 \circ c)$ if $X = j^1_0c$. This canonical form decomposes as $\theta^{(2)} = \theta_{-1} \oplus \theta_0$ where θ_{-1} is the pullback of the soldering form θ on P^1M, $\theta_{-1} = (\pi^2_1)^*\theta$, while θ_0 is $\mathfrak{g}^1_m$-valued.

It is well known, see e.g. in [Kobayashi, 72] that this is in fact a soldering form with zero torsion, and that for any reduction as assumed the pullback of θ is again a soldering form, which clearly has trivial torsion, too. □

3.6. A B–structure (P, θ) is related to a rich underlying structure. First, we can form the bundle $P_0 := P/B_1 \to M$, which is clearly a principal bundle with group B_0, and $P \to P/B_1$ is a principal B_1–bundle. Now consider the component θ_{-1} of the soldering form. By property (3) in the definition of the soldering form it is B_1–invariant and clearly it is horizontal as a form on $P \to P_0$, so it passes down to a well defined form in $\Omega^1(P_0, \mathfrak{g}_{-1})$, which we again denote by θ_{-1}. One easily verifies that this form is B_0–equivariant and its kernel on each tangent space is precisely the vertical tangent space of $P_0 \to M$. Then for each $u \in P_0$, θ_{-1} induces a linear isomorphism $T_uP_0/V_uP_0 \simeq \mathfrak{g}_{-1}$ and composing the inverse of this map with the tangent map of the projection $p : P_0 \to M$, we associate to each $u \in P_0$ a linear isomorphism $\mathfrak{g}_{-1} \simeq T_{p(u)}M$, thus obtaining a reduction $P_0 \to P^1M$ of the frame bundle of M to the group B_0, where B_0 is mapped to $\mathrm{GL}(m,\mathbb{R}) \simeq \mathrm{GL}(\mathfrak{g}_{-1})$ via the adjoint action.

In particular this shows that one can view the tangent bundle TM of M as the associated bundle $P_0 \times_{\mathrm{Ad}} \mathfrak{g}_{-1}$. Since $P_0 = P/B_1$, we can as well identify TM with $P \times_{(\mathrm{Ad},\mathrm{id})} \mathfrak{g}_{-1}$. Here (Ad,id) means the adjoint action of B_0 and the trivial action of B_1.

Lemma. *Let (P,θ) be a B–structure on M, $P_0 := P/B_1$. Then there exists a global smooth B_0–equivariant section $P_0 \to P$, and if σ is any such section we have:*

1. $\gamma := \sigma^*\theta_0 \in \Omega^1(P_0,\mathfrak{g}_0)$ *is a principal connection on* P_0.
2. $\omega := \sigma^*\theta$ *is a Cartan connection on P_0 with $\mathfrak{g}_{-1}$–component equal to the form θ_{-1} from above.*
3. *The invariant differential $\nabla^\omega : C^\infty(P_0,V) \to C^\infty(P_0,\mathfrak{g}^*_{-1}\otimes V)$ coincides with the usual covariant (exterior) differential $d^\gamma : \Omega^0(P_0,V) \to \Omega^1(P_0,V)$ viewed as $d^\gamma : C^\infty(P_0,V)^{B_0} \to C^\infty(P_0,\mathfrak{g}^*_{-1}\otimes V)^{B_0}$.*
4. *The components of the curvature $K = K_{-1} \oplus K_0$ of ω are just the torsion and the curvature of the principal connection γ.*

The space of all equivariant sections σ as above is an affine space modeled on the space $\Omega^1(M)$ of one–forms on M.

Proof. Starting from a principal bundle atlas for $P \to M$, we see that we can find a covering $\{U_\alpha\}$ of M such that the bundle $p : P \to P_0$ is trivial over any of the sets $\pi^{-1}(U_\alpha) \subset P_0$, where $\pi : P_0 \to M$ is the projection. Since B is the semidirect product of B_0 and B_1 we can choose a local B_0–equivariant section s_α of $P \to P_0$ over each of these subsets.

Next, there is a smooth mapping $\chi : P \times_{P_0} P \to \mathfrak{g}_1$ determined by the equation $v = u{\cdot}\exp(\chi(u,v))$. If $U_\alpha \cap U_\beta \neq \emptyset$ then we have a well defined smooth map $\chi_{\alpha\beta} : \pi^{-1}(U_\alpha \cap U_\beta) \to \mathfrak{g}_1$ given by $\chi_{\alpha\beta}(u) := \chi(s_\alpha(u), s_\beta(u))$. Since the sections are B_0–equivariant one easily verifies that $\chi_{\alpha\beta}(u{\cdot}b) = \mathrm{Ad}(b^{-1}){\cdot}\chi_{\alpha\beta}(u)$ for all $b \in B_0$. Let $\{f_\alpha\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$ of M. For $u \in P_0$ define $s(u) \in P$ as follows: Choose an α with $\pi(u) \in U_\alpha$ and put

$$s(u) := s_\alpha(u){\cdot}\exp(\textstyle\sum_\beta f_\beta(\pi(u))\chi_{\alpha\beta}(u)).$$

Clearly this expression makes sense, although the $\chi_{\alpha\beta}$ are only locally defined. Since B_1 is abelian, it is easily seen that $\chi_{\alpha\gamma}(u) = \chi_{\alpha\beta}(u) + \chi_{\beta\gamma}(u)$, whenever all terms are defined. Now if γ is another index such that $\pi(u) \in U_\gamma$, we get:

$$\begin{aligned} &s_\gamma(u){\cdot}\exp(\textstyle\sum_\beta f_\beta(\pi(u))\chi_{\gamma\beta}(u)) = \\ &\quad = s_\alpha(u){\cdot}\exp(\chi_{\alpha\gamma}(u)){\cdot}\exp(\textstyle\sum_\beta f_\beta(\pi(u))(\chi_{\gamma\alpha}(u) + \chi_{\alpha\beta}(u))) = \\ &\qquad = s_\alpha(u){\cdot}\exp(\chi_{\alpha\gamma}(u) + \chi_{\gamma\alpha}(u) + \textstyle\sum_\beta f_\beta(\pi(u))\chi_{\alpha\beta}(u)), \end{aligned}$$

so $s(u)$ is independent of the choice of α, and thus $s : P_0 \to P$ is a well defined smooth global section. Moreover, for $b \in B_0$

$$s(u{\cdot}b) = s_\alpha(u){\cdot}b{\cdot}\exp(\mathrm{Ad}(b){\cdot}\textstyle\sum_\beta f_\beta(\pi(u))\chi_{\alpha\beta}(u)) = s(u){\cdot}b,$$

so s is B_0–equivariant, too.

Now if s and σ are two global equivariant sections, then $u \mapsto \chi(s(u), \sigma(u))$ is the frame form of a smooth one–form on M. On the other hand if $\varphi : P_0 \to \mathfrak{g}_1$ is the frame form of a one–form, then $u \mapsto s(u)\cdot \exp(\varphi(u))$ is again a smooth equivariant section.

(1) and (2) are easily verified directly, and (3) was shown in 2.6. (4) follows immediately from (3) and (2) since $\omega = \theta_{-1} \oplus \gamma$ and the torsion and curvature of γ are by definition just $d^\gamma\theta_{-1}$ and $d^\gamma\gamma$, respectively. □

3.7. Induced Cartan connections. We have seen in 3.6 that the soldering form on $P \to M$ leads via B_0–equivariant sections $\sigma : P_0 \to P$ to a distinguished class of principal connections on the bundle $P_0 \to M$, which can be canonically extended to Cartan connections on the latter bundle. Next we show that to any principal connection γ from this distinguished class, i.e. to each equivariant section σ as above, we can construct an induced Cartan connection $\tilde{\gamma}$ on P, which is σ–related to the Cartan connection $\theta_{-1} \oplus \gamma$.

Lemma. *For each B_0-equivariant section $\sigma\colon P_0 \to P$, there is a uniquely defined Cartan connection $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$ satisfying $\omega_1|(T\sigma(TP_0)) = 0$.*

Proof. Using condition (4) of 3.1 we see that the section σ induces an isomorphism of P with $P_0 \times \mathfrak{g}_1$ defined by $u \mapsto (p(u), \tau(u))$ where $p : P \to P_0$ is the projection and the mapping $\tau\colon P \to \mathfrak{g}_1$ is defined by the equality $u = \sigma(p(u)).\exp(\tau(u))$. Now we define ω_1 on $\sigma(P_0)$ by $\omega_1|_{\sigma(P_0)} := d\tau$. Since $\tau \circ \sigma = 0$ we clearly have $\omega_1|(T\sigma(TP_0)) = 0$, and obviously for any $u \in \sigma(P_0)$, ω induces a bijection $T_uP \to \mathfrak{g}$.

Next, since τ is identically zero on $\sigma(P_0)$ and $\tau(u\cdot\exp tX) = \tau(u) + tX$ for $X \in \mathfrak{g}_1$ it follows from 3.4.(2) that on $\sigma(P_0)$ the form $\omega = \theta \oplus \omega_1$ reproduces the generators of fundamental fields.

Now one easily checks that this form can be uniquely extended using the equivariancy properties which are required for a Cartan connection. □

3.8. Lemma. *In the situation of 3.7, denote by $p : P \to P_0$ the projection and let V be any representation of B_0. Let $u \in P$ and $X, Y \in \mathfrak{g}_{-1}$, $b = \exp(\tau(u))$, where τ is the mapping from the proof of 3.7, and $s \in C^\infty(P_0, V)$. Then we have:*

1. $(\nabla^{\tilde{\gamma}} \circ p^* - p^* \circ \nabla^\gamma)s(u)(X) = \zeta_{[\tau(u),X]}(\sigma(p(u))).(s \circ p)$.
2. *Let $h \in B$ be arbitrary. The curvature $\kappa \in C^\infty(P, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g})$ of any Cartan connection satisfies $\kappa(X, Y)(u.h) = \mathrm{Ad}(h^{-1}).\kappa(\mathrm{Ad}(h).X, \mathrm{Ad}(h).Y)(u)$.*
3. *The curvature K of $\tilde{\gamma}$ and the curvature R of γ satisfy $d^\gamma\theta_{-1} \oplus R = \sigma^*K$. In particular $R = \sigma^*K_0$.*
4. *The curvature components of an arbitrary Cartan connection satisfy*

$$\begin{aligned}
\kappa_{-1}(u)(X, Y) &= \kappa_{-1}(\sigma(p(u)))(X, Y) \\
\kappa_0(u)(X, Y) &= \kappa_0(\sigma(p(u)))(X, Y) - [\tau(u), \kappa_{-1}(\sigma(p(u)))(X, Y)] \\
\kappa_1(u)(X, Y) &= \kappa_1(\sigma(p(u)))(X, Y) - [\tau(u), \kappa_0(\sigma(p(u)))(X, Y)] + \\
&\qquad \frac{1}{2}[\tau(u), [\tau(u), \kappa_{-1}(\sigma(p(u)))(X, Y)]].
\end{aligned}$$

5. *If the B–structure has zero torsion, then the curvature of the induced Cartan*

connection $\tilde{\gamma}$ satisfies

$$\begin{aligned}\kappa_{-1}(u)(X,Y) &= 0\\ \kappa_0(u)(X,Y) &= \kappa_0(\sigma(p(u)))(X,Y)\\ \kappa_1(u)(X,Y) &= [\kappa_0(\sigma(p(u)))(X,Y),\tau(u)].\end{aligned}$$

In particular, the component κ_1 vanishes on $\sigma(P_0)$.

