INVARIANT OPERATORS ON MANIFOLDS WITH ALMOST HERMITIAN SYMMETRIC STRUCTURES, II. NORMAL CARTAN CONNECTIONS

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Abstract. We construct explicitly the canonical principal B-bundles P and their canonical Cartan connections for all AHS-structures. Our methods are different from the development in [Tanaka, 79] or [Baston, 91], in particular they are simpler, more explicit and transparent. We also compute explicite formulae for the canonical Cartan connections in terms of the underlying distinguished linear connections.

In the first part of this series, [Čap, Slovák, Souček], we defined almost Hermitian symmetric structures as 'second order structures'. Now, we will first show that any first order structure with the 'right' structure group gives rise to an almost Hermitian symmetric structure in this sense. Basically, the construction is just the standard first prolongation of G–structures, see [Kobayashi] or [Sternberg]. Due to the special situation, there is a canonical prolongation which admits the structure of a principal bundle with the structure group B. Moreover, it turns out that for all almost Hermitian structures, there exists a unique normal Cartan connection. We shall present the explicit construction in section 2. Thus, the calculus developed in Part I of this series will yield natural operators in all these cases. Furthermore, we compute explicitly the corresponding deformation tensors Γ in the last section.

Our results extend those by [Ochiai, 70], but his methods using the vanishing torsion assumption restrict in fact the considerations to the locally flat structures in many cases, cf. [Baston, 91] or [Čap, Slovák, 95]. The general Tanaka's development certainly covers our existence result, however his approach and aims are so far different from ours, that we believe to need much more space to refer to [Tanaka, 79] than to present our simple independent proof. Another approach to the construction of the canonical Cartan connections on certain auxiliary vector bundles, thus avoiding the construction of the prolongation, can be found in [Baston, 91].

In the sequel, we shall use the notation and results from Part I of this series of papers, the citations like I.2.3 mean the corresponding items in that part.

1. The prolongation of first order structures

1.1. Let us recall the setting we work in from the first part of this series of papers: We start from a connected semisimple real Lie group G whose Lie algebra $\mathfrak{g}$ is equipped with a grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. By B we denote the closed (parabolic)

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subgroup corresponding to the Lie subalgebra $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, further we have the closed subgroup $B_0 \subset B$ corresponding to $\mathfrak{g}_0$ and the closed normal subgroup B_1 of B corresponding to $\mathfrak{g}_1$. Then it is known that

(1) $\mathfrak{g}_0$ is reductive with one–dimensional center
(2) the map $\mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_{-1})$ induced by the adjoint representation is the inclusion of a subalgebra and an irreducible representation
(3) the Killing form identifies $\mathfrak{g}_1$ as a $\mathfrak{g}_0$ module with the dual of $\mathfrak{g}_{-1}$.
(4) the restrictions of the exponential map to $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are diffeomorphisms onto the corresponding closed subgroups of G.
(5) $B_0 \cap B_1 = \{e\}$
(6) B is the semidirect product of B_0 and B_1,

see [Ochiai, Sections 3 and 6].

Examples of such Lie algebras and the corresponding structures can be found in I.3.3. In particular recall that there are the classical projective structures, which occur in this picture as the extremal case of an almost Grassmannian structure. The projective structures behave rather exceptionally and we will have to treat them separately.

1.2. Our starting point is a first order B_0–structure on a smooth manifold M of dimension $m = \dim(\mathfrak{g}_{-1})$, so assume that we have given a principal B_0 bundle $P_0 \to M$ together with a *soldering form* $\theta_{-1} \in \Omega^1(P_0, \mathfrak{g}_{-1})$ which is strictly horizontal, i.e. its kernel in each tangent space is precisely the vertical tangent space, and B_0 equivariant, so $(r^b)^*\theta_{-1} = \mathrm{Ad}(b^{-1}) \circ \theta_{-1}$. This is equivalent to P_0 being a reduction of the (first order) frame bundle P^1M of M, cf. I.3.6. Now consider the tangent bundle TP_0, the vertical subbundle VP_0 and the quotient bundle TP_0/VP_0. The fundamental vector field map gives a trivialization $VP_0 \simeq P_0 \times \mathfrak{g}_0$, while the soldering form induces a trivialization $TP_0/VP_0 \simeq P_0 \times \mathfrak{g}_{-1}$.

For a point $u \in P_0$ consider a linear isomorphism $\varphi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_uP_0$ which is compatible with the two trivializations from above, i.e. such that $\varphi(0, A) = \zeta_A(u)$ and $\theta_{-1}(u)(\varphi(X, A)) = X$. Via φ the exterior derivative $d\theta_{-1}(u)$ gives rise to a mapping $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$, defined by $(X, Y) \mapsto d\theta_{-1}(u)(\varphi(X, 0), \varphi(Y, 0))$, and we view this mapping as $t_\varphi \in \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$, and call it the *torsion* of φ. Now let $\bar\varphi$ be another isomorphism compatible with the trivializations. Then there is a linear map $\psi : \mathfrak{g}_{-1} \to \mathfrak{g}_0$ such that $\bar\varphi(X, A) - \varphi(X, A) = \zeta_{\psi(X)}(u)$. The difference between the corresponding maps constructed using $d\theta_{-1}(u)$ can be easily computed:

Lemma. *In this situation we have:*

$$d\theta_{-1}(u)(\bar\varphi(X, 0), \bar\varphi(Y, 0)) - d\theta_{-1}(u)(\varphi(X, 0), \varphi(Y, 0)) = -[\psi(X), Y] + [\psi(Y), X].$$

Proof. Using bilinearity of $d\theta_{-1}(u)$ and the fact the $\bar\varphi(X, A) = \varphi(X, A) + \zeta_{\psi(X)}(u)$ the difference can be expressed as

$$d\theta_{-1}(u)(\zeta_{\psi(X)}, \varphi(Y, 0)) + d\theta_{-1}(u)(\varphi(X, 0), \zeta_{\psi(Y)}) + d\theta_{-1}(u)(\zeta_{\psi(X)}, \zeta_{\psi(Y)}).$$

Since θ_{-1} is horizontal and the Lie bracket of vertical vector fields is vertical, the last term vanishes. On the other hand, the infinitesimal version of the B_0–equivariancy of θ_{-1} is clearly $\mathcal{L}_{\zeta_A}\theta_{-1} = -\mathrm{ad}(A)\circ\theta_{-1}$, and again by horizontality this reduces to $i_{\zeta_A} d\theta_{-1} = -\mathrm{ad}(A)\circ\theta_{-1}$. Applying this we see that the first term from above reduces to $-[\psi(X), \theta_{-1}(\varphi(Y,0))] = -[\psi(X), Y]$ and similarly for the second term. □

There is a canonical map ∂ from $L(\mathfrak{g}_{-1}, \mathfrak{g}_0) \simeq \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ to $\mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$, the composition of the alternation in the first two factors with the map induced by the inclusion $\mathfrak{g}_0 \to \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$ obtained from 1.1.(2). Using this map, the lemma above just says that

$$d\theta_{-1}(u)(\bar\varphi(X,0), \bar\varphi(Y,0)) - d\theta_{-1}(u)(\varphi(X,0), \varphi(Y,0)) = -\partial(\psi)(X,Y).$$

Thus the above construction gives rise to a well defined function

$$P_0 \to (\mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1})/\partial(\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0),$$

called the *structure function* of the B_0–structure.

1.3. The map $\partial : \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 \to \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$ from above is the differential in the Spencer cohomology, the cohomology of the abelian Lie algebra $\mathfrak{g}_{-1}$ with values in the representation $\mathfrak{g}$. It is a crucial fact for the computation of this cohomology that there is an adjoint $\partial^* : \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1} \to \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$, defined by $(\partial^*\varphi)(X) = \sum_i [Z^i, \varphi(X_i, X)]$, where $\{X_i\}$ is a basis of $\mathfrak{g}_{-1}$ and Z^i is the dual basis of $\mathfrak{g}_1$, see 1.1.(3). It turns out that there is an inner product on $\mathfrak{g}$ such that ∂^* is the adjoint of ∂, see [Ochiai, Proposition 4.2]. Thus the kernel $\mathrm{Ker}(\partial^*)$ is a complementary subspace to the image of ∂.

Note that all spaces occurring in the above considerations are in fact $\mathfrak{g}_0$–modules. It is easy to verify that both ∂ and ∂^* are in fact homomorphisms of $\mathfrak{g}_0$–modules. In particular, this implies that $\mathrm{Ker}(\partial^*)$ is even a complementary $\mathfrak{g}_0$–module to the image of ∂. This will be crucial in the sequel.

1.4. Now we define P to be the set of all linear isomorphisms $\varphi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_u P_0$ as in 1.2 such that $\partial^*(t_\varphi) = 0$. It is easy to see that for each $u \in P_0$ such φ actually exist as follows: Take any φ satisfying the conditions of 1.2. Then, as $\mathrm{Ker}(\partial^*)$ is complementary to $\mathrm{Im}(\partial)$, there is a linear map $\psi \in \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0$ such that $\partial^*\big(d\theta_{-1}(u)(\varphi(X,0), \varphi(Y,0)) + (\partial\psi)(X,Y)\big) = 0$. (In fact the image of ψ under ∂ is uniquely determined.) Then one immediately verifies that $\bar\varphi(X,A) := \varphi(X,A) + \zeta_{\psi(X)}(u)$ satisfies the condition.

Next take an element $b \in B$. Viewing b as an element of G we have the adjoint action $\mathrm{Ad}(b) : \mathfrak{g} \to \mathfrak{g}$, and since $\mathrm{Ad}(\exp(Z)){\cdot}X = X + [Z,X] + 1/2[Z,[Z,X]] + \dots$ (cf. I.3.8), we see that $\mathfrak{g}_1$ is stable under this adjoint action, so we get an induced linear automorphism $\mathrm{Ad}(b)$ of the space $\mathfrak{g}/\mathfrak{g}_1 \simeq \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$.

For $b \in B$ denote by b_0 the class of b in $B/B_1 \simeq B_0$. Then for an element $\varphi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_u P_0$ of P we define $\varphi{\cdot}b : \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_{u\cdot b_0} P_0$ by $\varphi{\cdot}b := Tr^{b_0} \circ \varphi \circ \mathrm{Ad}(b)$, where r^{b_0} denotes the principal right action of b_0 on P_0.

1.5. Proposition. *This defines a free right action of B on P. In each case except the one of projective structures this action is also transitive on each fiber of the obvious projection $P \to M$.*

Proof. Let us first verify that $\varphi{\cdot}b$ is again in P. So we have to compute

$$d\theta_{-1}(u{\cdot}b_0)((\varphi{\cdot}b)(X,0), (\varphi{\cdot}b)(Y,0)),$$

for elements $X, Y \in \mathfrak{g}_{-1}$. By B_0–equivariancy of θ_{-1} this equals

$$\mathrm{Ad}(b_0^{-1})(d\theta_{-1}(u)(\varphi(\mathrm{Ad}(b)\cdot(X,0)), \varphi(\mathrm{Ad}(b)\cdot(Y,0)))).$$

Now we may write $b = b_0 b_1$ for some $b_1 \in B_1$ and by 1.1.(4) there is a $Z \in \mathfrak{g}_1$ such that $b_1 = \exp(Z)$. Using the formula for the adjoint action of an exponential from above we see that

$$\mathrm{Ad}(b)\cdot(X,0) = \mathrm{Ad}(b_0)\cdot\mathrm{Ad}(\exp(Z))(X,0) = (\mathrm{Ad}(b_0)\cdot X, \mathrm{Ad}(b_0)\cdot[Z,X]),$$

and thus $\varphi(\mathrm{Ad}(b)\cdot(X,0)) = \varphi(\mathrm{Ad}(b_0)\cdot X, 0) + \zeta_{\mathrm{Ad}(b_0)\cdot[Z,X]}(u)$. The same computation as in the proof of lemma 1.2 then shows that

$$\begin{aligned} &d\theta_{-1}(u)(\varphi(\mathrm{Ad}(b)\cdot(X,0)), \varphi(\mathrm{Ad}(b)\cdot(Y,0))) = \\ &\qquad = d\theta_{-1}(u)(\varphi(\mathrm{Ad}(b_0)\cdot X,0), \varphi(\mathrm{Ad}(b_0)\cdot Y,0)) + \\ &\qquad\quad \mathrm{Ad}(b_0)\cdot([[Z,X],Y] - [[Z,Y],X]) \\ &\qquad = d\theta_{-1}(u)(\varphi(\mathrm{Ad}(b_0)\cdot X,0), \varphi(\mathrm{Ad}(b_0)\cdot Y,0)). \end{aligned}$$

This shows that $t_{\varphi\cdot b} = b_0\cdot t_\varphi$, so $\partial^*(t_{\varphi\cdot b}) = b_0\cdot\partial^*(t_\varphi) = 0$, and hence $\varphi\cdot b \in P$.

Next, let us assume that $\varphi\cdot b = \varphi$ for some $\varphi \in P$ and $b \in B$. Then obviously $b \in B_1$, since B_0 acts freely on P_0. So as before we may write $b = \exp(Z)$. But then $\varphi\cdot b = \varphi$ implies that $[Z,X] = 0$ for all $X \in \mathfrak{g}_{-1}$, which implies $Z = 0$ by 1.1.(3).

Finally, to prove transitivity of the action it suffices to show that B_1 acts transitive on each fiber of $P \to P_0$, since B_0 acts transitive on each fiber of $P_0 \to M$. But for two maps $\varphi, \bar\varphi$ in the same fiber we see from 1.2 that $\bar\varphi(X,A) = \varphi(X,A) + \zeta_{\psi(X)}(u)$ for some $\psi \in \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$, and lemma 1.2 shows that if both maps are in P we must have $\partial(\psi) = 0$. But now in all cases except the projective one the corresponding Spencer cohomology group $H^{1,1}(\mathfrak{g})$ is trivial, so there is a $Z \in \mathfrak{g}_1$ such that $\psi = \mathrm{ad}_Z$, see [Ochiai, Proposition 7.3]. Thus $\bar\varphi = \varphi\cdot\exp(Z)$. $\square$

1.6. The soldering form. From now on we exclude the projective case which we will discuss separately later. So $P \to M$ is a principal B–bundle, and the proof of 1.5 also shows that $p : P \to P_0$ is a principal B_1–bundle. Now we define on P a one–form θ with values in $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ as follows: For a point $\varphi \in P$ consider a tangent vector $\xi \in T_\varphi P$. Then $Tp\cdot\xi$ is a tangent vector in $T_{p(\varphi)}P_0$ and by definition φ is a linear isomorphism from $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ to this tangent space, so we may define $\theta(\xi) := \varphi^{-1}(Tp\cdot\xi)$. This form is called the *soldering form* or *displacement form* on P. The *torsion* T of θ is defined by the structure equation

$$d\theta_{-1} = -[\theta_0, \theta_{-1}] + T.$$

Lemma. *The one form θ has the following properties:*

(1) *the component θ_{-1} is the pullback of the form from 1.2.*
(2) *$\theta_0(\zeta_{Y+Z}) = Y$ for all $Y \in \mathfrak{g}_0$, $Z \in \mathfrak{g}_1$.*
(3) *θ is B–equivariant, i.e. $(r^b)^*\theta = \mathrm{Ad}(b^{-1}) \circ \theta$, where* Ad *is the action from* 1.4
(4) *The torsion T is horizontal over M and can be viewed as a function in $C^\infty(P, \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1})$. Moreover, $\partial^* \circ T = 0$.*

In particular (P, θ) is a B–structure on M in the sense of I.3.4.