Proof. By the definition of the Cartan connections, the formula

$$Tr^b(\tilde{\gamma}^{-1}(X)(u)) = \tilde{\gamma}^{-1}(\mathrm{Ad}(b^{-1}).X)(u.b) \tag{6}$$

holds for all $u \in P$, $b \in B$. Since $\gamma = \sigma^*(\tilde{\gamma})_0$, the horizontal lift of the vector $\xi \in T_xM$ corresponding to $X \in \mathfrak{g}_{-1}$ and $p(u) \in P_0$ with respect to γ is just $Tp(\tilde{\gamma}^{-1}(X)(\sigma(p(u))))$, see 2.6. The definition of the tangent mapping then yields

$$\begin{aligned}(\nabla^{\tilde{\gamma}} \circ p^* - p^* \circ \nabla^{\gamma})s(u)(X) &= \tilde{\gamma}^{-1}(X)(u).(s\circ p) - Tp\big(\tilde{\gamma}^{-1}(X)(\sigma\circ p(u))\big).s\\ &= Tr^b(\tilde{\gamma}^{-1}(\mathrm{Ad}b.X)(\sigma\circ p(u))).(s\circ p) - Tp\big(\tilde{\gamma}^{-1}(X)(\sigma\circ p(u))\big).s\\ &= \tilde{\gamma}^{-1}(X+[\tau(u),X])(\sigma\circ p(u)).(s\circ p) - Tp\big(\tilde{\gamma}^{-1}(X)(\sigma\circ p(u))\big).s\\ &= \zeta_{[\tau(u),X]}(\sigma\circ p(u)).(s\circ p)\end{aligned}$$

where the last but one equality is obtained using the fact that for $Z \in \mathfrak{g}_1$, $X \in \mathfrak{g}_{-1}$ we have:

$$\begin{aligned}(\mathrm{Ad}(\exp Z)).X &= X + [Z,X] + \frac{1}{2}[Z,[Z,X]] + \frac{1}{6}[Z,[Z,[Z,X]]] + \dots\\ &= X + [Z,X] + \frac{1}{2}[Z,[Z,X]].\end{aligned} \tag{7}$$

The next claim also follows from the formula (6) and from the fact that the Lie bracket of f-related vector fields is f-related:

$$\begin{aligned}\kappa(X,Y)(u.b) &= K(\tilde{\gamma}^{-1}(X),\tilde{\gamma}^{-1}(Y))(u.b)\\ &= -\tilde{\gamma}([\tilde{\gamma}^{-1}(X),\tilde{\gamma}^{-1}(Y)](u.b))\\ &= -\mathrm{Ad}(b^{-1})\circ\tilde{\gamma}(Tr^{b^{-1}}.([\tilde{\gamma}^{-1}(X),\tilde{\gamma}^{-1}(Y)](u.b)))\\ &= -\mathrm{Ad}(b^{-1})\circ\tilde{\gamma}([\tilde{\gamma}^{-1}(\mathrm{Ad}b.X),\tilde{\gamma}^{-1}(\mathrm{Ad}b.Y)](u))\\ &= \mathrm{Ad}(b_{-1}).\kappa(\mathrm{Ad}b.X,\mathrm{Ad}b.Y)(u).\end{aligned}$$

(3) follows immediately from the fact that $\tilde{\gamma}$ and $\theta_{-1}\oplus\gamma$ are σ–related.

If $b \in \mathfrak{g}_1$, the horizontal part of $\tilde{\gamma}^{-1}(\mathrm{Ad}b.X)$ is just $\tilde{\gamma}^{-1}(X)$. The curvature of a Cartan connection is a horizontal form and so (2) implies that $\kappa(X,Y)(u) = \mathrm{Ad}b^{-1}\kappa(X,Y)(\sigma(p(u)))$. Now 3.8.(7) implies directly the relations (4).

Once we prove that $\kappa_1|\sigma(P_0) = 0$, (5) will follow directly form (4) since in this case κ_{-1} is just the torsion. But according to the definition of $\tilde{\gamma}$, the vector fields $\tilde{\gamma}^{-1}(X)$ are tangent to $\sigma(P_0)$ for all $X \in \mathfrak{g}_{-1}$. Consequently also the Lie brackets of such fields are tangent to $\sigma(P_0)$ and thus $\tilde{\gamma}_1([\tilde{\gamma}^{-1}(X),\tilde{\gamma}^{-1}(Y)]) = 0$. □

3.9. Admissible Cartan connections. Let (P,θ) be a B–structure on M. A Cartan connection ω on P is called *admissible* if and only if it is of the form $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$. Thus in particular the induced connections from 3.7 are admissible. Moreover, by definition the $\mathfrak{g}_{-1}$ component of the curvature of any admissible Cartan connection is given by the torsion of the B–structure.

Let us now consider two admissible Cartan connections ω, $\bar\omega$, so that they differ only in the $\mathfrak{g}_1$-component. Then there is a function $\Gamma \in C^\infty(P, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1)$ such that $\bar\omega = \omega - \Gamma \circ \theta_{-1}$. Indeed, $\omega - \bar\omega$ has values in $\mathfrak{g}_1$ and vanishes on vertical vectors.

The function Γ can be viewed as an expression for the 'deformation' of ω into $\bar\omega$ and in view of its properties proved below, we call it the *deformation tensor*.

3.10. Lemma. (1) $\Gamma(u.b) = \mathrm{Ad}(b^{-1}) \circ \Gamma(u) \circ \mathrm{Ad}(b)$ *for all* $b \in B_0$
(2) $\Gamma(u.b) = \Gamma(u)$ *for all* $b \in B_1$
(3) $\bar\omega^{-1}(X)(u) = \omega^{-1}(X)(u) + \zeta_{\Gamma(u).X}(u)$ *for all* $X \in \mathfrak{g}_{-1}$
(4) $(\kappa_0 - \bar\kappa_0)(u)(X,Y) = [X, \Gamma(u).Y] + [\Gamma(u).X, Y]$
(5) $(\kappa_1 - \bar\kappa_1)(u)(X,Y) = \nabla^\omega_X \Gamma(u).Y - \nabla^\omega_Y \Gamma(u).X + \Gamma(u)(\kappa_{-1}(X,Y))$
(6) $(\kappa_{-1} - \bar\kappa_{-1})(u)(X,Y) = 0.$

Proof. By definition, $(r^b)^*(\Gamma \circ \theta_{-1}) = \mathrm{Ad}(b^{-1}) \circ (\Gamma \circ \theta_{-1})$ and the adjoint action is trivial if $b \in B_1$. Since $(r^b)^*\theta_{-1} = \theta_{-1}$ for $b \in B_1$ too, the second claim has been proved. If $b \in B_0$ then $\Gamma \circ \theta_{-1}(Tr^b.\xi)(u.b) = \mathrm{Ad}(b^{-1})(\Gamma \circ \theta_{-1})(\xi)(u)$ and the left hand side is $\Gamma(u.b) \circ \mathrm{Ad}(b^{-1}) \circ \theta_{-1}(u)(\xi)$ by the equivariancy of θ_{-1}. Comparing the results, we obtain just the required formula (1).

In order to obtain (3), we compute $\bar\omega(\omega^{-1}(X)) = X - \Gamma \circ \theta_{-1}(\omega^{-1}(X))$ and so $\bar\omega^{-1}(X) = \omega^{-1}(X) + \bar\omega^{-1}(\Gamma \circ \theta(\omega^{-1}(X))) = \omega^{-1}(X) + \zeta_{\Gamma.X}$.

In order to verify (4) and (5), let us compute (we use just the definition of the frame form of the curvature)

$$\begin{aligned}(\kappa - \bar\kappa)(X,Y) &= \bar\omega([\bar\omega^{-1}(X), \bar\omega^{-1}(Y)]) - \omega([\omega^{-1}(X), \omega^{-1}(Y)])\\ &= (\omega - \Gamma \circ \theta)([\omega^{-1}(X) + \zeta_{\Gamma.X}, \omega^{-1}(Y) + \zeta_{\Gamma.Y}]) - \omega([\omega^{-1}(X), \omega^{-1}(Y)])\\ &= \omega([\zeta_{\Gamma.X}, \omega^{-1}(Y)]) + \omega([\omega^{-1}(X), \zeta_{\Gamma.Y}]) + \omega([\zeta_{\Gamma.X}, \zeta_{\Gamma.Y}]) + \Gamma(\kappa_{-1}(X,Y)) -\\ &\quad \Gamma \circ \omega_{-1}([\zeta_{\Gamma.X}, \omega^{-1}(Y)]) - \Gamma \circ \omega_{-1}([\omega^{-1}(X), \zeta_{\Gamma.Y}]) - \Gamma \circ \omega_{-1}([\zeta_{\Gamma.X}, \zeta_{\Gamma.Y}]).\end{aligned}$$

We have to notice that the fields $\zeta_{\Gamma.X}(u) = \omega^{-1}(\Gamma(u).(X))(u)$ are defined by means of the fundamental field mapping, but with arguments varying from point to point in P. To resolve the individual brackets, we shall evaluate the curvature K of ω on the corresponding fields:

$$\begin{aligned}&d\omega(\omega^{-1}(\Gamma.X), \omega^{-1}(Y)) =\\ &\quad = \mathcal{L}_{\omega^{-1}(\Gamma.X)}\omega(\omega^{-1}(Y)) - \mathcal{L}_{\omega^{-1}(Y)}\omega(\omega^{-1}(\Gamma(X))) - \omega([\omega^{-1}(\Gamma(X)), \omega^{-1}(Y)])\\ &\quad = -[\omega(\omega^{-1}(\Gamma(X))), \omega(\omega^{-1}(Y))] + K(\omega^{-1}(\Gamma(X)), \omega^{-1}(Y)).\end{aligned}$$

Since K is a horizontal 2-form it evaluates to zero and $\omega(\omega^{-1}(Y)) = Y$ is constant. Thus we obtain

$$\omega([\omega^{-1}(\Gamma(X)), \omega^{-1}(Y)]) = [\Gamma.X, Y] - \mathcal{L}_{\omega^{-1}(Y)}\Gamma.X.$$

Now we can decompose this equality into the individual components.

$$\begin{aligned} \omega_{-1}([\omega^{-1}(\Gamma(X)),\omega^{-1}(Y)] &= 0 \\ \omega_0([\omega^{-1}(\Gamma(X)),\omega^{-1}(Y)] &= [\Gamma.X,Y] \\ \omega_1([\omega^{-1}(\Gamma(X)),\omega^{-1}(Y)] &= -\nabla^\omega_Y\Gamma.X. \end{aligned}$$

It remains to evaluate the structure equation on the fields $\zeta_{\Gamma.X}$, $\zeta_{\Gamma.Y}$. Since $\mathfrak{g}_1$ is abelian and $K(\zeta_{\Gamma.X},\zeta_{\Gamma.Y})=0$, we obtain

$$\omega([\omega^{-1}(\Gamma(X)),\omega^{-1}(\Gamma(Y))])=\mathcal{L}_{\omega^{-1}(\Gamma(X))}\Gamma.Y-\mathcal{L}_{\omega^{-1}(\Gamma(Y))}\Gamma.X.$$

But the Lie derivatives of Γ depend only on the value of the vector field in the point in question. However, we have already proved that Γ is B_1-invariant and consequently the derivative is zero. Now we can insert the expressions for the brackets into the above expression for the difference $\kappa-\bar\kappa$ and we get exactly the required formulae. □

In particular (1) and (2) show that Γ is always a pullback of a tensor on M. This fact is of basic importance for our approach.

4. Formulae for the iterated invariant differential

As before, we shall consider sections $s\in C^\infty(P_0,V_\lambda)^{B_0}$ of associated bundles induced by representations of B_0 and we shall view them as equivariant mappings $p^*s\in C^\infty(P,V_\lambda)^B$. We shall develop a recurrence procedure which expands the iterated differentials of such sections with respect to any admissible connection in terms of the underlying linear connections. This expression splits the invariant derivatives into the equivariant part (thus a section) and the obstruction parts (which concentrates the failure to the B_1-invariance). Thus, after having the canonical Cartan connections, this will provide us with a direct method of constructing the invariant operators.

4.1. In view of the results of the preceding section, the comparison of the iterated covariant differential with respect to the principal connection $\gamma=\sigma^*\omega_0$ on P_0, with the invariant differential ∇^ω with respect to the admissible Cartan connection ω becomes quite algorithmic. Indeed we can write

$$\begin{aligned} (\nabla^\omega)^k\circ p^*-p^*\circ(\nabla^\gamma)^k &= \nabla^\omega\circ((\nabla^\omega)^{k-1}\circ p^*-p^*\circ(\nabla^\gamma)^{k-1})+ \\ &\quad(\nabla^\omega\circ p^*-p^*\circ\nabla^\gamma)\circ(\nabla^\gamma)^{k-1} \\ &= \nabla^\omega\circ((\nabla^\omega)^{k-1}\circ p^*-p^*\circ(\nabla^\gamma)^{k-1})+ \\ &\quad(\nabla^\omega\circ p^*-\nabla^{\tilde\gamma}\circ p^*)\circ(\nabla^\gamma)^{k-1}+(\nabla^{\tilde\gamma}\circ p^*-p^*\circ\nabla^\gamma)(\nabla^\gamma)^{k-1}. \end{aligned}$$

Thus, we have to start an induction procedure. Let us remind, that the deformation tensor $\Gamma\in\mathcal{C}^\infty(P,\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_1)$ transforming $\tilde\gamma$ into ω is a pullback of a tensor on M, see Lemma 3.10. We shall work in the setting of 3.6-3.10 with $s\in C^\infty(P_0,V_\lambda)^{B_0}$, where V_λ is the representation space for $\lambda\colon B_0 \to GL(V_\lambda)$. In particular, we know from Lemmas 3.8, 3.10 that for $u \in P$, X, $Y \in \mathfrak{g}_{-1}$

$$\nabla^\omega_X(p^*s)(u) - \nabla^{\tilde\gamma}_X(p^*s)(u) = \zeta_{\Gamma(u).X}(u).(p^*s) = 0$$
$$(\nabla^{\tilde\gamma} \circ p^* - p^* \circ \nabla^\gamma)s(u)(X) = \zeta_{[\tau(u),X]}(\sigma(p(u))).(p^*s) = \lambda([X,\tau(u)])(s(p(u))).$$

Consequently, the middle term in the above inductive formula vanishes and the last one yields always the induced action of the bracket on the target space of the iterated covariant differential $(\nabla^\gamma)^{k-1}$. In particular, we have already deduced the general formula for the first order operators:

4.2. Proposition. *Let ω be an admissible Cartan connection, γ be the linear connection corresponding to an equivariant section $\sigma\colon P_0 \to P$. For all $X \in \mathfrak{g}_{-1}$, $s \in C^\infty(P_0, V_\lambda)^{B_0}$, $u \in P$ we have*

$$(\nabla^\omega \circ p^* - p^* \circ \nabla^\gamma)s(u)(X) = \lambda([X,\tau(u)])(s(p(u))).$$

In particular, the difference is zero if evaluated at points with $\tau(u) = 0$.