Proof. (1) is clear since $\theta_{-1}(\varphi(X, A)) = X$. For (2) note that ζ_Z lies in the kernel of Tp, while ζ_Y is mapped by Tp to the fundamental vector field on P_0 corresponding to Y. Next, (3) follows immediately from the definition of the B–action on P, and the fact that $p \circ r^b = r^{b_0} \circ p$. Finally, i_{ζ_X} with $X \in \mathfrak{g}_1$, applied to any of the terms in the structure equation yields zero, while for $X \in \mathfrak{g}_0$ we obtain

$$i_{\zeta_X}(d\theta_{-1} + [\theta_0, \theta_{-1}]) = \mathcal{L}_{\zeta_X}\theta_{-1} + [i_{\zeta_X}\theta_0, \theta_{-1}] = 0$$

by the equivariancy of θ_{-1}. Now, we can define $T(X, Y)(u)$ by evaluating the structure equation on arbitrary vectors ξ, $\eta \in T_uP$ such that $\theta_{-1}(\xi) = X$ and $\theta_{-1}(\eta) = Y$. It remains to prove $\partial^* \circ T = 0$ which can be done pointwise. So take $\varphi \in P$ and choose $\xi, \eta \in T_\varphi P$ so that $Tp.\xi = \varphi(X, 0)$ and $Tp.\eta = \varphi(Y, 0)$. Then $\theta_0(\xi) = \theta_0(\eta) = 0$ by the construction. Using (1), we see that

$$d\theta_{-1}(\xi, \eta) + [\theta_0, \theta_{-1}](\xi, \eta) = t_\varphi(X, Y) + 0. \quad \square$$

1.7. Consider a principal connection γ on P_0. Then at each point $u \in P_0$ we get an isomorphism $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_uP_0$ as in 1.2, defined by the soldering form θ_{-1} and the connection form of γ. Thus we have the torsion $t_\gamma : P_0 \to \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$, which is in fact the frame form of the usual torsion of γ.

The connection γ is called *harmonic* if $\partial^* \circ t_\gamma = 0$. (The name harmonic is due to the fact that the Spencer coboundary operator ∂ is trivial on the space in question so that our condition is equivalent to harmonicity of the torsion.)

Proposition. *There is a B_0–equivariant section $\sigma : P_0 \to P$, and the space of all such sections is in bijective correspondence with the space of all harmonic principal connections on P_0. Moreover, it is an affine space modeled on $\Omega^1(M)$, the space of one–forms on M.*

Proof. We have already shown in I.3.6 that a global B_0-equivariant section σ always exists, but now we shall supply another simple (and more geometric) argument.

Note first that any principal connection γ on P_0 splits the exact sequence

$$0 \to VP_0 \to TP_0 \to TP_0/VP_0 \to 0$$

and thus gives rise to a linear isomorphism $\varphi_u \colon \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_uP_0$, which satisfies the conditions of 1.2, in each point u.

Further, let us choose a B_0-module homomorphism ψ which is a right inverse of $\partial \colon \mathfrak{g}_{-1} \otimes \mathfrak{g}_0 \to \mathrm{Im}(\partial) \subset \mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$. Starting from a chosen principal connection γ, let f be the $\mathrm{Im}(\partial)$–part of the torsion t_γ, and consider the smooth map $u \mapsto \varphi_u + \zeta_{\psi(f(u))}(u)$. By the construction, this has values in P, since by Lemma 1.2 $t_{\varphi+\zeta_{\psi\circ f}} = t_\varphi - \partial \circ \psi \circ f = t_\varphi - f$. Due to the equivariancy of ψ, this defines a B_0–equivariant section of $P \to P_0$. If the original connection γ was harmonic, then $f = 0$ and the mapping $u \mapsto \varphi_u$ itself is a B_0-equivariant section.

Any B_0-equivariant section $\sigma \colon P_0 \to P$ can clearly be interpreted as a principal connection γ on P_0. For each point $u \in P_0$ and $\xi \in T_uP_0$, we have

$$(\sigma^*\theta)(u)(\xi) = \theta(\sigma(u))(T\sigma.\xi) = \sigma(u)^{-1}(\xi) \in \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$$

and the $\mathfrak{g}_0$–part of this expression is just the connection form of the connection γ. Applying σ^* to the structure equation from 1.6.(4) we obtain (using 1.6.(1))

$$d\theta_{-1} = -[\gamma, \theta_{-1}] + \sigma^* T,$$

so that $\sigma^* T$ is the torsion of the principal connection γ. Thus γ is a harmonic connection.

Finally, if σ and $\bar{\sigma}$ are two B_0-equivariant sections of $P \to P_0$, then there is a unique smooth map $\tau \colon P_0 \to \mathfrak{g}_1$ such that $\bar{\sigma}(u) = \sigma(u).\exp\tau(u)$. Since the sections are B_0-equivariant, we obtain $\tau(u.b_0) = \mathrm{Ad}(b_0^{-1}).\tau(u)$, so that τ is a frame form of a one-form on M. $\square$

1.8. The bundle P can be viewed as a subbundle of the frame bundle $P^1 P_0$ of P_0. In fact, a point $\varphi \in P$ is by definition an isomorphism $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \to T_{p(\varphi)} P_0$. Moreover, taking into account that P_0 is a reduction of $P^1 M$ to the group B_0, we can view P as a reduction of $P^1(P^1 M)$ to the group B. In fact, it can be shown that this reduction has values in the second order frame bundle $P^2 M$ of M, if and only if the torsion of θ vanishes, cf. [Slovák, 94], but we will not pursue this point of view.

1.9. The projective case. In this case the underlying first order structure is the whole $P^1 M$, so it carries no information. Thus to get a B–structure in the sense of I.3.4 with harmonic torsion, one has to choose a reduction of the second order frame bundle $P^2 M$ to the appropriate group B. (Note that in this case $\partial : \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \to \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$ is surjective, so the harmonic connections are exactly the torsion free ones.)

2. Canonical Cartan connections

Our next task is to prove that in all but the very low dimensional cases, on all prolongations as constructed in the previous section, there is a canonical Cartan connection. Basically, this is a consequence of the fact that in these cases the next prolongation is trivial, so its soldering form is a Cartan connection.

2.1. Assume we have constructed the B–bundle $P \to M$ with the soldering form $\theta = \theta_{-1} \oplus \theta_0$ for a B_0–bundle $P_0 \to M$ as above. As we have seen in 1.8 this is in fact a B_1 structure on P_0, so we can try to apply the same construction as above to this structure using the additional information we have in this case.

The starting point is to consider for $\varphi \in P$ linear isomorphisms

$$\Phi : \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \to T_\varphi P$$

such that $\theta(\Phi(X, A, Z)) = (X, A)$ and $\Phi(0, A, Z) = \zeta_{A+Z}(\varphi)$ (here we use the finer structure and do not only fix $\Phi(0, 0, Z)$). Having given such a Φ we have to consider its torsion

$$\begin{gathered} t_\Phi \in (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)^* \wedge (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)^* \otimes (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0) \\ ((X, A), (Y, B)) \mapsto d\theta(\varphi)(\Phi(X, A, 0), \Phi(Y, B, 0)). \end{gathered}$$

In fact, several parts of this mapping are independent of Φ. For later use we prove a slightly more general result than we need here:

Lemma. *Let $pr : \mathfrak{g} \to \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$ be the obvious projection. Then for all (X, A, Z), $(Y, B, W) \in \mathfrak{g}$ we have*

$$d\theta(\Phi(X, A, Z), \Phi(Y, B, W)) = d\theta(\Phi(X, 0, 0), \Phi(Y, 0, 0)) - pr([X+A+Z, Y+B+W]).$$

Proof. The infinitesimal version of the equivariancy of θ gives

$$i_{\zeta_{A+Z}} d\theta = -\mathrm{ad}(A+Z) \circ \theta,$$

where ad is the composition of *pr* with the adjoint action on $\mathfrak{g}$. Now the result follows easily using bilinearity of $d\theta$ and the fact that $\Phi(X, A, Z) = \Phi(X, 0, 0) + \zeta_{A+Z}(\varphi)$. □

Consequently, the torsion of Φ is determined by its component in $\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$.

2.2. The next step is to compute the change of the torsion if one replaces Φ by another isomorphism satisfying the above conditions. As in the proof of 1.2 one verifies that in fact the change lies in the image of $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1$ under the composition of the alternation with the map induced by the inclusion

$$\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1 \to \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0 \subset (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)^* \otimes (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)^* \otimes (\mathfrak{g}_{-1} \oplus \mathfrak{g}_0).$$

To get a well defined structure function as in section 1 we have to factor the latter space by the image of $\partial : \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1 \to \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$. As before this is the differential in the Spencer cohomology and it has an adjoint ∂^* defined by the same formula as in 1.3.

2.3. Theorem. *In all cases but the one of $\mathfrak{g} = \mathfrak{sl}(2)$, for each $\varphi \in P$ there is a unique linear isomorphism Φ as in 2.1 such that $\partial^* \circ t_\Phi = 0$. The inverses of these can be viewed as a smooth one form $\omega \in \Omega^1(P, \mathfrak{g})$ with the following properties:*

(1) $\omega(\zeta_X) = X$ *for all* $X \in \mathfrak{b}$
(2) $(r^b)^*\omega = Ad(b^{-1}) \circ \omega$ *for all* $b \in B$

Proof. First, since the kernel of ∂^* is a complement to the image of ∂, we can construct such a Φ in a point φ like in 1.4. Moreover, it is clear that the set of all such Φ is parameterized by the kernel of $\partial : \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1 \to \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_0$. This coincides with the Spencer cohomology group $H^{2,1}(\mathfrak{g})$ which is trivial for all cases in question, see [Ochiai, Proposition 7.1], so Φ is unique.

Let us verify the properties of ω. For $A \in \mathfrak{g}_0$ and $Z \in \mathfrak{g}_1$ we have $\Phi(0, A, Z) = \zeta_{A+Z}(\varphi)$, so ω reproduces the generators of fundamental vector fields. Finally, we have to verify the equivariancy of ω. Put $\Phi = \omega(\varphi)^{-1} : \mathfrak{g} \to T_\varphi P$ and consider $\Phi{\cdot}b := Tr^b \circ \Phi \circ \mathrm{Ad}(b) : \mathfrak{g} \to T_{\varphi \cdot b}P$ for $b \in B$. If we verify that $\Phi{\cdot}b$ satisfies the conditions of 2.1 and that $\partial^* \circ t_{\Phi \cdot b} = 0$, then the uniqueness proved above concludes the proof.

For $A \in \mathfrak{g}_0$ and $Z \in \mathfrak{g}_1$ we have

$$(\Phi{\cdot}b)(0, A, Z) = Tr^b \zeta_{\mathrm{Ad}(b)\cdot(0,A,Z)}(\varphi) = \zeta_{A+Z}(\varphi{\cdot}b).$$

Further, $\theta(\varphi{\cdot}b)((\Phi{\cdot}b)(X, A, Z)) = \mathrm{Ad}(b^{-1})\theta(\varphi)(\Phi(\mathrm{Ad}(b)\cdot(X, A, Z))) = (X, A)$, since $\mathrm{Ad}(b)\cdot(X, A)$ is by definition just the first two components of $\mathrm{Ad}(b)\cdot(X, A, Z)$.

It remains to check the condition on the torsion. For $b \in B$, we write $b = b_0\exp(W)$ (see 1.5). Using the equivariancy of θ and lemma 2.1 we compute:

$$\begin{aligned}
&d\theta(\varphi{\cdot}b)(Tr^b\Phi(\mathrm{Ad}(b){\cdot}X), Tr^b\Phi(\mathrm{Ad}(b){\cdot}Y)) = \\
&\quad = \mathrm{Ad}(b^{-1})(d\theta(\varphi)(\Phi(\mathrm{Ad}(b){\cdot}X), \Phi(\mathrm{Ad}(b){\cdot}Y))) \\
&\quad = \mathrm{Ad}(b^{-1})(d\theta(\varphi)(\Phi(\mathrm{Ad}(b_0){\cdot}X), \Phi(\mathrm{Ad}(b_0){\cdot}Y))) + pr([\mathrm{Ad}(b){\cdot}X, \mathrm{Ad}(b){\cdot}Y])
\end{aligned}$$

The second term in this expression vanishes since $\mathrm{Ad}(b)$ is a Lie algebra homomorphism, so for the $\mathfrak{g}_0$–component we get

$$\begin{aligned}
&\mathrm{Ad}(b_0^{-1})(d\theta_0(\varphi)(\Phi(\mathrm{Ad}(b_0){\cdot}X), \Phi(\mathrm{Ad}(b_0){\cdot}Y)))+ \\
&\qquad\qquad + \mathrm{Ad}(b_0^{-1})([W, d\theta_{-1}(\varphi)(\Phi(\mathrm{Ad}(b_0){\cdot}X), \Phi(\mathrm{Ad}(b_0){\cdot}Y))]).
\end{aligned}$$

The first term lies in the kernel of ∂^* since this is a B_0–submodule. The second one lies in this kernel since by definition of ∂^* we have $\partial^* \circ \mathrm{ad}(W) = \mathrm{ad}(W) \circ \partial^*$ (cf. 1.3). $\square$

Since the restriction of the one form ω to any tangent space $T_\varphi P$ is an isomorphism, ω is a Cartan connection on P, see the definition in I.2.1. Moreover, the first condition put on Φ in 2.1 implies that the $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$–part of ω coincides with θ. Thus, ω is an admissible Cartan connection in the sense of I.3.9.

Let us remark that another approach to the construction of canonical prolongations equipped with canonical Cartan connections can be found in [Alekseevsky, Michor, 95].

2.4. Let us return to the point of view of G–structures and compute the structure function of the last prolongation. Clearly this is induced by

$$(X, A, Z), (Y, B, W) \mapsto d\omega(\omega^{-1}(X, A, Z), \omega^{-1}(Y, B, W)).$$

Using the B–equivariancy of ω one proves precisely as in lemma 2.1 that

$$\begin{aligned}
&d\omega(\omega^{-1}(X, A, Z), \omega^{-1}(Y, B, W)) = \\
&\qquad\qquad = d\omega(\omega^{-1}(X, 0, 0), \omega^{-1}(Y, 0, 0)) + [X + A + Z, Y + B + W].
\end{aligned}$$

By definition the first term of the right hand side is just the curvature $\kappa(X, Y)$, see I.2.1. Thus, if this curvature vanishes then the structure function is constant (and equal to the Lie bracket, viewed as an element of $(\mathfrak{g}^* \wedge \mathfrak{g}^*) \otimes \mathfrak{g}$), independent of the manifold under consideration.

In this situation, taking into account that the components ω_{-1} and ω_0 coincide with the respective components of θ, we see from 2.1 that also the next "lower" structure function is constant and independent of the manifold. Similarly, one shows that the same is true for the first structure function constructed in 1.2 .

In the flat case $M = G/B$ the cananonical Cartan connection is just the Maurer Cartan form, and the Maurer Cartan equation means just that $\kappa = 0$ in this case. Thus we see that a B_0–structure $P_0 \to M$ has the structure functions of all prolongations constant and equal to those of the flat model if and only if the curvature of the canonical Cartan connection vanishes. From [Sternberg, p. 339] we conclude:

Proposition. *$P_0 \to M$ is locally isomorphic to the flat model if and only if the canonical Cartan connection has zero curvature.*

2.5. Using the properties of Cartan connections derived in [Čap, Slovák, Souček] it is quite easy to compute explicitely the obstructions against flatness of the canonical Cartan connection in terms of any of the underlying linear connections. The main step is to understand the link of the second cohomology $H^2(\mathfrak{g}_{-1}, \mathfrak{g})$ to the curvature. Then it is easy to determine, which parts of the curvature are the true obstructions, and which vanish automatically. Essentially, this can be found implicitly also in [Tanaka, 79], and quite explicitly in [Baston, 91] for the four main series of complex simple groups. This is also worked out in our approach in [Čap, Slovák], using the results on cohomologies listed in [Baston, 91]. However, an easy computation using the Kostant's version of the Bott-Borel-Weil theorem yields that the cohomology is sitting only in the torsion part for the two exceptional simple groups. Thus the only obstructions are the torsions in both these cases.