In order to continue to higher orders, we need to know how to differentiate the expressions which will appear. Thus let us continue with two technical lemmas.

4.3. Lemma. *Let $X \in \mathfrak{g}_{-1}$, $Z \in \mathfrak{g}_1$, $u \in P$. Then*

$$\mathcal{L}_{\tilde\gamma^{-1}(X)}\tau(u) = \tfrac{1}{2}[\tau(u),[\tau(u),X]] \tag{1}$$
$$\mathcal{L}_{\omega^{-1}(Z)}\tau(u) = Z \tag{2}$$

Proof. The definition of τ can be written as $\tau(\sigma(p(u)) = 0$, $\tau \circ r^{\exp Z}(u) = \tau(u) + Z$, $Z \in \mathfrak{g}_1$. Thus in order to get (1), we can compute for $u = \sigma(p(u)).b$

$$\begin{aligned}\nabla^{\tilde\gamma}_X\tau(u) &= T\tau.(\tilde\gamma^{-1}(X)(u)) = T\tau \circ Tr^b.\tilde\gamma^{-1}(\mathrm{Ad}b.X)(\sigma(p(u)))\\ &= T(\tau \circ r^b)(\tilde\gamma^{-1}(X + [\tau(u),X] + \tfrac{1}{2}[\tau(u),[\tau(u),X]])(\sigma(p(u))))\\ &= T\tau(\tilde\gamma^{-1}(X + [\tau(u),X])(\sigma(p(u)))) + \tfrac{1}{2}T\tau(\tilde\gamma^{-1}([\tau(u),[\tau(u),X]])(\sigma(p(u))))\\ &= \tfrac{1}{2}[\tau(u),[\tau(u),X]].\end{aligned}$$

Next let us compute $\zeta_Z.\tau(u)$ for $Z \in \mathfrak{g}_1$.

$$T\tau(\zeta_Z)(u) = \tfrac{\partial}{\partial t}\big|_0 \big(\tau(u.\exp tZ\big) = \tfrac{\partial}{\partial t}\big|_0 \big(\tau(u) + tZ\big) = Z$$

i.e. (2) holds. $\square$

4.4. Lemma. *Let $f\colon P \to V_\lambda$ be a mapping defined by*

$$f(u) = \tilde f(p(u))(\tau(u), \dots, \tau(u)),$$

where $\tilde{f}\colon P_0 \to \otimes^k \mathfrak{g}_1^* \otimes V_\lambda$ *is* $\mathfrak{g}_0$*-equivariant with respect to the canonical action* $\tilde{\lambda}$ *on the tensor product. Then*

$$\begin{aligned}\nabla^\omega_Y f(u) = \lambda([Y,\tau(u)])(f(u)) - &\\ &\tfrac{1}{2}\textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,[\tau(u),[\tau(u),Y]],\dots,\tau(u)) +\\ &(p^*(\nabla^\gamma_Y \tilde{f}))(u)(\tau(u),\dots,\tau(u)) +\\ &\textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,\Gamma(u).Y,\dots,\tau(u)).\end{aligned}$$

Moreover, all the terms in the above expression for $\nabla^\omega f\colon P \to \mathfrak{g}_{-1}^* \otimes V_\lambda$ *satisfy the assumptions of this lemma with the corresponding canonical representation on* $\otimes^t \mathfrak{g}_1^* \otimes \mathfrak{g}_{-1}^* \otimes V_\lambda$*, where* t *is the number of* τ*'s entering the term in question.*

Proof. Let us compute using the chain rule, Proposition 4.2, Lemma 4.3 and 4.1

$$\begin{aligned}(\nabla^\omega_Y f)(u) &= (\nabla^\omega_Y (p^*\tilde{f})(u))(\tau(u),\dots,\tau(u)) +\\ &\quad \textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,\nabla^\omega_Y\tau(u),\dots,\tau(u))\\ &= (p^*(\nabla^\gamma_Y\tilde{f})(u))(\tau(u),\dots,\tau(u)) +\\ &\quad (\tilde{\lambda}([Y,\tau(u)])(p^*\tilde{f})(u))(\tau(u),\dots,\tau(u)) +\\ &\quad \tfrac{1}{2}\textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,[\tau(u),[\tau(u),Y]],\dots,\tau(u)) +\\ &\quad \textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,\Gamma(u).Y,\dots,\tau(u))\\ &= p^*(\nabla^\gamma_Y\tilde{f})(u)(\tau(u),\dots,\tau(u)) + \lambda([Y,\tau(u)])(f(u)) -\\ &\quad \textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,[[Y,\tau(u)],\tau(u)],\dots,\tau(u)) +\\ &\quad \tfrac{1}{2}\textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,[[Y,\tau(u)],\tau(u)],\dots,\tau(u)) +\\ &\quad \textstyle\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,\Gamma(u).Y,\dots,\tau(u)).\end{aligned}$$

It remains to prove that the resulting expressions satisfy once more the assumptions of the lemma. Let us show the argument on the first term $f_1(u)(Y) := \lambda([Y,\tau(u)])f(u)$. We have $f_1(u)(Y) = \tilde{f}_1(p(u))(\tau(u),\dots,\tau(u))(Y)$ with $\tilde{f}_1\colon P_0 \to \otimes^{k+1}\mathfrak{g}_1^* \otimes \mathfrak{g}_{-1}^* \otimes V_\lambda$, $\tilde{f}_1(p(u))(Z_0,\dots,Z_k,Y) = \lambda([Y,Z_0])(\tilde{f}(p(u))(Z_1,\dots,Z_k))$. The evaluation of $\tilde{f}_1$ on Z_0 and Y can be written as the composition

$$(\mathrm{id}\otimes\lambda)\circ(\tilde{f}\otimes\mathrm{ad})\colon P_0 \times \mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \to \otimes^k \mathfrak{g}_1^* \otimes V_\lambda$$

of equivariant mappings, so $\tilde{f}_1$ is equivariant as well. Similarly one can write down explicitly the terms in the second and the third part of the expression. The equivariancy of the terms with Γ follows from 3.10. □

4.5. The second order. Now we have just to apply the above Lemma to the first order formula. Let us write $\lambda^{(k)}$ for the canonical representation on $\otimes^k \mathfrak{g}_{-1}^* \otimes V_\lambda$.

$$\begin{aligned}
&((\nabla^\omega)^2 \circ p^* - p^*(\nabla^\gamma)^2)s(u)(X,Y) = \\
&\quad = \nabla^\omega_Y(\lambda([_,\tau(u)]) \circ (p^*s)(u))(X) + (\lambda^{(1)}([Y,\tau(u)])(p^*\nabla^\gamma s)(u))(X) \\
&\quad = \lambda([X,\Gamma(u).Y])(p^*s(u)) + \\
&\qquad \lambda^{(1)}([Y,\tau(u)])(\lambda([_,\tau(u)])(p^*s)(u))(X) - \\
&\qquad \tfrac{1}{2}\lambda([X,[\tau(u),[\tau(u),Y]]])(p^*s)(u) + \\
&\qquad \lambda([X,\tau(u)])(p^*(\nabla^\gamma_Y s))(u) + \\
&\qquad (\lambda^{(1)}([Y,\tau(u)])(p^*(\nabla^\gamma s))(u))(X)
\end{aligned}$$

Altogether we have got

4.6. Proposition. *For each admissible Cartan connection ω, B_0-equivariant section σ and for each B_0-equivariant function $s\colon P_0 \to V_\lambda$*

$$\begin{aligned}
((\nabla^\omega)^2 \circ p^* - p^* \circ (\nabla^\gamma)^2)s(u)(X,Y) = \lambda([X,\Gamma(u).Y])(p^*s(u)) + &\\
\lambda^{(1)}([Y,\tau(u)])(\lambda([_,\tau(u)])(p^*s)(u))(X) - &\\
\tfrac{1}{2}\lambda([X,[\tau(u),[\tau(u),Y]]])(p^*s)(u) + &\\
\lambda([X,\tau(u)])(p^*(\nabla^\gamma_Y s))(u) + &\\
(\lambda^{(1)}([Y,\tau(u)])(p^*(\nabla^\gamma)s)(u))(X) &
\end{aligned}$$

holds for all $u \in P$. In particular, vanishing of τ yields

$$((\nabla^\omega)^2 \circ p^* - p^* \circ (\nabla^\gamma)^2)s(u)(X,Y) = \lambda([X,\Gamma(u).Y])(p^*s(u)).$$

4.7. The third order. Exactly in the same way, we use the second order formula to compute the next one. Let us write briefly $\mathrm{ad}^2_{\tau(u)}X := [\tau(u),[\tau(u),X]]$. Furthermore, we shall write the arguments X_i on the places where they have to be evaluated, the order of the evaluation is clear from the context. In fact, whenever $\lambda^{(k)}$ appears, the arguments $X_1,\dots,X_k$ are evaluated after this action. We obtain

$$\begin{aligned}
&((\nabla^\omega)^3 \circ p^* - p^*(\nabla^\gamma)^3)s(u)(X_1,X_2,X_3) = \\
&\quad = \nabla^\omega_{X_3}(\text{2nd order difference}) + \zeta_{[\tau(u),X_3]}(\sigma(p(u))).(p^*(\nabla^\gamma)^2 s)(X_1,X_2) \\
&\quad = \lambda^{(2)}([X_3,\tau(u)])\lambda([X_1,\Gamma(u).X_2])(p^*s)(u) + \\
&\qquad \lambda([X_1,(\nabla^\gamma_{X_3}\Gamma)(u).X_2])(p^*s(u)) + \\
&\qquad \lambda([X_1,\Gamma(u).X_2])((p^*\nabla^\gamma_{X_3}s)(u)) + \\
&\qquad \lambda^{(2)}([X_3,\tau(u)])\lambda^{(1)}([X_2,\tau(u)])\lambda([X_1,\tau(u)])(p^*s)(u) - \\
&\qquad \tfrac{1}{2}\lambda^{(1)}([X_2,\tau(u)])\lambda([X_1,\mathrm{ad}^2_{\tau(u)}X_3])(p^*s)(u) - \\
&\qquad \tfrac{1}{2}\lambda^{(1)}([X_2,\mathrm{ad}^2_{\tau(u)}X_3])\lambda([X_1,\tau(u)])(p^*s)(u) + \\
&\qquad \lambda^{(1)}([X_2,\tau(u)])\lambda([X_1,\tau(u)])(p^*\nabla^\gamma_{X_3}s)(u) + \\
&\qquad \lambda^{(1)}([X_2,\Gamma(u).X_3]) \circ \lambda([X_1,\tau(u)])(p^*s)(u) + \\
&\qquad \lambda^{(1)}([X_2,\tau(u)]) \circ \lambda([X_1,\Gamma(u).X_3])(p^*s)(u) -
\end{aligned}$$

$$
\begin{aligned}
&\tfrac{1}{2}\lambda^{(2)}(X_3,\tau(u))\lambda([X_1,\operatorname{ad}^2_{\tau(u)}X_2])(p^*s)(u)+\\
&\tfrac{1}{4}\lambda([X_1,[\operatorname{ad}^2_{\tau(u)}X_3,\operatorname{ad}_{\tau(u)}X_2]])(p^*s)(u)+\\
&\tfrac{1}{4}\lambda([X_1,[\tau(u),[\operatorname{ad}^2_{\tau(u)}X_3,X_2]]])(p^*s)(u)-\\
&\tfrac{1}{2}\lambda([X_1,\operatorname{ad}^2_{\tau(u)}X_2])(p^*\nabla^\gamma_{X_3}s)(u)-\\
&\tfrac{1}{2}\lambda([X_1,[\Gamma(u).X_3,[\tau(u),X_2]]])(p^*s)(u)-\\
&\tfrac{1}{2}\lambda([X_1,[\tau(u),[\Gamma(u).X_3,X_2]]])(p^*s)(u)+
\end{aligned}
$$

$$
\begin{aligned}
&\lambda^{(2)}([X_3,\tau(u)])\lambda([X_1,\tau(u)])(p^*\nabla^\gamma_{X_2}s)(u)-\\
&\tfrac{1}{2}\lambda([X_1,\operatorname{ad}^2_{\tau(u)}X_3])(p^*(\nabla^\gamma_{X_2}s))(u)+\\
&\lambda([X_1,\tau(u)])p^*(\nabla^\gamma_{X_3}\nabla^\gamma_{X_2}s)(u)+\\
&\lambda([X_1,\Gamma(u).X_3])(p^*(\nabla^\gamma_{X_2}s))(u)+
\end{aligned}
$$

$$
\begin{aligned}
&\lambda^{(2)}([X_3,\tau(u)])\lambda^{(1)}([X_2,\tau(u)])(p^*\nabla^\gamma_{X_1}s)(u)-\\
&\tfrac{1}{2}\lambda^{(1)}([X_2,\operatorname{ad}^2_{\tau(u)}X_3])(p^*\nabla^\gamma_{X_1}s)(u)+\\
&\lambda^{(1)}([X_2,\tau(u)])(p^*(\nabla^\gamma_{X_3}\nabla^\gamma_{X_1}s)(u))+\\
&\lambda^{(1)}([X_2,\Gamma(u).X_3])(p^*(\nabla^\gamma_{X_1}s)(u))+
\end{aligned}
$$

$$
\lambda^{(2)}([X_3,\tau(u)])(p^*(\nabla^\gamma)^2 s(u))(X_1,X_2)
$$

where the horizontal rules indicate the relation to the individual terms in the second order difference. Collecting the terms without τ we obtain the universal formula for the third order correction terms.