3. Explicit formulae for the canonical Cartan connections

Let us consider a B-structure $P \to P_0 \to M$, its soldering form θ with a harmonic torsion T, and the canonically defined Cartan connection ω on P, as constructed in section 2. Note that the canonical Cartan connection is characterized by the fact the the component κ_0 of its curvature is in the kernel of ∂^*.

For each global B_0–equivariant section $\sigma\colon P_0 \to P$ there is the principal connection $\sigma^*\theta_0$ on P_0, the induced admissible Cartan connection $\tilde{\gamma}$ on P, and the difference between the canonical Cartan connection and the latter one is described by the so called deformation tensor Γ, see I.3.9. In this section, we shall compute explicitly the deformation of a chosen induced admissible Cartan connection which leads to the canonical one. It turns out, that for each of the structures in question, there is a universal formula for Γ in terms of the curvature tensor of the chosen underlying connection γ.

Since the computations are quite elementary and in fact an explicit use of the general result from section 2 does not spare much work, we prefer to recover completely also the existence and uniqueness of the canonical Cartan connection in this way. Thus a part of the next considerations will be redundant, but on the other hand, this will also provide the link to the traditional concept of the normal Cartan connection, see e.g. [Kobayashi, 72].

3.1. The trace of the curvature. Let ω be an admissible Cartan connection on a B-structure $P \to M$, i.e. $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$. Let us recall the definition of the trace of the $\mathfrak{g}_0$-component κ_0 of the curvature function κ of ω. We can view the values of κ_0 as elements in $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1} = \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{1}$.

There are three possible evaluations in the target space. The evaluation over the last two entries is just the trace in $\mathfrak{g}_0$, the other two possibilities coincide up to a sign. By definition, the *trace* $\operatorname{Tr}\kappa_0$ of the curvature function κ_0 is the evaluation over the first and the last entry.

Lemma. *For all $X \in \mathfrak{g}_{-1}$ we have $(\partial^*\kappa_0)(X) = (\operatorname{Tr}\kappa_0)(X, \_) \in \mathfrak{g}_1$. In particular, $\operatorname{Tr}\kappa_0 = 0$ if and only if $\partial^*\kappa_0 = 0$.*

Proof. By the definition above, $(\operatorname{Tr}\kappa_0)(X,Y) = \sum_i \kappa_0(e_i,X)(Y)(e^i)$, where e_i is a basis in $\mathfrak{g}_{-1}$ while e^i is its dual basis in $\mathfrak{g}_1$. If we take Y as a free argument, we obtain $\operatorname{Tr}\kappa_0(X,\_) \in \mathfrak{g}_1$, $\operatorname{Tr}\kappa_0(X,\_) = \sum_i [e^i, \kappa_0(e_i,X)]$. But the latter is exactly the formula for $(\partial^*\kappa_0)(X)$, see 1.3. $\square$

3.2. Definition. A *normal Cartan connection* $\omega \in \Omega^1(P,\mathfrak{g})$ is an admissible connection with the curvature satisfying $\operatorname{Tr}\kappa_0 = 0$.

3.3. Lemma. *Let $P \to M$ be a B structure with harmonic torsion, P_0 be the underlying first order structure. Then for each admissible Cartan connection ω on P, $\partial^*\kappa_0$ is constant on the fibers of $P \to P_0$.*

Proof. By the formula I.3.8.(4), for each section σ of $P \to P_0$ and $u \in P$ we have

$$\kappa_0(u)(X,Y) = \kappa_0(\sigma(p(u)))(X,Y) - [\tau(u), \kappa_{-1}(\sigma(p(u)))(X,Y)],$$

where τ is the mapping introduced in the proof of I.3.7.

Further, $\partial^*\kappa_{-1} = 0$ since the torsion κ_{-1} is harmonic, and we obtain

$$\begin{aligned}
\partial^*\kappa_0(u)(X) &= [e^i, \kappa_0(\sigma(p(u)))(e_i,X)] - [e^i, [\tau(u), \kappa_{-1}(\sigma(p(u)))(e_i,X)]] \\
&= \partial^*\kappa_0(\sigma(p(u)))(X) - [\tau(u), [e^i, \kappa_{-1}(\sigma(p(u)))(e_i,X)]] \\
&= \partial^*\kappa_0(\sigma(p(u)))(X) - [\tau(u), \partial^*\kappa_{-1}(\sigma(p(u)))(X)] \\
&= \partial^*\kappa_0(\sigma(p(u)))(X). \quad \square
\end{aligned}$$

We shall also need another technical lemma. In view of lemma 3.1, it is a direct consequence of the uniqueness result from the previous section, the elementary argument used here is an easy application of the Bianchi identity for general Cartan connections.

3.4. Lemma. *Let ω and $\bar\omega$ be two normal Cartan connections on a B-structure P, κ and $\bar\kappa$ be their curvatures. Then the trace $\operatorname{Tr}_{\mathfrak{g}_0}(\bar\kappa_0 - \kappa_0)$ within $\mathfrak{g}_0$ vanishes.*

Proof. Let us write $\delta = \bar\kappa_0 - \kappa_0$, and let e_i and e^i be the dual bases in $\mathfrak{g}_{\pm1}$. According to the Bianchi identity (proved in I.2.4) we have for $X, Z \in \mathfrak{g}_{-1}$

$$\begin{aligned}
[\delta(X,Z), e_i] = {} & [\delta(X,e_i), Z] + [\delta(e_i,Z), X] + \\
& \nabla^\omega_Z \kappa_{-1}(X,e_i) + \nabla^\omega_X \kappa_{-1}(e_i,Z) + \nabla^\omega_{e_i} \kappa_{-1}(Z,X) + \\
& \kappa_{-1}(\kappa_{-1}(X,e_i),Z) + \kappa_{-1}(\kappa_{-1}(e_i,Z),X) + \kappa_{-1}(\kappa_{-1}(Z,X),e_i) - \\
& \nabla^{\bar\omega}_Z \bar\kappa_{-1}(X,e_i) - \nabla^{\bar\omega}_X \bar\kappa_{-1}(e_i,Z) - \nabla^{\bar\omega}_{e_i} \bar\kappa_{-1}(Z,X) - \\
& \bar\kappa_{-1}(\bar\kappa_{-1}(X,e_i),Z) - \bar\kappa_{-1}(\bar\kappa_{-1}(e_i,Z),X) - \bar\kappa_{-1}(\bar\kappa_{-1}(Z,X),e_i).
\end{aligned}$$

Since $\bar\kappa_{-1} = \kappa_{-1}$ and the torsion κ_{-1} is constant on the fibers of $P \to P_0$, all lines except the first one vanish, see I.3.8.(4), I.3.10.(3) and the definition of ∇^ω in I.2.3. Now, $\operatorname{Tr}_{\mathfrak{g}_0}(\delta)(X,Z) = \sum_i [\delta(X,Z), e_i](e^i)$ while $(\operatorname{Tr}\delta)(X,Z) = [\delta(e_i,X), Z](e^i) = 0$. Thus the above computation shows that the traces inside of $\mathfrak{g}_0$ coincide as required. $\square$

3.5. Remark. If the torsion of a B-structure P vanishes, then all the admissible Cartan connections have vanishing $\mathfrak{g}_{-1}$-part of the curvature. Then the Bianchi identity implies directly that $\partial\kappa_0$ vanishes for all admissible Cartan connections. Thus, in the language of the Hodge theory for the corresponding cohomologies, this means just that the normal Cartan connections are exactly those admissible Cartan connections for which κ_0 is harmonic. As discussed in 1.8, if there is a torsion free connection on a reduction P_0 of P^1M to the structure group B_0, then there is the canonical B-structure P over P_0 with vanishing torsion and a normal Cartan connection on P is then an admissible Cartan connection with a harmonic $\mathfrak{g}_0$-part of the curvature. This is the point of view adopted in [Ochiai, 70] where the torsion–free case is discussed. However this cannot yield a canonical Cartan connection in the cases of non vanishing torsion in view of the results of the previous section.

3.6. The conformal case. Since there is always a torsion–free linear connection on each Riemannian manifold, the canonical prolongation P of a first order conformal Riemannian structure $P_0 \to M$ is always a reduction of P^2M and so we reproduce the classical construction in this case, cf. [Kobayashi, 72]. We already deduced in I.6.3 the existence and uniqueness of the normal Cartan connections, and the corresponding explicit deformation tensors Γ, on all manifolds M with conformal Riemannian structures, $\dim M \geq 3$. Let us recall the final formula: Starting with a torsion–free connection γ on P_0 with curvature tensor R^i_{jkl}, Ricci tensor R_{ij} and scalar curvature R, the necessary deformation tensor $\Gamma \in C^\infty(P_0, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1})$ is given by

$$\Gamma_{ij} = \tfrac{-1}{m-2}\big(R_{ij} - \tfrac{\delta_{ij}}{2(m-1)}R\big).$$

3.7. The almost Grassmannian case. Now we shall construct the normal Cartan connections on manifolds with almost Hermitian symmetric structures corresponding to the algebras $\mathfrak{g} = \mathfrak{sl}(p+q, \mathbb{R})$. The description of the algebra $\mathfrak{g} = \mathrm{Mat}_{q,p}(\mathbb{R}) \oplus (\mathfrak{sl}(p,\mathbb{R}) \oplus \mathfrak{sl}(q,\mathbb{R}) \oplus \mathbb{R}) \oplus \mathrm{Mat}_{p,q}(\mathbb{R})$ yields easily the formulas for the bracket. Let us use the generators e^α_β of the vector spaces of matrices, the matrices with all entries zero except a 1 in the β–th line and the α–th column. We shall use the letters $a, b, c, \dots$ for the indices between 1 and p, the letters $i, j, k, \dots$ will indicate indices running between 1 and q. For example, e^a_i means one of the generators in $\mathrm{Mat}_{q,p}(\mathbb{R})$. Using the fact that the Killing form of $\mathfrak{g} = \mathfrak{sl}(p+q,\mathbb{R})$ is a scalar multiple of the trace form one easily see that the bases $\{e^i_a\}$ and $\{e^a_i\}$ are also dual with respect to the Killing form, up to a fixed scalar multiple, and this suffices for our purposes. Then we have

$$[e^a_i, e^j_b] = \delta^a_b e^j_i - \delta^j_i e^a_b,\ [e^k_a, e^b_c] = -\delta^b_a e^k_c,\ [e^k_a, e^j_l] = \delta^k_l e^j_a.$$

Let us fix the sizes p and q, and consider an almost Grassmannian structure $P \to M$ with a harmonic torsion. Let $P_0 \to M$ be the underlying first order structure with the distinguished class of the harmonic connections.

The deformation tensor Γ is expressed through functions $\Gamma_{\substack{b\,a\\ j\,i}}$ defined by $\Gamma(e^a_i) = \Gamma_{\substack{b\,a\\ j\,i}} e^j_b$. The possible deformations $\delta\kappa_0$ of the curvature are described in I.3.10.(4) The trace of the curvature is obtained through evaluation in $\mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_{-1}$

over the first and the fourth entry, however according to Lemma 3.1, we can compute $\partial^*(\kappa_0)$ instead. Let us first evaluate $[e^i_b,[\Gamma.e^b_i,Y]-[\Gamma.Y,e^b_i]]$ on the generators.

$$
\begin{aligned}
{}[e^i_b,[\Gamma.e^a_k,e^b_i]-[\Gamma.e^b_i,e^a_k]] &= [e^i_b,[\Gamma_{\substack{d\,a\\s\,k}}e^s_d,e^b_i]-[\Gamma_{\substack{d\,b\\s\,i}}e^s_d,e^a_k]] \\
&= [e^i_b,-\Gamma_{\substack{b\,a\\s\,k}}e^s_i+\Gamma_{\substack{d\,a\\i\,k}}e^b_d+\Gamma_{\substack{a\,b\\s\,i}}e^s_k-\Gamma_{\substack{d\,b\\k\,i}}e^a_d] \\
&= (-\delta^i_i\Gamma_{\substack{b\,a\\s\,k}}+\delta^i_k\Gamma_{\substack{a\,b\\s\,i}})e^s_b+(-\delta^b_b\Gamma_{\substack{d\,a\\i\,k}}+\delta^b_a\Gamma_{\substack{d\,b\\k\,i}})e^i_d
\end{aligned}
$$

The application of the formula for ∂^* from 1.3 yields

(1)
$$
\begin{aligned}
\partial^*(\delta\kappa_0)(e^a_k) &= \textstyle\sum_{s=1}^q\sum_{b=1}^p(-q\Gamma_{\substack{b\,a\\s\,k}}+\Gamma_{\substack{a\,b\\s\,k}})e^s_b+\sum_{d=1}^p\sum_{i=1}^q(-p\Gamma_{\substack{d\,a\\i\,k}}+\Gamma_{\substack{d\,a\\k\,i}})e^i_d \\
&= \textstyle\sum_{l=1}^q\sum_{c=1}^p(-q\Gamma_{\substack{c\,a\\l\,k}}-p\Gamma_{\substack{c\,a\\l\,k}}+\Gamma_{\substack{a\,c\\l\,k}}+\Gamma_{\substack{c\,a\\k\,l}})e^l_c.
\end{aligned}
$$

According to 3.1, the trace of κ_0 evaluated on the base elements e^a_k, e^c_l is exactly the expression inside the brackets in the last sum.

For each admissible Cartan connection ω on P there are the two parts $\kappa_{0,1}$, $\kappa_{0,2}$ of κ_0, corresponding to the decomposition of $\mathfrak{g}_0$ into two components. They are given by functions $K^a_{d\substack{b\,c\\k\,l}}$, $K^i_{j\substack{b\,c\\k\,l}}$, one set for each of the two blocks in the matrices in $\mathfrak{g}_0$. From the second line of the above computation, we can read the formulae for the deformation of these functions achieved by the chosen deformation tensor Γ

$$
\delta K^l_{s\substack{b\,a\\i\,k}}=\Gamma_{\substack{b\,a\\s\,k}}\delta^l_i-\Gamma_{\substack{a\,b\\s\,i}}\delta^l_k,\qquad \delta K^d_{c\substack{b\,a\\i\,k}}=\Gamma_{\substack{d\,b\\k\,i}}\delta^a_c-\Gamma_{\substack{d\,a\\i\,k}}\delta^b_c.
$$

Consequently, the deformation of the traces $\mathrm{Tr}_{\mathfrak{g}_0}(\delta(\kappa_{0,1}))$, $\mathrm{Tr}_{\mathfrak{g}_0}(\delta(\kappa_{0,2}))$ of these two components within $\mathfrak{g}_0$ are $\mp(\Gamma_{\substack{a\,b\\k\,i}}-\Gamma_{\substack{b\,a\\i\,k}})$.