4.8. Proposition. *For each admissible Cartan connection ω, B_0-equivariant section σ for each function $s\colon P_0\to V_\lambda$ and for all $u\in\sigma(P_0)$ we have*

$$
\begin{aligned}
((\nabla^\omega)^3\circ p^*-p^*\circ(\nabla^\gamma)^3)s(u)(X,Y,Z)={}&\lambda([X,(\nabla^\gamma_Z\Gamma)(p(u)).Y])(s(p(u)))+\\
&\lambda([X,\Gamma(p(u)).Y])((\nabla^\gamma_Z s)(p(u)))+\\
&\lambda([X,\Gamma(p(u)).Z])((\nabla^\gamma_Y s)(p(u)))+\\
&(\lambda^{(1)}([Y,\Gamma(p(u)).Z])((\nabla^\gamma s(p(u))))(X).
\end{aligned}
$$

4.9. Higher orders. We have seen that the computation of the full formulae goes quickly out of hands, but it is algorithmic enough to be a good task for computers.

Algorithm. *The difference $F^k s:=(\nabla^\omega)^k(p^*s)-p^*((\nabla^\gamma)^k s)$ is given by the recursive formula*

$$
\begin{aligned}
F^0 s(u)&=0\\
F^k s(u)(X_1,\dots,X_k)&=\lambda^{(k-1)}([X_k,\tau(u)])(F^{k-1}s(u))(X_1,\dots,X_{k-1})+\\
&\quad S_\tau(F^{k-1}s(u))(X_1,\dots,X_{k-1})+\\
&\quad S_\nabla(F^{k-1}s(u))(X_1,\dots,X_{k-1})+\\
&\quad S_\Gamma(F^{k-1}s(u))(X_1,\dots,X_{k-1})+\\
&\quad \lambda^{(k-1)}([X_k,\tau(u)])(p^*((\nabla^\gamma)^{k-1}s)(u))(X_1,\dots,X_{k-1}).
\end{aligned}
$$

This expression expands into a sum of terms of the form

$$a\lambda^{(t_1)}(\beta_1)\dots\lambda^{(t_i)}(\beta_i)p^*(\nabla^\gamma)^j s$$

where a is a scalar coefficient, the β_ℓ are iterated brackets involving some arguments X_ℓ, the iterated invariant differentials $(\nabla^\gamma)^r\Gamma$ evaluated on some arguments X_ℓ, and τ. Exactly the first t_j arguments $X_1,\dots,X_{t_j}$ are evaluated after the action of $\lambda^{(t_j)}(\beta_j)$, the other ones appearing on the right are evaluated before. The individual transformations in S_τ, S_∇ and S_Γ act as follows.

(1) *The action of S_τ replaces each summand $a\lambda^{(t_1)}(\beta_1)\dots\lambda^{(t_i)}(\beta_i)p^*(\nabla^\gamma)^j s$ by a sum with just one term for each occurrence of τ where this τ is replaced by $[\tau,[\tau,X_k]]$ and the coefficient a is multiplied by $-1/2$.*
(2) *S_∇ replaces each summand in F^{k-1} by a sum with just one term for each occurrence of Γ and its differentials, where these arguments are replaced by their covariant derivatives $\nabla^\gamma_{X_k}$, and with one additional term where $(\nabla^\gamma)^j s$ is replaced by $\nabla^\gamma_{X_k}((\nabla^\gamma)^j s)$.*
(3) *S_Γ replaces each summand by a sum with just one term for each occurrence of τ where this τ is replaced by $\Gamma(u).X_k$.*

If we want to compute the correction terms in order k, then during the expansion of $F^{k-\ell}$ we can omit all terms which involve more then ℓ occurrences of τ.[1]

Proof. The algorithm is fully based on Lemma 4.4 and the initial discussion in 4.1. The last term uses just the equivariancy of the $(k-1)$st covariant differential. All the other terms correspond exactly to the four groups of terms in Lemma 4.4. Since we have proved already in 4.4 that an application of this lemma brings always sums of terms with the required equivariancy properties, it remains only to verify that the rules deduced in 4.4 yield exactly our formulae.

The first two terms are just in the form derived in 4.4. The third one is obtained by the differentiation of the induced mapping $\tilde f$ defined on P_0. But this means that we have to differentiate it like a matrix valued function, i.e. we can first evaluate in τ's and then differentiate them as constants. Since the whole expression is multilinear in the arguments involving Γ, the final form of the transformation follows from the chain rule. The fourth term is also precisely that one from 4.4. □

4.10. Let us now consider the sections $s\in C^\infty(P,V_\lambda)^B$ for an irreducible B-representation λ as before, another irreducible B-representation space V_μ, and a linear zero order operator $\Phi\in\operatorname{Hom}(\otimes^k\mathfrak{g}^*_{-1}\otimes V_\lambda,V_\mu)^{B_0}$. Our formula for the iterated invariant differential yields

$$\Phi\circ(\nabla^\omega)^k s=\Phi\circ(\nabla^\gamma)^k s+D_0(\gamma,\Gamma)s+D_1(\gamma,\Gamma,\tau)s+\dots+D_k(\gamma,\Gamma,\tau)s$$

[1]Some formulae were computed using MAPLEV3. The number of terms in low order formulae are

Order	1	2	3	4	5	6
Full formula	1	5	24	134	900	7184
Correction terms	0	1	4	16	67	328
Linear obstruction terms	1	2	8	30	153	830

where D_j collects just those terms which involve precisely j occurrences of τ. We call D_0 the *correction term* while D_j, $j > 1$, are called the *obstruction terms* of degree j (they are j-linear in τ). Let us underline, that the correction terms and the obstruction terms are built by the universal recursive formula based on 4.4, by means of the same linear mapping Φ. Their values depend on the initial choice of the equivariant section $\sigma\colon P_0 \to P$, however they turn out to be universal polynomial expressions in ∇^γ, Γ and τ (but τ itself depends on the chosen σ). Of course, the composition $D = \Phi \circ (\nabla^\omega)^k$ is a differential operator transforming $C^\infty(P, V_\lambda)^B$ into $C^\infty(P, V_\mu)^B$ if and only if all obstruction terms vanish independently of the choice of σ.

Lemma. *The obstruction terms $D_1, \dots, D_k$ vanish for all choices of equivariant sections $\sigma\colon P_0 \to P$ if and only if the first degree obstruction term D_1 vanishes for all choices of σ.*

Proof. Let us consider a B_0–homomorphism $\Phi\colon \otimes^k \mathfrak{g}_{-1}^* \otimes V_\lambda \to V_\mu$. The obstruction terms vanish for all choices of σ if and only if $\Phi \circ (\nabla^\omega)^k(p^*s)$ is B_1-invariant for all $s \in C^\infty(P_0, V_\lambda)^{B_0}$. This is equivalent to the vanishing of the derivative $\zeta_Z(u).(\Phi \circ (\nabla^\omega)^k)(p^*s)$ for all $Z \in \mathfrak{g}_1$ and $u \in P$. Let us fix $u_0 \in P$ and the section $\sigma\colon P_0 \to P$ with $u_0 \in \sigma(P_0)$, set $\gamma = \sigma^*\theta_0$, and let Γ be the (unique) deformation tensor corresponding to $\tilde\gamma$ and ω. Then each of the obstruction terms is expressed as

$$D_j(\gamma, \Gamma, \tau)s(u) = f_j(u) = \tilde f_j(p(u))(\tau(u), \dots, \tau(u))$$

where $f_j(p(u)) \in S^j\mathfrak{g}_1^* \otimes V$ is a homogeneous polynomial mapping with values in V_μ. Of course, $\tilde f_j \circ p$ are constant in the $\mathfrak{g}_1$ directions and according to our choices $\tau(u_0) = 0$. Now we can compute

$$\begin{aligned} &\zeta_Z(u_0).(\Phi \circ (\nabla^\omega)^k(p^*s)) = \\ &\qquad = \zeta_Z(u_0).p^*(\Phi \circ (\nabla^\gamma)^k s + \tilde f_0) + \sum_{j=1}^k \zeta_Z(u_0).((\tilde f_j \circ p)(\tau, \dots, \tau)) \\ &\qquad = (\tilde f_1 \circ p)(Z) \end{aligned}$$

where the last equality follows from Lemma 4.3. Thus if the first degree obstruction terms vanishes for all choices of σ, then $\Phi \circ (\nabla^\omega)^k(p^*s)$ is B_1-invariant as required. □

4.11. Remark. The above calculus for admissible connections gives a unified way how to compute the variation of an expression given in terms of covariant derivatives with respect to γ_0 and its curvature tensor, caused by the replacement of γ_0 by another connection γ from the distinguished class. Let σ_0 and σ be the B_0–equivariant sections corresponding to the connections γ_0 and γ. Then there is the one form $\Upsilon \in C^\infty(P_0, \mathfrak{g}_1)^{B_0}$ defined by $\sigma(u) = \sigma_0(u).\exp\Upsilon(u)$, see 3.6. Now, we can use the calculus for the admissible Cartan connections to compare the induced connections $\tilde\gamma_0$ and $\tilde\gamma$ and it turns out that the above expansions in terms of the covariant derivatives and τ's yield exactly the variations of the covariant parts. We shall only comment on these topics here, the details are worked out in [Slovák, 96].

The relation between the covariant derivatives is

$$\nabla^{\gamma} s(u)(X) = \nabla^{\gamma_0} s(u)(X) + \lambda([X, \Upsilon(u)]) \circ s(u), \quad X \in \mathfrak{g}_{-1},\ u \in P_0$$

while the change of the deformation tensors Γ_0 and Γ transforming $\tilde{\gamma}_0$ and $\tilde{\gamma}$ into another fixed admissible Cartan connection ω (e.g. the canonical one) is

$$\Gamma(u)(X) = \Gamma_0(u)(X) - \nabla^{\gamma_0}\Upsilon(u)(X) - \tfrac{1}{2}[\Upsilon(u), [\Upsilon(u), X]], \quad X \in \mathfrak{g}_{-1},\ u \in P_0.$$

For the proof see [Slovák, 96, Theorem 1].

The covariant part of the expansion of $(\nabla^{\omega})^k s$ in terms of the connection γ_0 is $(\nabla^{\gamma_0})^k s + D_0(\gamma_0, \Gamma_0)s$. In terms of the other connection γ, the evaluation on the section σ_0 yields

$$\begin{aligned}(\nabla^{\omega})^k p^* s(\sigma_0(u)) &= (\nabla^{\gamma_0})^k s(u) + D_0(\Gamma_0, \gamma_0)s(u) + 0\\ &= (\nabla^{\gamma})^k s(u) + D_0(\Gamma, \gamma)s(u) + D_1(\Gamma, \gamma, \Upsilon)s(u) + \cdots + D_k(\Gamma, \gamma, \Upsilon)s(u).\end{aligned}$$

Thus, the difference of the covariant parts (i.e. the variation of this expression under the change of the underlying connection) is exactly the sum of the obstruction terms with Υ substituted for τ. The striking consequence of this observation is that the covariant parts of the expansions of the iterated differentials do not involve any derivatives of the Υ's in their variations under the change of the connection.

Moreover, we can apply our formulae to any linear combination $D = \sum_{\ell=1}^{k} A_\ell \circ (\nabla^{\omega})^{\ell}$ where the zero order operators A_ℓ may be allowed to depend on the curvature of the canonical Cartan connection ω and its iterated invariant derivatives. Such an expression defines a natural differential operator if and only if all the obstruction terms vanish. A more detailed discussion based on our recurrence procedure and 3.10, 3.8, 3.6 shows that we can find all differential operators built of the covariant derivatives and the curvatures of the underlying connections γ and independent on the particular choice of γ in this way, see [Slovák, 96, Theorem 2].