Now, given a connection γ in the distinguished class on P_0, we shall compute the deformation tensor Γ which deforms the induced admissible Cartan connection $\tilde\gamma$ into a normal Cartan connection ω with curvature $\bar\kappa=\kappa-\delta\kappa$. Let κ be the curvature function of $\tilde\gamma$, and write $\delta\kappa$ for its change achieved by the choice of Γ. We have

$$
\begin{aligned}
\mathrm{Tr}_{\mathfrak{g}_0}(\delta(\kappa_{0,2}))_{\substack{c\,a\\l\,k}} &= \Gamma_{\substack{c\,a\\l\,k}}-\Gamma_{\substack{a\,c\\k\,l}} \\
(p+q)\mathrm{Tr}(\delta(\kappa_0))_{\substack{c\,a\\l\,k}} &= -(p+q)^2\Gamma_{\substack{c\,a\\l\,k}}+2(p+q)\Gamma_{\substack{a\,c\\l\,k}}-(p+q)\mathrm{Tr}_{\mathfrak{g}_0}(\delta(\kappa_{0,2}))_{\substack{a\,c\\l\,k}} \\
2\mathrm{Tr}(\delta(\kappa_0))_{\substack{a\,c\\l\,k}} &= -2(p+q)\Gamma_{\substack{a\,c\\l\,k}}+4\Gamma_{\substack{c\,a\\l\,k}}-2\mathrm{Tr}_{\mathfrak{g}_0}(\delta(\kappa_{0,2}))_{\substack{c\,a\\l\,k}}
\end{aligned}
$$

where the aim of our manipulation is to get rid of the interchanging indices in the formula for the trace of κ_0.

Let $\sigma\colon P_0\to P$ be the section corresponding to the connection γ. The curvature $R=R_1+R_2\colon P_0\to\mathfrak{g}^*_{-1}\wedge\mathfrak{g}^*_{-1}\otimes\mathfrak{g}_0$ of γ is σ-related to $\kappa_0=\kappa_{0,1}+\kappa_{0,2}$. In particular, on the image $\sigma(P_0)\subset P$, we can achieve the vanishing of the trace of $\bar\kappa_0$ by the following choice of the deformation

(2) $$\Gamma_{\substack{c\,a\\l\,k}}=\tfrac{-1}{4-(q+p)^2}\big((p+q)\mathrm{Tr}(R)_{\substack{c\,a\\l\,k}}+2\mathrm{Tr}(R)_{\substack{a\,c\\l\,k}}+(p+q)\mathrm{Tr}_{\mathfrak{g}_0}(R_2)_{\substack{a\,c\\l\,k}}+2\mathrm{Tr}_{\mathfrak{g}_0}(R_2)_{\substack{c\,a\\l\,k}}\big).$$

Since the torsion of the B-structure P is harmonic, this choice of Γ leads to a normal Cartan connection according to Lemma 3.3.

The results of the previous section assure that there is a unique normal Cartan connection on P, however it is easy to verify this directly. Indeed, it is equivalent to prove, that if ω and $\bar{\omega}$ are two normal Cartan connections on P, then the (uniquely defined) deformation tensor Γ is identically zero. In fact, we have computed above a tensor Γ deforming a given ω in such a way, that on the image of a section $\sigma\colon P_0 \to P$ the achieved deformation of the trace of $\mathfrak{g}_0$-part of the curvature of γ reaches a value prescribed in advance. But Lemma 3.4 states that the traces of κ_0 and $\bar{\kappa}_0$ inside of $\mathfrak{g}_0$ coincide. In view of our computation this means, that the deformation tensor Γ satisfies $\Gamma_{\substack{a\,b\\k\,i}} = \Gamma_{\substack{b\,a\\i\,k}}$ and so for any two normal Cartan connections ω, $\bar{\omega}$, the corresponding deformation tensor Γ is symmetric. Further, the achieved deformation of the trace of κ_0 by means of Γ has to vanish too and since we can use the equality $\Gamma_{\substack{a\,b\\k\,i}} = \Gamma_{\substack{b\,a\\i\,k}}$ we obtain $2\Gamma_{\substack{c\,a\\k\,l}} = (p+q)\Gamma_{\substack{c\,a\\l\,k}}$. Applying the latter equality twice, we get

$$2(p+q)\Gamma_{\substack{c\,a\\l\,k}} = 4\Gamma_{\substack{c\,a\\k\,l}} = (p+q)(p+q)\Gamma_{\substack{c\,a\\k\,l}}.$$

Thus, if $q \geq p \geq 1$, $q+p \geq 3$ then $\Gamma_{\substack{c\,a\\k\,l}} = 0$ for all c, a, k, l and so there is at most one normal Cartan connection ω on P.

Thus, we can formulate the final result of our computations.

3.8. Theorem. *Let $P \to M$ be a real almost Grassmannian structure with a harmonic torsion, on a smooth manifold M and assume $q \geq p \geq 1$, $q+p \geq 3$. Then there is a uniquely defined normal Cartan connection ω on P and for each linear harmonic connection γ on the underlying first order structure P_0 with curvature $R = R_1 + R_2$, $\omega = \tilde{\gamma} - \Gamma \circ \theta_{-1}$, where the corresponding deformation tensor Γ is given by the formula* 3.7.(2).

3.9. Corollary. *Let $P \to M$ be a projective structure on a smooth manifold M,* $\dim(M) = q > 1$. *Then there is a uniquely defined normal Cartan connection ω on P and for each linear torsion–free connection γ from the underlying class on the first order structure P_0 with curvature $R = (R^i_{jkl})$, we obtain $\omega = \tilde{\gamma} - \Gamma \circ \theta_{-1}$, where the corresponding deformation tensor Γ is given by*

$$\Gamma_{jk} = \tfrac{1}{(q-1)}(R^l_{jlk} + R^l_{ljk}).$$

3.10. The almost Lagrangian case. We have to deal with $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_{-1} = S^2\mathbb{R}^m$, $\mathfrak{g}_1 = S^2\mathbb{R}^{m*}$, $\mathfrak{g}_0 = \mathfrak{gl}(m,\mathbb{R})$, cf. I.3.3. Let us fix the base $e_k \odot e_l$ consisting of symmetric matrices with entries $a^i_j = \frac{1}{2}(\delta^i_k\delta_{jl} + \delta^i_l\delta_{jk})$. Let $e^s \odot e^t$ be the dual base of $\mathfrak{g}_1$ and let e^i_j be the usual base of $\mathfrak{g}_0$. The commutators of the base elements are

$$[e^s \odot e^t, e_k \odot e_l] = -\frac{1}{4}(\delta^s_k e^t_l + \delta^s_l e^t_k + \delta^t_k e^s_l + \delta^t_l e^s_k) \tag{1}$$

$$[e^s \odot e^t, e^p_w] = \delta^t_w e^p \odot e^s + \delta^s_w e^p \odot e^t. \tag{2}$$

We shall express the deformation tensor Γ by its values on the generators, so we write $\Gamma.(e_i \odot e_j) =: \sum_{s,t} \Gamma_{(st)(ij)} e^s \odot e^t$. Similarly to the above cases we compute the deformation of the curvature.

$$
\begin{aligned}
&[\Gamma.(e_k \odot e_l), e_i \odot e_j] - [\Gamma.(e_i \odot e_j), e_k \odot e_l] = \\
&\quad = \textstyle\sum_{s,t} \big(\Gamma_{(st)(kl)}.[e^s \odot e^t, e_i \odot e_j] - \Gamma_{(st)(ij)}[e^s \odot e^t, e_k \odot e_l]\big) \\
&\quad = -\frac{1}{4} \textstyle\sum_{s,t} \big(\Gamma_{(st)(kl)} (\delta_i^s e_j^t + \delta_j^s e_i^t + \\
&\qquad \delta_i^t e_j^s + \delta_j^t e_i^s) - \Gamma_{(st)(ij)} (\delta_k^s e_l^t + \delta_l^s e_k^t + \delta_k^t e_l^s + \delta_l^t e_k^s)\big) \\
&\quad = \frac{1}{2} \textstyle\sum_{p,w} \big(\delta_l^w \Gamma_{(kp)(ij)} + \delta_k^w \Gamma_{(lp)(ij)} - \delta_j^w \Gamma_{(ip)(kl)} - \delta_i^w \Gamma_{(jp)(kl)}\big) e_w^p
\end{aligned} \tag{3}
$$

In order to get the deformation of the trace we compute $\partial^*(\kappa_0)(e_k \odot e_l)$:

$$
\begin{aligned}
&[e^i \odot e^j, [\Gamma.(e_k \odot e_l), e_i \odot e_j] - [\Gamma.(e_i \odot e_j), e_k \odot e_l]] = \\
&\qquad = \frac{1}{2} \textstyle\sum_{p,w} (\text{the above coefficient at } e_w^p)(\delta_w^j e^p \odot e^i + \delta_w^i e^p \odot e^j) \\
&\qquad = \textstyle\sum_{p,q} \big(\Gamma_{(kp)(ql)} + \Gamma_{(lp)(qk)} - (m+1)\Gamma_{(pq)(kl)}\big) e^p \odot e^q
\end{aligned} \tag{4}
$$

and so the value of the deformation of the trace on the generators is

$$
\delta \operatorname{Tr}(\kappa_0)_{(pq)(kl)} = \delta \operatorname{Tr}(\kappa_0)(e_k \odot e_l, e_p \odot e_q) = \Gamma_{(kp)(ql)} + \Gamma_{(lp)(qk)} - (m+1)\Gamma_{(pq)(kl)}. \tag{5}
$$

Now, similarly to the Grassmannian case, we have to consider a suitable combination. Surprisingly enough, we do not need the traces inside of $\mathfrak{g}_0$ in order to express the tensor Γ. If we substitute (5) into

$$
m\delta \operatorname{Tr}(\kappa_0)_{(pq)(kl)} + \delta \operatorname{Tr}(\kappa_0)_{(pk)(ql)} + \delta \operatorname{Tr}(\kappa_0)_{(pl)(qk)},
$$

we are left with $(2 - m(m+1))\Gamma_{(pq)(kl)}$ on the right hand side. Thus if we start with a linear harmonic connection γ on M and κ is the curvature of $\tilde{\gamma}$, then we can achieve vanishing of $\partial^* \bar{\kappa}_0$ on the section which corresponds to γ by the choice

$$
\Gamma_{(pq)(kl)} = \tfrac{1}{m(m+1)-2} (m \operatorname{Tr}(R)_{(pq)(kl)} + \operatorname{Tr}(R)_{(pk)(ql)} + \operatorname{Tr}(R)_{(pl)(qk)}) \tag{6}
$$

where $\operatorname{Tr}(R)$ is the Ricci curvature of γ. In view of Lemma 3.3, this deformation tensor leads to a normal Cartan connection. Moreover, this deformation is uniquely determined by our computation. Thus, we have proved

3.11. Theorem. *Let $P \to M$ be an almost Lagrangian structure with a harmonic torsion, on a smooth manifold M with dimension greater then 2. Then there is a uniquely defined normal Cartan connection ω on P and for each linear harmonic connection γ on the underlying first order structure P_0 with curvature R, $\omega = \tilde{\gamma} - \Gamma \circ \theta_{-1}$, where the corresponding deformation tensor Γ is given by the formula 3.10.(6).*

3.12. The almost spinorial case. The situation is very similar to the almost Lagrangian case. We have to proceed quite analogously with the symmetric matrices replaced by the antisymmetric ones.

We have $\mathfrak{g}_{-1} = \Lambda^2\mathbb{R}^m$, $\mathfrak{g}_1 = \Lambda^2\mathbb{R}^{m*}$, $\mathfrak{g}_0 = \mathfrak{gl}(m,\mathbb{R})$. The commutators of the base elements are

$$[e^s \wedge e^t, e_k \wedge e_l] = \tfrac{1}{4}(-\delta^s_l e^t_k + \delta^t_l e^s_k + \delta^s_k e^t_l - \delta^t_k e^s_l) \tag{1}$$

$$[e^s \wedge e^t, e^p_w] = \delta^t_w e^s \wedge e^p - \delta^s_w e^t \wedge e^p. \tag{2}$$

We write, $\Gamma.(e_i \wedge e_j) =: \sum_{s,t} \Gamma_{[st][ij]} e^s \wedge e^t$. Similarly as before we compute the deformation of the curvature.

$$\begin{aligned}
&[\Gamma.(e_k \wedge e_l), e_i \wedge e_j] - [\Gamma.(e_i \wedge e_j), e_k \wedge e_l] = \\
&\quad= \textstyle\sum_{s,t}\big(\Gamma_{[st][kl]}.[e^s \wedge e^t, e_i \wedge e_j] - \Gamma_{[st][ij]}[e^s \wedge e^t, e_k \wedge e_l]\big) \\
&\quad= \tfrac{1}{4}\textstyle\sum_{s,t}\big(\Gamma_{[st][kl]}(-\delta^s_j e^t_i + \delta^t_j e^s_i + \delta^s_i e^t_j - \delta^t_i e^s_j) - \\
&\quad\quad \Gamma_{[st][ij]}(-\delta^s_l e^t_k + \delta^t_l e^s_k + \delta^s_k e^t_l - \delta^t_k e^s_l)\big) \\
&\quad= \tfrac{1}{2}\textstyle\sum_{p,w}\big(\delta^w_i \Gamma_{[pj][kl]} + \delta^w_j \Gamma_{[ip][kl]} + \delta^w_k \Gamma_{[lp][ij]} + \delta^w_l \Gamma_{[pk][ij]}\big) e^p_w
\end{aligned} \tag{3}$$

Further we compute

$$\begin{aligned}
\partial^*(\kappa_0)(e_k \wedge e_l) &= [e^i \wedge e^j, [\Gamma.(e_k \wedge e_l), e_i \wedge e_j] - [\Gamma.(e_i \wedge e_j), e_k \wedge e_l]] = \\
&= \tfrac{1}{2}\textstyle\sum_{p,w}(\text{the above coefficient at } e^p_w)(\delta^j_w e^i \wedge e_p - \delta^i_w e^j \wedge e^p) \\
&= \textstyle\sum_{p,q}\big(\Gamma_{[pk][ql]} + \Gamma_{[lp][qk]} + (1-m)\Gamma_{[pq][kl]}\big) e^q \wedge e^p \\
\delta\,\mathrm{Tr}(\kappa_0)(e_k \wedge e_l, e_p \wedge e_q) &= \Gamma_{[pk][ql]} + \Gamma_{[lp][qk]} - (m-1)\Gamma_{[pq][kl]}
\end{aligned} \tag{4}$$

Now, we have to find a suitable combination. Let us substitute (4) into

$$m\delta\,\mathrm{Tr}(\kappa_0)_{[pq][kl]} + \delta\,\mathrm{Tr}(\kappa_0)_{[pk][ql]} - \delta\,\mathrm{Tr}(\kappa_0)_{[pl][qk]}.$$

Then only $(2 - m(m-1))\Gamma_{[pq][kl]}$ remains on the right hand side. Thus if we start with a linear harmonic connection γ on M and κ is the curvature of $\tilde{\gamma}$, then we can achieve global vanishing of $\partial^*\bar{\kappa}_0$ by the choice (cf. Lemma 3.3)

$$\Gamma_{[pq][kl]} = \tfrac{1}{m(m-1)-2}(m\,\mathrm{Tr}(R)_{[pq][kl]} + \mathrm{Tr}(R)_{[pk][ql]} - \mathrm{Tr}(R)_{[pl][qk]}) \tag{5}$$

where $\mathrm{Tr}(R)$ is the Ricci curvature of γ. Since this deformation is uniquely determined by our computation, we have proved:

3.13. Theorem. *Let $P \to M$ be an almost spinorial structure with a harmonic torsion, on a smooth manifold M with dimension greater then 2. Then there is a uniquely defined normal Cartan connection ω on P and for each linear harmonic connection γ on the underlying first order structure P_0 with curvature R, $\omega = \tilde{\gamma} - \Gamma \circ \theta_{-1}$, where the corresponding deformation tensor Γ is given by* 3.12.(5).

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