Thus, our procedure extends the methods due to Wünsch and Günther (developed originally for conformal Riemannian manifolds of dimensions $m \geq 4$) to all AHS structures. More discussion on various links to classical methods can be found in [Slovák, 96].

5. Invariant jets and natural operators

In this section we discuss the concepts of natural bundles and natural operators on manifolds equipped with B–structures. We show how to interpret invariant derivatives with respect to Cartan connections as sections of bundles, and how to naturally construct operators from them. Since there are canonical Cartan connections on the AHS structures, this will lead to natural operators.

5.1. Natural bundles and operators. We shall write $\mathcal{M}f_m(G)$ for the category of m-dimensional manifolds with almost Hermitian symmetric structures corresponding to the Lie group G with Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as defined in 3.4. The morphisms in the category $\mathcal{M}f_m(G)$ are just principal bundle homomorphisms which cover locally invertible smooth maps between the bases and preserve the soldering forms.

For each representation $\lambda\colon B \to GL(V_\lambda)$ and each object $(P, M, \theta) \in \mathcal{M}f_m(G)$ there is the associated vector bundle $E_\lambda M$ to the principal bundle $P \to M$ defining the B–structure on M. This construction is functorial, we obtain the so called *natural vector bundle* E_λ on $\mathcal{M}f_m(G)$. Classically, one is mainly interested in representations of the first order part B_0, which are trivially extended to the whole B. We will devote special attention to this case, too.

A *natural operator* $D\colon E_\lambda \to E_\mu$ between two natural vector bundles is a system of operators $D_M\colon C^\infty(E_\lambda M) \to C^\infty(E_\mu M)$ such that for all morphisms f covering a smooth map $\underline{f}\colon M \to N$ and sections s_1, $s_2 \in C^\infty(E_\lambda M)$, the right-hand square commutes whenever the left-hand one does

$$\begin{array}{ccccc} E_\lambda M & \xleftarrow{\;s_1\;} & M & \xrightarrow{\;Ds_1\;} & E_\mu M \\ {\scriptstyle E_\lambda f}\downarrow & & \downarrow{\scriptstyle \underline{f}} & & \downarrow{\scriptstyle E_\mu f} \\ E_\lambda N & \xleftarrow{\;s_2\;} & N & \xrightarrow{\;Ds_2\;} & E_\mu N \end{array}$$

Notice, that the latter definition implies the locality of all operators D_M. A general approach to natural bundles and operators is developed in [Kolář, Michor, Slovák, 93].

These general definitions of natural bundles and natural operators work well for each category of manifolds with structures of a fixed type, however, in our cases the naturality requirements are very weak. The reason is, that there are nearly no morphisms on general manifolds with AHS structures. Thus, a stronger restriction of the class of operators under study is specified by most authors. Mostly one is interested in operators built from the distinguished linear connections and their curvatures by means of the covariant derivatives which are independent of any particular choice. Such operators are usually called *invariant* and obviously they are natural in the above sense.

5.2. The homogeneous case. There is the subcategory $\mathcal{M}f_m^{\text{flat}}(G) \subset \mathcal{M}f_m(G)$ of spaces locally isomorphic to the homogeneous space $M = G/B$. We can apply the same definition of the natural operators to this subcategory. Due to the homogeneity of the objects, each natural operator on $\mathcal{M}f_m^{\text{flat}}(G)$ is completely determined by $D_{G/B}$ and the latter is in turn determined by its action on germs of sections in one point of G/B. The action of the automorphisms of G/B on the corresponding structure bundle $G \to G/B$ is given by the left multiplication by the individual elements of G. The sections of the bundles $E_\lambda(G/B)$ are identified with B-equivariant V_λ-valued functions on G and the induced action of the automorphisms is just the composition of these functions with the left multiplications by the inverse element. Thus the operators $D_{G/B}$ with the invariance properties of our natural operators are exactly the so called translational invariant operators, cf. [Baston, 90].

This observation suggests another problem on the invariant operators: What are the invariant operators whose restrictions to the locally flat spaces coincide with a given natural operator on $\mathcal{M}f_m^{\text{flat}}(G)$?

The invariant derivatives are manifestly natural operations depending on the Cartan connection, but they do not map sections of bundles to sections of bundles.

However, we shall build a modification of the standard jet prolongation and this will lead to operators depending naturally on a Cartan connection which will play the role of the universal invariant kth order operators.

5.3. The jet prolongation of a representation. As noted above, we would like to view the invariant derivatives of a section of a natural bundle again as sections of natural bundles. For the individual derivatives this is impossible, but we can define some sort of jets. The general idea is to use the invariant differential to identify the standard first jet prolongation of a natural bundle E_λ with the associated bundle induced by an appropriate B-representation. In fact, this can be done in the general setting, where $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{b}$ is any Lie algebra, which linearly splits into the direct sum of an abelian subalgebra $\mathfrak{g}_{-1}$ and a subalgebra $\mathfrak{b}$. So we return to the setting of chapter 2 for the next three subsections.

Assume that we have given a principal B–bundle $P \to M$ with a Cartan connection $\omega = \omega_{-1} \oplus \omega_{\mathfrak{b}} \in \Omega^1(P, \mathfrak{g})$. Moreover assume that $\lambda : B \to \mathrm{GL}(V_\lambda)$ is a representation of B and $s \in C^\infty(P, V_\lambda)$ is a smooth map, and consider the smooth map $(s, \nabla^\omega s) : P \to V_\lambda \oplus (\mathfrak{g}^*_{-1} \otimes V_\lambda)$. Then for each $Z \in \mathfrak{b}$, we obtain

$$\begin{aligned}\zeta_Z.(s, \nabla^\omega_X s) &= (\mathcal{L}_{\omega^{-1}(Z)} s, \mathcal{L}_{\omega^{-1}(Z)} \circ \mathcal{L}_{\omega^{-1}(X)} s) \\ &= (\zeta_Z.s, \nabla^\omega_X(\zeta_Z.s) + \mathcal{L}_{\omega^{-1}([Z,X])}.s),\end{aligned}$$

where we have essentially used the horizontality of the curvature of any Cartan connection. Assume now that $s \in C^\infty(P, V_\lambda)^B$ is equivariant. Then $\zeta_Z.s = -\lambda(Z) \circ s$ and we get

$$-\zeta_Z.(s, \nabla^\omega_X s) = (\lambda(Z) \circ s, \lambda(Z) \circ (\nabla^\omega_X s) - \nabla^\omega_{[Z,X]_{-1}} s + \lambda([Z, X]_{\mathfrak{b}}) \circ s),$$

where we have split $[Z, X] = [Z, X]_{-1} + [Z, X]_{\mathfrak{b}}$ according to the decomposition of $\mathfrak{g}$.

Thus we define the space $\mathcal{J}^1(V_\lambda) := V_\lambda \oplus (\mathfrak{g}^*_{-1} \otimes V_\lambda)$ and the mapping $\tilde\lambda \colon \mathfrak{b} \times \mathcal{J}^1(V_\lambda) \to \mathcal{J}^1(V_\lambda)$ by the formula

$$\tilde\lambda(Z)(v, \varphi) = (\lambda(Z)(v), \lambda(Z) \circ \varphi - \varphi \circ \mathrm{ad}_{-1}(Z) + \lambda(\mathrm{ad}_{\mathfrak{b}}(Z)(\_))(v))$$

where $\mathrm{ad}_{-1}(Z) : \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ is the map $X \mapsto [Z, X]_{-1}$ and $\lambda(\mathrm{ad}_{\mathfrak{b}}(Z)(\_))(v) \colon \mathfrak{g}_{-1} \to V_\lambda$ is defined by $\lambda(\mathrm{ad}_{\mathfrak{b}}(Z)(\_))(v)(X) = \lambda([Z, X]_{\mathfrak{b}})(v)$.

Lemma. *The mapping $\tilde\lambda$ is an action of $\mathfrak{b}$ on $\mathcal{J}^1(V_\lambda)$. For each $\mathfrak{b}$–equivariant element $s \in C^\infty(P, V_\lambda)$, the mapping $(s, \nabla^\omega s) \colon P \to \mathcal{J}^1(V_\lambda)$ is $\mathfrak{b}$-equivariant with respect to this action.*

Proof. For Z, $W \in \mathfrak{b}$ and $(v, \varphi) \in \mathcal{J}^1(V_\lambda)$ we compute:

$$\begin{aligned}\tilde\lambda(W)\tilde\lambda(Z)(v, \varphi) = (&\lambda(W)\lambda(Z)(v), \lambda(W) \circ \lambda(Z) \circ \varphi - \lambda(W) \circ \varphi \circ \mathrm{ad}_{-1}(Z) + \\ &+ \lambda(W) \circ \lambda(\mathrm{ad}_{\mathfrak{b}}(Z)(\_))(v) - \lambda(Z) \circ \varphi \circ \mathrm{ad}_{-1}(W) + \varphi \circ \mathrm{ad}_{-1}(Z) \circ \mathrm{ad}_{-1}(W) - \\ &- \lambda(\mathrm{ad}_{\mathfrak{b}}(Z)(\_))(v) \circ \mathrm{ad}_{-1}(W) + \lambda(\mathrm{ad}_{\mathfrak{b}}(W)(\_))(\lambda(Z)(v))).\end{aligned}$$

Now when forming the commutator of $\tilde\lambda(W)$ and $\tilde\lambda(Z)$ the second and fourth term in the second component do not contribute, so we get

$$(\tilde\lambda(W)\tilde\lambda(Z) - \tilde\lambda(Z)\tilde\lambda(W))(v, \varphi) = (\lambda([W, Z])(v), \lambda([W, Z]) \circ \varphi + \Phi),$$

where Φ is the linear map defined by

$$\begin{aligned}\Phi(X) = \lambda(W)\lambda([Z,X]_{\mathfrak{b}})(v) - \lambda(Z)\lambda([W,X]_{\mathfrak{b}})(v) + \varphi([Z,[W,X]_{-1}]_{-1}) - \\ - \varphi([W,[Z,X]_{-1}]_{-1}) - \lambda([Z,[W,X]_{-1}]_{\mathfrak{b}})(v) + \lambda([W,[Z,X]_{-1}]_{\mathfrak{b}})(v) + \\ + \lambda([W,X]_{\mathfrak{b}})\lambda(Z)(v) - \lambda([Z,X]_{\mathfrak{b}})\lambda(W)(v).\end{aligned}$$

Now using the Jacobi identity and the fact that $\mathfrak{b}$ is a subalgebra while $\mathfrak{g}_{-1}$ is abelian, one immediately verifies that

$$\begin{aligned}[[W,Z],X]_{-1} &= [W,[Z,X]_{-1}]_{-1} - [Z,[W,X]_{-1}]_{-1} \\ [[W,Z],X]_{\mathfrak{b}} &= [W,[Z,X]_{\mathfrak{b}}] - [Z,[W,X]_{\mathfrak{b}}] + \\ &\quad + [W,[Z,X]_{-1}]_{\mathfrak{b}} - [Z,[W,X]_{-1}]_{\mathfrak{b}}.\end{aligned}$$

Using this one immediately sees that

$$\Phi(X) = -\varphi([[W,Z],X]_{-1}) + \lambda([[W,Z],X]_{\mathfrak{b}})(v).$$

The rest of the lemma is a consequence of our definition. □

Thus we can consider the mapping $s \mapsto (s, \nabla^\omega s)$ as a section of the associated bundle to P induced by the $\mathfrak{b}$-module $\mathcal{J}^1 V_\lambda$.

In fact, the action $\tilde{\lambda}$ coincides with the canonical action of $\mathfrak{b}$ on the standard fiber of the usual first jets of sections of $E_\lambda(G/B)$ (which also could be used as a geometrical argument for the proof of the above Lemma). Thus our construction can be understood as identifications of the standard first jet prolongations $J^1 E_\lambda M$ with the associated bundles $P \times_B \mathcal{J}^1 V_\lambda$, determined by the Cartan connection ω. The section $(s, \nabla^\omega s)$ of this bundle is then called the *invariant one–jet* of the section s of $E_\lambda M$.

5.4. The second jet prolongation of a representation. Observe that it is easy to extend $\mathcal{J}^1(\_)$ to a functor on the category of $\mathfrak{b}$–representations. In fact for a homomorphism $f\colon V \to W$ of $\mathfrak{b}$–modules we just define $\mathcal{J}^1(f)\colon \mathcal{J}^1(V) \to \mathcal{J}^1(W)$ by $\mathcal{J}^1(f)(v,\varphi) := (f(v), f \circ \varphi)$. One easily computes directly that $\mathcal{J}^1(f)$ is a module homomorphism.

Second, it is clear that by projecting onto the first component we get a module homomorphism $\mathcal{J}^1(V) \to V$, and actually these homomorphisms constitute a natural transformation between $\mathcal{J}^1(\_)$ and the identity functor.

Next, let us consider $\mathcal{J}^1(\mathcal{J}^1(V))$. There are two natural homomorphism from this space to $\mathcal{J}^1(V)$: First, we have the above mentioned natural projection, and second, there is the first jet prolongation of the projection $\mathcal{J}^1(V) \to V$, and we define $\mathcal{J}^2(V)$ to be the submodule of $\mathcal{J}^1(\mathcal{J}^1(V))$ on which these two homomorphisms coincide. The underlying vector space of $\mathcal{J}^1(\mathcal{J}^1(V))$ is just $(V \oplus (\mathfrak{g}_{-1}^* \otimes V)) \oplus \mathfrak{g}_{-1}^* \otimes (V \oplus (\mathfrak{g}_{-1}^* \otimes V)) \cong V \oplus (\mathfrak{g}_{-1}^* \otimes V) \oplus (\mathfrak{g}_{-1}^* \otimes V) \oplus (\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes V)$, and under this identification $\mathcal{J}^2(V)$ is just the submodule of those elements where the two middle components are equal. One immediately verifies that $\mathcal{J}^2(\_)$ is again a functor and that projecting out the first two components gives a natural transformation to $\mathcal{J}^1(\_)$.

5.5. Higher jet prolongations. We can iterate the above procedure as follows: Suppose we have already constructed functors $\mathcal{J}^i(\_)$ for $i \leq k$ such that $\mathcal{J}^i(V)$ is a submodule in $\mathcal{J}^1(\mathcal{J}^{i-1}(V))$ and such that for each i there is a natural transformation $\mathcal{J}^i(\_) \to \mathcal{J}^{i-1}(\_)$ induced by the projection $\mathcal{J}^1(\mathcal{J}^{i-1}(V)) \to \mathcal{J}^{i-1}(V)$, i.e. by the natural transformation from $\mathcal{J}^1(\_)$ to the identity.

Then consider $\mathcal{J}^1(\mathcal{J}^k(V))$ for some module V. We have two natural homomorphisms $\mathcal{J}^1(\mathcal{J}^k(V)) \to \mathcal{J}^1(\mathcal{J}^{k-1}(V))$, namely the natural projection $\mathcal{J}^1(\mathcal{J}^k(V)) \to \mathcal{J}^k(V)$ followed by the inclusion of the latter space into $\mathcal{J}^1(\mathcal{J}^{k-1}(V))$ and the first jet prolongation of the natural map $\mathcal{J}^k(V) \to \mathcal{J}^{k-1}(V)$, and we define $\mathcal{J}^{k+1}(V)$ to be the submodule where these two module homomorphisms coincide. Moreover for a module homomorphism $f\colon V \to W$ we define $\mathcal{J}^{k+1}(f)$ as the homomorphism induced by $\mathcal{J}^1(\mathcal{J}^k(f))$. Finally, from the obvious projection $\mathcal{J}^{k+1}(V) \to \mathcal{J}^k(V)$ we clearly get a natural transformation $\mathcal{J}^{k+1}(\_) \to \mathcal{J}^k(\_)$.

Also by induction, it is easy to see that as a vector space we always have $\mathcal{J}^k(V) \cong \oplus_{i=0}^k(\otimes^i\mathfrak{g}_{-1}^* \otimes V)$. Moreover starting from lemma 5.3 it is again clear by induction that for any B–equivariant function $s : P \to V$ and any Cartan connection ω on P, the mapping

$$j^k_\omega s := (s, \nabla^\omega s, \dots, (\nabla^\omega)^k s) : P \to \mathcal{J}^k(V)$$

is equivariant, too. This map is called the *invariant k–jet of s with respect to ω.*

5.6. The AHS case. Since $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, there are a few simplifications in the construction of the jet prolongations. First of all for $A \in \mathfrak{g}_0$ we clearly have $\mathrm{ad}_\mathfrak{b}(A) = 0$, while for $Z \in \mathfrak{g}_1$ we have $\mathrm{ad}_{-1}(Z) = 0$, so the action on the first jet prolongation becomes easier for each case. In particular, we see that in fact the action of $\mathfrak{g}_0$ is just the tensorial one. Thus, the isomorphism $\mathcal{J}^k(V) \cong \oplus_{i=0}^k(\otimes^i\mathfrak{g}_{-1}^*\otimes V)$ is not only an isomorphism of vector spaces but also of $\mathfrak{g}_0$–modules.

Moreover, in this case we have the additional information that the group B is the semidirect product of the contractible subgroup B_1 and the subgroup B_0. If we start with a B–representation and form the first jet prolongation of the corresponding $\mathfrak{b}$–representation, then this will always integrate to a B–representation. This is due to the fact that the restriction to $\mathfrak{g}_1$ integrates by contractibility of B_1, while the action of $\mathfrak{g}_0$ is the tensor product of the original action with the adjoint action, and both of these integrate.

Therefore in this case, for each representation $\lambda : B \to GL(V_\lambda)$ we obtain the jet prolongations $\mathcal{J}^k(\lambda) : B \to GL(\mathcal{J}^k(V_\lambda))$. This in turn implies that for each natural bundle E_λ on $\mathcal{M}f_m(G)$ there is the natural bundle $\mathcal{J}^k(E_\lambda)$. By the construction, this bundle coincides with the so called kth semi-holonomic jet prolongation of E_λ.

5.7. There is now a simple procedure how to use the invariant jets with respect to a Cartan connection for the constructions of differential operators. Suppose that $\lambda\colon B \to GL(V_\lambda)$, is a representation of B, and suppose that for some k and another such representation μ on V_μ, there is a B–equivariant (even nonlinear) mapping $\Phi\colon \mathcal{J}^k(V_\lambda) \to V_\mu$. Now, for each $P \to M$ with a Cartan connection ω, we can define a k–th order differential operator $D_M : C^\infty(E_\lambda M) \to C^\infty(E_\mu M)$ by putting

$$D_M(s)(u) := \Phi(j^k_\omega s(u)) \qquad \forall s \in C^\infty(P, V)^B, u \in P.$$

The associated bundles $E_\lambda M$, $E_\mu M$ are functorial in $P \to M$, and so by the construction the operators defined in this way intertwine the actions of all morphisms of the B–structures on the sections, which preserve the Cartan connections. In particular, if there is a canonical Cartan connection, which is preserved under the action of all morphisms, then the operator constructed in this way will be natural. More generally, one can also interpret these operators as natural operators which also depend on the Cartan connections, but we will not work out this point of view here.

5.8. Let us discuss in more detail now, how to find the $\mathfrak{b}$-module homomorphisms $\Phi\colon \mathcal{J}^k V_\lambda \to V_\mu$ for irreducible representations λ and μ of B_0 on V_λ and V_μ, viewed as irreducible representations of $\mathfrak{b}$. Let us recall that $\mathcal{J}^k(V) = \oplus_{i=0}^k(\otimes^i \mathfrak{g}_{-1}^* \otimes V)$ as $\mathfrak{g}_0$-module.

Lemma. *Let $\pi\colon \mathcal{J}^k(V_\lambda) \to \otimes^k \mathfrak{g}_{-1}^* \otimes V_\lambda$ be the $\mathfrak{g}_0$–homomorphism corresponding to the decomposition of $\mathcal{J}^k(V_\lambda)$ and let $\Phi\colon \mathcal{J}^k V_\lambda \to V_\mu$ be a $\mathfrak{g}_0$-module homomorphism whose restriction to $\otimes^k \mathfrak{g}_{-1}^* \otimes V \subset \mathcal{J}^k V_\lambda$ does not vanish. Then Φ is a $\mathfrak{b}$-module homomorphism if and only if it factors through π and Φ vanishes on the image of $\otimes^{k-1}\mathfrak{g}_{-1}^* \otimes V_\lambda \subset \mathcal{J}^k(V_\lambda)$ under the action of $\mathfrak{b}_1$.*

Proof. Let $\mathbb{I}$ be a generator of the center of $\mathfrak{g}_0$. Then by Schur's lemma $\mathbb{I}$ acts by a scalar on every irreducible representation of $\mathfrak{g}_0$. Moreover, for the adjoint representation these scalars are just given by the grading. Now the action of $\mathfrak{g}_0$ on each of the components $\pi_i^k(\mathcal{J}^k(V_\lambda))$ is the tensorial one, so $\mathbb{I}$ acts by different scalars on each of them. Moreover, any $\mathfrak{g}_0$-module homomorphism Φ defined on the top part of $\mathcal{J}^k(V_\lambda)$ is a $\mathfrak{b}$-module homomorphism if and only if Φ is $\mathfrak{g}_1$-invariant. □

5.9. Lemma. *The action of an element $Z \in \mathfrak{g}_1$ on $Y_1 \otimes \cdots \otimes Y_{k-1} \otimes v_{k-1} \in \otimes^{k-1}\mathfrak{g}_{-1}^* \otimes V_\lambda \subset \mathcal{J}^k V_\lambda$ yields*

$$\sum_{i=0}^{k-1}\Big(\sum_\alpha Y_1 \otimes \cdots \otimes Y_i \otimes \eta_\alpha \otimes \big([Z,\xi_\alpha].(Y_{i+1} \otimes \cdots \otimes Y_{k-1} \otimes v_{k-1})\big)\Big) \in \otimes^k \mathfrak{g}_{-1}^* \otimes V_\lambda$$

where η_α and ξ_α are dual basis of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ with respect to the Killing form and the dot means the canonical action of the element in $\mathfrak{g}_0$ on the argument.

Proof. The statement follows easily from the definition of $\mathcal{J}^k V_\lambda$ by induction on the order k. □

5.10. Remark. A reformulation of the preceding Lemma reads: *A $\mathfrak{g}_0$-module homomorphism $\Phi\colon \otimes^k \mathfrak{g}_{-1}^* \otimes V \to W$ can be considered as a $\mathfrak{b}$-module homomorphisms $\mathcal{J}^k(V_\lambda) \to V_\mu$ if and only if*

$$\Phi\Big(\sum_{i=1}^k \lambda^{(i-1)}([Z,X_i])\psi(X_1,\dots,X_{i-1},X_{i+1},\dots,X_k)\Big) = 0$$

for all elements $\psi \in \otimes^{k-1}\mathfrak{g}_{-1}^* \otimes V$, all $X_1,\dots,X_k \in \mathfrak{g}_{-1}$, and all $Z \in \mathfrak{g}_1$.

This expression can be also found among the obstruction terms in the expansion of the kth iterated invariant differential $(\nabla^\omega)^k$. Indeed, the linear obstruction terms involve in particular the terms with highest order derivatives of the section, i.e. those of order $k-1$ in s, and a simple check shows that they are of the above form. Let us call this part the *algebraical obstruction term*. Now, the above Lemma implies that if this algebraical obstruction vanishes 'algebraically', i.e. before substitution of the values of the invariant jets, then all other obstruction terms vanish as well and we have got a natural operator in this way.

Once we have a correspondence between the $\mathfrak{b}$-module homomorphisms of the jets and the natural operators, we should try to extend the algebraic methods leading to the well known classification of all linear natural operators on the locally flat spaces to the general setting. We shall come back to this point in the fourth part of this series. A more straightforward generalization of the Verma module technique can be found in the forthcoming paper [Eastwood, Slovák].

An important observation is, that not all operators are created in such an algebraic way, there are also examples of operators where the algebraic obstruction does vanish only after the substitution of the invariant jets. The simplest example is the second power of the Laplacian on the four dimensional conformal Riemannian manifolds. A more detailed discussion on such cases can be found in [Eastwood, Slovák].

6. Remarks on applications

6.1. Let us indicate now in more detail how the theory developed so far applies to the study of natural operators. The simplest possibility is the one discussed in the end of the previous section:

(1) Starting with irreducible representations λ and μ of B, we consider all the compositions $\Phi\circ(\nabla^\omega)^k$, where $\Phi\colon\ \otimes^k\mathfrak{g}^*_{-1}\otimes V_\lambda\to V_\mu$ is $\mathfrak{g}_0$–equivariant and linear.
(2) Such an expression yields a differential operator on sections if and only if $\Phi\circ(\nabla^\omega)^k s$ is B_1–invariant for each $s\in C^\infty(P,V_\lambda)^B$. In view of the expansion of the iterated differential in terms of the underlying connections, this is equivalent to the vanishing of the linear obstruction terms after substitution of the invariant jets. Moreover, the algebraic vanishing of the algebraic obstruction terms suffices, see 4.10 and 5.10.
(3) There are the canonical Cartan connections ω on all manifolds with AHS structures and so the differential operators obtained in (2) with help of ω turn out to be natural.
(4) If we choose a linear connection γ in the distinguished class, then there is the unique deformation tensor Γ transforming the induced Cartan connection $\tilde\gamma$ into the canonical one. Thus the formulae from Section 4 express the natural operators by means of the covariant derivatives and curvatures of the linear connection γ.

The general construction of the canonical connection ω and the deformation tensors Γ are postponed to the next part of the series. However, in order to have some concrete examples, we present the computations in the conformal Riemannian case below.

6.2. Conformal Riemannian structures. The existence of the principal bundle $P\to M$ with a canonical Cartan connection is well known in the conformal case, see [Kobayashi, 72]. But we prefer to present an explicit construction here to illustrate the links of our concepts and formulae to the the classical approach.

Let us start with a manifold M of dimension $m \geq 3$ equiped with a conformal class of Riemannian metrics or equivalently with a reduction of the first order frame bundle P^1M to the group $B_0 = CSO(m) \simeq SO(m) \rtimes \mathbb{R}$. Any metric in the conformal class has its Levi–Cività connection, which is torsion free. There is a bijective correspondence between torsion–free connections on M and $GL(m)$–equivariant sections of $P^2M \to P^1M$, see [Kobayashi, 72, Proposition 7.1]. Our group B can be viewed as a subgroup of G^2_m, see Lemma 3.2. It turns out that the orbit of the images of the Levi–Cività connections under the group B coincide, and actually give a reduction of P^2M to the group B, and thus a torsion free B–structure on M by 3.5.

If we start with a reduction $\varphi : P_0 \to P^1M$ in the sense of 3.2 to the group $B_0 = Spin(m) \rtimes \mathbb{R}$ then one can still construct a subbundle $\tilde{P}$ of P^2M as above, and $P := \varphi^*\tilde{P}$ is again a torsion free B–structure with the induced soldering form. Thus we can include the Spin representations in our approach.

To obtain a canonical Cartan connection on such B–structures we proceed as follows: The values of the $\mathfrak{g}_0$-component κ_0 of the curvature function κ of a Cartan connection ω can be viewed as elements in $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$, cf. 3.1.(2). There are three possible evaluations in the target space. The evaluation over the last two entries is just the trace in $\mathfrak{g}_0$, the other two possibilities coincide up to a sign since κ is a two form.

Definition. The *trace of the curvature* κ_0 is the composition of κ_0 with the evaluation over the first and the last entry. A *normal Cartan connection* $\omega \in \Omega^1(P, \mathfrak{g})$ is an admissible connection with a trace-free curvature κ_0.

The general obstruction to the existence of a normal Cartan connection is in certain cohomology group, we shall not discuss this point here, cf. [Ochiai, 70], [Baston, 90]. But we shall use the formula 3.10.(4) for the deformation of the curvature in order to compute explicitly the necessary deformation tensor Γ for a given admissible connection. It turns out that the result is uniquely determined by the initial data.

6.3. We have to use a coordinate notation for the values of Γ and κ_0 in order to handle the proper evaluations in the trace. So let e_i be the standard basis of the vector space $\mathfrak{g}_{-1}$, e^i the standard dual basis in $\mathfrak{g}^*_{-1}$ and e^i_j the standard basis of $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$. Note that the bases e_i and e^i are in fact dual with respect to the Killing form, up to a fixed scalar multiple. Then $\Gamma(u)(e_i) = \sum_j \Gamma_{ji}(u).e^j$, $\kappa_0(u)(e_i, e_j) = \sum_{k,l} K^k_{lij}(u).e^l_k$. In the sequel, we shall not always indicate explicitly sums over repeated indices. If we restrict the manipulations with these symbols to permutations of indices, contractions and similar invariant tensorial operations, our computations will be manifestly independent of the choice of the basis. In particular, the trace of κ_0 is expressed by the functions K^i_{lij}.

The brackets of the generators of $\mathfrak{so}(m, \mathbb{R})$, $m > 2$, are computed easily from the block-wise representation in 3.3:

$$[e_i, e^j] = e^j_i - e^i_j + \delta^j_i \mathbb{I}_m, \qquad [e^i_j, e_k] = \delta^i_k e_j$$

where $\mathbb{I}_m$ stands for the unit matrix. Now we evaluate the formula for the deformation $\bar\kappa_0 - \kappa_0 =: \delta(K^k_{lij})$ of the curvature caused by a choice of Γ, see 3.10.(4).

$$[\Gamma.e_j, e_i] - [\Gamma.e_i, e_j] = \textstyle\sum_p \Gamma_{pj}(-e^p_i + e^i_p - \delta^i_p \mathbb{I}_m) - \sum_p \Gamma_{pi}(-e^p_j + e^j_p - \delta^j_p \mathbb{I}_m)$$
$$= (-\Gamma_{kj}\delta^l_i + \Gamma_{lj}\delta^i_k - \Gamma_{ij}\delta^l_k + \Gamma_{ki}\delta^l_j - \Gamma_{li}\delta^j_k + \Gamma_{ji}\delta^l_k)e^k_l$$

Thus, the deformation of the trace achieved by Γ is

$$\delta(K^l_{klj}) = (m-3)\Gamma_{kj} + \Gamma_{jk} + \delta^k_j \textstyle\sum_i \Gamma_{ii}$$
$$\delta(K^k_{kij}) = m(\Gamma_{ji} - \Gamma_{ij})$$
$$\textstyle\sum_j \delta(K^i_{jij}) = 2(m-1)\sum_i \Gamma_{ii}.$$

We need the third 'contraction' for technical reasons.

Now, assume first we have two normal Cartan connections and let Γ be the corresponding deformation tensor. Since the torsion is zero, the Bianchi identity shows that for any normal Cartan connection, not only the trace defined in 6.2 but also the trace inside $\mathfrak{g}_0$ vanishes. Thus the resulting deformation of all three contractions above must be zero. So in particular, $\sum_i \Gamma_{ii} = 0$ and the functions Γ_{ij} are symmetric in i, j. But then the first equation yields $0 = (m-2)\Gamma_{lj}$. Thus if there is a normal Cartan connection, it is unique.

Let γ be the Riemannian connection of an arbitrary metric from the conformal class on M. Then it induces an admissible connection $\tilde\gamma$ on P, see 3.8. Moreover, the $\mathfrak{g}_0$-component of the curvature of the induced connection is just the pullback of the Riemannian curvature to P. Let us try to deform $\tilde\gamma$ by means of symmetric functions Γ_{ij}.

The deformation is expressed above in the form $\mathrm{Tr}(\kappa_0 - \bar\kappa_0)$, where $\bar\kappa_0$ is the 'new one'. Thus we have just to solve the above equations with respect to Γ with $\delta(K^k_{lij})$ replaced by $-R^k_{lij}$, the Riemannian curvature. We obtain easily

$$\Gamma_{ij} = \frac{-1}{m-2}\big(R_{ij} - \frac{\delta_{ij}}{2(m-1)}R\big), \tag{1}$$

where R_{ij} and R are the pullbacks of the Ricci tensor and the scalar curvature to P (expressed in the frame form, i.e. as functions on P). Let us notice that the above deformation tensor Γ is exactly the so called 'rho–tensor' used extensively in conformal geometry because of its 'beautiful transformation properties'.

Altogether we have reproved, even for conformal Spin structures:

6.4. Theorem. *Let M be a connected smooth manifold,* $\dim M \geq 3$, *with a conformal structure $P_0 \to M$. Then there is a unique normal Cartan connection ω on $P \to M$ which is expressed by means of any Riemannian connection γ from the conformal class by the formula $\omega = \tilde\gamma - \Gamma \circ \theta_{-1}$ with Γ defined by 6.3.(1).*

6.5. Operators on locally flat manifolds. Now we can apply the canonical normal Cartan connections in the construction from 5.7. In view of the next lemma, this procedure yields at least all natural operators 'visible' on the locally flat manifolds.

Let us fix two representations λ and μ of B_0 and let E_λ and E_μ be the corresponding natural bundles on the manifolds with the conformal (Spin) structures.

Further let us consider the locally flat structures $P \to P_0 \to M$. This means, we assume that there are (locally defined) connections in the distinguished class with vanishing curvature, or equivalently, $P \to M$ is locally isomorphic to the homogeneous space $G \to G/B$.

Lemma. *Suppose that the family of operators $D_M\colon C^\infty(E_\lambda M) \to C^\infty(E_\mu M)$ is a natural operator on the category of locally flat conformal (Spin) structures and let $\Pi \circ (\nabla^\gamma)^k$ be its expression in the (locally defined) flat connection γ in the distinguished class on M. Then the operator $\tilde{D} = \Pi \circ (\nabla^\omega)^k$ defined by means of the invariant differential with respect to the unique normal Cartan connection ω on $P \to M$ transforms B-equivariant functions into B-equivariant functions and equals to D_M.*

Proof. Since the operator D is natural, $D_M\colon C^\infty(PM, V_\lambda) \to C^\infty(PM, V_\mu)$ commutes with the induced action of the morphisms which is given by the composition with the inverses. On the other hand, $\tilde{D}$ commutes with these actions as well and since the structure in question is locally flat, the automorphisms of $P \to M$ act transitively. Thus, if we show that $\tilde{D}$ coincides on PM with D_M in one point of PM, then they must coincide globally. But if we choose a flat local connection γ and the corresponding (local) B_0-equivariant section $\sigma\colon P_0 \to P$, then the unique normal Cartan connection ω equals to the induced admissible Cartan connection $\tilde{\gamma}$, in particular the corresponding deformation tensor Γ is zero. Thus, according to the preceding section, the iterations of the invariant derivative with respect to ω and the pullbacks of the iterations of the covariant derivative with respect to γ coincide on $\sigma(P_0)$. In particular, the operator $\tilde{D}$ transforms sections into sections. □

6.6. Remark. By virtue of the general theory of natural operators on Riemannian manifolds, the naturality assumption in the previous lemma means just that the operator D is defined by a universal expression in terms of the underlying Riemannian connections in the conformal class, see e.g. [Kolář, Michor, Slovák, 93]. Thus our result shows that the 'conformally invariant operators' in the usual sense (see e.g. [Branson, 82], [Wünsch, 86], [Baston, Eastwood, 90]) are all obtained by our procedure, at least in the conformally flat case. Moreover, if we allow more general linear combinations of the iterated invariant differential (involving the iterated invariant differentials of the Weyl curvature in dimensions $m \geq 4$, or the invariant differentials of the Cotton-York tensor for $m - 3$), then we can achieve all the invariant operators mentioned above, cf. Remark 4.11.

Furthermore, the lemma is not restricted to linear operators, on the contrary, the same arguments apply if the expression for the operator D_M is a polynomial in the covariant derivatives.

6.7. Examples. To illustrate the use of the general formulae, let us consider now some special cases. As before, we shall restrict the attention here to the conformal case.

Consider an irreducible representation $\lambda\colon B_0 \to GL(V_\lambda)$ and let us write λ' for its restriction to the semisimple part of B_0. Each λ is given by λ' and the scalar action of the center. On the Lie algebra level this means $\lambda(\mathbb{I}_m)(v) = -w{\cdot}v$. The scalar w is called the *conformal weight* of λ.

According to the above discussion, if there is a $\mathfrak{g}_0$–homomorphism $\Phi : \otimes^k \mathfrak{g}^*_{-1} \otimes V_\lambda \to V_\rho$ onto an irreducible representation V_ρ such that $\Phi \circ (\nabla^\omega)^k$ is a natural operator then the formulae obtained in Section 4 yield its expression by means of a universal formula in terms of the underlying linear connections. Recall that we denoted by Γ the deformation tensor determined by a choice of a metric in the conformal class.

We shall look first at the second order operators. For each irreducible representation V_λ of B_0, the tensor product $\mathfrak{g}^*_{-1} \otimes V_\lambda$ decomposes uniquely into irreducible representations V_ρ (i.e. there are no multiplicities in the decomposition), see e.g. [Fegan, 76]. Let us write $\mathrm{Id} = \sum_\rho \pi^{\lambda\rho}$ for this decomposition.

Let V_λ be an irreducible representation of $\mathfrak{g}_0$ and let $\pi^{\lambda\rho\sigma}$ be a projection of $\otimes^2 \mathfrak{g}^*_{-1} \otimes V_\lambda$ onto an irreducible representation V_σ given by $\pi^{\lambda\rho\sigma}(Z_1 \otimes Z_2 \otimes s) = \pi^{\rho\sigma}[Z_1 \otimes (\pi^{\lambda\rho}(Z_2 \otimes s))]$. Lemma 5.9 gives a possibility to prove that $\pi^{\lambda\rho\sigma} \circ (\nabla^\omega)^2$ is a natural operator for certain choices of λ, ρ and σ, and Proposition 4.6 is saying that the natural operator can be written (using the underlying linear connections) as $\pi^{\lambda\rho\sigma}\{(\nabla^\gamma)^2 + \lambda([X, \Gamma.Y])\}s$. This is a universal formula valid for any dimension, any representation and any projection (even for the other structures, not only for the conformal one).

Choosing a specific representation, the formula can be simplified further. Let us consider now for simplicity the case of an even dimension $m = 2k$ and let $e^i, i = 1, \dots, m$ be weights of the representation $\mathfrak{g}_1$.

1. Let us discuss a simple example - second order operators acting on functions (having possibly a conformal weight). Hence let $\lambda' = 0$ be the heighest weight of the trivial representation $V_{\lambda'} = \mathbb{C}$ and w its conformal weight. The tensor product $\otimes^2 \mathfrak{g}^*_{-1} \otimes V_\lambda$ decomposes into three irreducible parts, namely $S^2_0(\mathfrak{g}^*_{-1})$ (symmetric traceless tensors), the trivial representation and $\Lambda^2(\mathfrak{g}^*_{-1})$. Let π_1, π_2, π_3 denote the corresponding projections.

We can use now the algebraic conditions discussed in 5.9. So ξ_α, resp. η_α are dual bases in $\mathfrak{g}_{-1}$, resp. $\mathfrak{g}_1$. We have to consider elements of the form

$$\sum_\alpha \{\eta_\alpha \otimes [Z, \xi_\alpha]\cdot(Y \otimes v) + Y \otimes \eta_\alpha \otimes [Z, \xi_\alpha]\cdot v\} =$$

$$\sum_\alpha \{\eta_\alpha \otimes ([[Z, \xi_\alpha], Y] \otimes v) + w\eta_\alpha \otimes Y \otimes \xi_\alpha(Z)v + wY \otimes \eta_\alpha \otimes \xi_\alpha(Z)v\}\,.$$

Using $[[Z, \xi_\alpha], Y] = -\langle Z, Y\rangle\eta_\alpha + \xi_\alpha(Z)Y + \xi_\alpha(Y)Z$ and $\sum_\alpha \xi_\alpha(Z)\eta_\alpha = Z$, we get

$$(w + 1)[Z \otimes Y \otimes v + Y \otimes Z \otimes v] - \langle Z, Y\rangle(\sum_\alpha \eta_\alpha \otimes \eta_\alpha \otimes v).$$

The traceless piece of the sum is the traceless part of the first summand, while the trace part of the sum is

$$(\frac{2}{m}(w + 1) - 1)\langle Z, Y\rangle(\sum_\alpha \eta_\alpha \otimes \eta_\alpha \otimes v).$$

Consequently, $\pi_1 \circ (\nabla^\omega)^2$ is an invariant operator for $w = -1$, $\pi_2 \circ (\nabla^\omega)^2$ is invariant for $w = \frac{m-2}{2}$ and $\pi_3 \circ (\nabla^\omega)^2$ is invariant for any value of w.

We can now compute the form of those three invariant operators.

2. Let $\lambda' = 0, w = -1$, let $\rho' = e^1, \sigma' = 2e^1$, so $\pi^{\lambda\rho\sigma} = \pi_1$. Note that $[e_k, e^i] = e^i_k - e^k_i + \delta^i_k \mathbb{I}_m$; the semisimple part of $\mathfrak{g}_0$ is acting trivially and

$$\lambda([X, \Gamma.Y])s = (-w)\langle \Gamma Y, X\rangle s.$$

Hence the invariant operator can be written as

$$\pi_1[(\nabla^\gamma_a \nabla^\gamma_b + \Gamma_{ab})s] = [\nabla^\gamma_{(a}\nabla^\gamma_{b)_0} + \Gamma_{(ab)_0}]s,$$

where the brackets indicate the symmetrization and the subscript 0 means the trace free part.

3. Let $\lambda' = \sigma' = 0, \rho' = e^1; w = \frac{m-2}{2}$ then $\lambda([X, \Gamma.Y])s = \frac{2-m}{2}\sum_{ij}\Gamma_{ij}X^jY^i$. The corresponding projection π_2 is here just the trace and we can express the operator $\pi_2 \circ (\nabla^\omega)^2$ in a more standard form using

$$\operatorname{Tr}\left\{\frac{2-m}{2}\left[\frac{-1}{m-2}\left(R_{ab} - \frac{\delta_{ab}}{2(m-1)}R\right)\right]\right\} = \frac{m-2}{4(m-1)}R,$$

where we used formula 6.3.(1). Hence we get the conformally invariant Laplace operator

$$\pi_2 \circ (\nabla^\omega)^2 = g^{ab}\nabla^\gamma_a\nabla^\gamma_b + \frac{m-2}{4(m-1)}R.$$

This is an example of a so called nonstandard operator.

4. Let $\lambda' = 0, \rho' = e^1, \sigma' = e^1 + e^2$. Then $\pi^{\lambda\rho\sigma}$ is the projection to $\Lambda^2(\mathfrak{g}^*_{-1}) \otimes V_0$; i.e. the antisymmetrization. The tensor Γ is symmetric, so

$$\pi_3 \circ (\nabla^\omega)^2 = \nabla^\gamma_{[a}\nabla^\gamma_{b]}.$$

Hence we have got a zero order operator in this case given by the action of the curvature. In this case, however, it is the trivial operator, due to the fact that the action of $\mathfrak{g}_0$ on V_0 is trivial. But it shows a possibility that for more complicated representations V_λ, (e.g. for one forms), we could get in such a way nontrivial zero order action by the curvature.

5. To have a more complicated example, let us consider a simple third order operator. Take $\lambda' = 0, \rho' = e^1, \sigma' = 2e^1$ and $\tau' = 3e^1$. The projection $\pi^{\lambda\rho\sigma\tau}$ is uniquely defined by iterated projections to factors having the corresponding highest weights. The projection $\pi^{\lambda\rho\sigma\tau}$ is the projection to the traceless part of the third symmetric power.

We can now repeate the computation described in Example 1. The projection $\pi^{\lambda\rho\sigma\tau}$ factorizes through the projection to $S^3(\mathfrak{g}^*_{-1}) \otimes V_\lambda$, hence the order of the factors is irrelevant, moreover $\pi^{\lambda\rho\sigma\tau}$ kills all trace terms. Hence all elements used in Lemma 5.9 have the form

$$3(w+2)Z \otimes Y_1 \otimes Y_2 \otimes v.$$

The choice $w = -2$ anihilates them all and for this value of conformal weight, we get a conformally invariant operator.

Proposition 4.8 describes the form of the correction terms. Due to the fact that action of the orthogonal group is trivial, we get for the first term

$$\lambda([X,(\nabla^\gamma_Z\Gamma).Y])s=2(\nabla^\gamma_{(a}\Gamma_{bc)_0})s.$$

The next two terms $\lambda([X,\Gamma.Y])(\nabla^\gamma_Z s)+\lambda([X,\Gamma.Z])(\nabla^\gamma_Y s)$ lead (due to symmetrization) to the term $4\Gamma_{(ab}\nabla^\gamma_{c)_0}s$. The last term

$$(\lambda^{(1)}([Y,\Gamma.Z])((\nabla^\gamma s)(X)$$

can be written as $\lambda([Y,\Gamma.Z])(\nabla^\gamma s)(X)+\nabla^\gamma[X,[Y,\Gamma.Z]]s$.

Using $[X,[Y,\Gamma.Z]]=\langle X,Y\rangle\Gamma.Z-\langle X,\Gamma.Z\rangle Y-\langle\Gamma.Z,Y\rangle X$, we see that the first term on the right hand side will disappear due to the projection to the traceless part and the other two will cancel the contribution coming from the previous term $\lambda(...)$. Hence we get the operator

$$\nabla^\gamma_{(a}\nabla^\gamma_b\nabla^\gamma_{c)_0}s+4\Gamma_{(ab}\nabla^\gamma_{c)_0}s+2(\nabla^\gamma_{(a}\Gamma_{bc)_0})s.$$

The examples shown above illustrate possibilities of our approach to construct and to compute the form of invariant operators. To make computation effective for a general representations, it is necessary to use appropriate Casimir operators. In the next part of the series, we shall use this approach to describe explicitely the broad family of the so called standard operators for all AHS structures.

References

Bailey, T. N.; Eastwood, *Complex paraconformal manifolds: their differential geometry and twistor theory*, Forum Math. **3** (1991), 61–103.

Bailey, T. N.; Eastwood, M. G.; Gover, A. R., *Thomas's structure bundle for conformal, projective and related structures*, Rocky Mountain J. **24** (1994), 1191–1217.

Bailey, T. N.; Eastwood, M. G.; Graham, C. R., *Invariant theory for conformal and CR geometry*, Annals of Math. **139** (1994), 491-552.

Baston, R. J., *Verma modules and differential conformal invariants*, J. Differential Geometry **32** (1990), 851–898.

Baston, R. J., *Almost Hermitian symmetric manifolds, I: Local twistor theory; II: Differential invariants*, Duke Math. J. **63** (1991), 81–111, 113–138.

Baston, R.J.; Eastwood, M.G., *Invariant operators*, Twistors in mathematics and physics, Lecture Notes in Mathematics 156, Cambridge University Press, 1990.

Boe, B. D.; Collingwood, D. H., *A comparison theory for the structure of induced representations I.*, J. of Algebra **94** (1985), 511-545.

Boe, B. D.; Collingwood, D. H., *A comparison theory for the structure of induced representations II.*, Math. Z. **190** (1985), 1-11.

Branson, T. P., *Conformally covariant equations on differential forms*, Communications in PDE **7** (1982), 392-431.

Branson, T. P., *Differential operators canonically associated to a conformal structure*, Math. Scand. **57** (1985), 293–345.

Branson T. P., *Second-order conformal covariants I., II.*, Kobenhavns universitet matematisk institut, Preprint Series, No. 2, 3, (1989).

Bureš, J., *Special invariant operators I.*, ESI Preprint **192** (1995), ??.

Čap, A., *Translation of natural operators on manifolds with AHS-structures*, in electronic form on www.emis.de, Arch. Math. (Brno) **32** (1996), 249–266.

Čap, A.; Slovák, J., *On local flatness of manifolds with AHS–structures*, to appear in Rendiconti Circ. Mat. Palermo, Proceedings of the Winter School Geometry and Physics, Srní 1995, pp. ??.

Cartan, É., *Les sous-groupes des groupes continus de transformations*, Ann. École Norm. Supp **25** (1908), 57–194.

Eastwood, M. G., *Notes on conformal differential geometry*, to appear in Rendiconti Circ. Mat. Palermo, Proceedings of the 15th Winter School on Geometry an Physics, Srní, 1995, pp. ??.

Eastwood, M. G.; Graham, C. R., *Invariants of conformal densities*, Duke Math. J. **63** (1991), 633–671.

Eastwood, M. G.; Rice, J. W., *Conformally invariant differential operators on Minkowski space and their curved analogues*, Commun. Math. Phys. **109** (1987), 207–228.

Eastwood, M. G.; Slovák, J., *Semi-Holonomic Verma Modules*, to appear in J. of Algebra.

Fegan, H. D., *Conformally invariant first order differential operators*, Quart. J. Math. **27** (1976), 371–378.

Goncharov, A. B., *Generalized conformal structures on manifolds*, Selecta Math. Soviet. **6** (1987), 308–340.

Gover, A. R., *Conformally invariant operators of standard type*, Quart. J. Math. **40** (1989), 197–208.

Graham, C.R.; Jenne, R.; Mason L.J.; Sparling, G.A., *Conformally invariant powers of the Laplacian, I: Existence*, J. London Math. Soc. **46** (1992), 557–565.

Graham, C. R., *Conformally invariant powers of the Laplacian, II: Nonexistence*, J. London Math. Soc. **46** (1992), 566–576.

Günther, P.; Wünsch, V., *Contributions to a theory of polynomial conformal tensors*, Math. Nachr. **126** (1986), 83-100.

Jakobsen, H. P., *Conformal invariants*, Publ. RIMS, Kyoto Univ **22** (1986), 345–364.

Jakobsen H. P.; Vergne, M., *Wave and Dirac operators, and representations of the conformal group*, Jour. of Funct. Anal. **24** (1977), 52–106.

Kobayashi, S., *Transformation groups in differential geometry*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

Kobayashi, S.; Nagano, T., *On filtered Lie algebras and geometric structures I*, J. Math. Mech. **13** (1964), 875–907.

Kobayashi, S.; Nagano, T., *On filtered Lie algebras and geometric structures II*, J. Math. Mech. **14** (1965), 513–521.

Kolář, I., *Generalized G–Structures and G–Structures of Higher Order*, Boll. Un. Mat. Ital. **4** (12) Supp. fasc. 3 (1975), 245–256.

Kolář, I.; Michor, P. W.; Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin Heidelberg New York, 1993, pp. 437.

Ochiai, T., *Geometry associated with semisimple flat homogeneous spaces*, Trans. Amer. Math. Soc. **152** (1970), 159–193.

Ørsted, B., *Conformally invariant differential equations and projective geometry*, J. Funct. Anal. **44** (1981), 1–23.

Slovák, J., *On the geometry of almost Hermitian symmetric structures*, Proceedings of the Conference Differential Geometry and Applications, Brno, 1995, in electronic form on www.emis.de, Masaryk University in Brno, 1996, pp. 191–206.

Tanaka, N., *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.

Wünsch, V., *On conformally invariant differential operators*, Math. Nachr. **129** (1986), 269–281.

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