THE PRINCIPAL PROLONGATION OF FIRST ORDER G-STRUCTURES

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Abstract. In this short note we use the concept of the principal prolongation of principal fiber bundles to develop an alternative procedure for the construction of prolongations of a class of G-structures of the first order. The motivation comes from the so called almost Hermitian structures which can be defined either as standard first order structures, or higher order structures, but if they do not admit a connection without torsion, the classical constructions fail in general.
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In the sequel, we shall always work with smooth finite dimensional manifolds and Lie groups. First we shall recall the standard procedure of the principal prolongation, due to [Libermann, 71] and [Kolář, 71], the details can be found in [Kolář, Michor, Slovák, 93, sections 15, 16]. This generalization of the concept of frame bundles will enable us to construct prolongations of the first order structures which are principal fiber bundles over the original base manifold with nice properties.

The motivation comes from the study of the so called almost Hermitian structures in [Baston, 91] and from the study of the natural operators on them. In particular, the calculus for the Cartan connections developed in the forthcoming series of papers [Čap, Slovák, Souček] requires just the properties of our prolonged structures.

During the work on this paper the author discussed extensively the topics with the coauthors of the above mentioned series, A. Čap and V. Souček, and he likes very much to acknowledge their influence.

1. The principal prolongation

1.1. Let us consider an arbitrary principal fiber bundle P with an arbitrary structure group B_0.

Each jet $j^1\varphi(0,e)$ of an invertible principal fiber bundle morphism $\varphi\colon \mathbb{R}^m\times B_0 \to P$ determines the jets $j^1\varphi(0,g)$ for all $g \in B_0$. Thus the space $W^1P := \{j^1\varphi(0,e)\}$ of all such jets can be identified with the space of certain frames of the tangent bundle TP. There is also the Lie group $W^1_m B_0$ consisting of the invertible jets of morphisms of the trivial principal fiber bundle $\mathbb{R}^m \times B_0 \to \mathbb{R}^m \times B_0$ at $(0,e)$ preserving the fiber

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over $0 \in \mathbb{R}^m$, which acts on the right on W^1P. The action is defined by forming the jets of the composition of the corresponding morphisms.

For each $j^1\varphi(0,e) \in W^1_mB_0$, the morphism φ is fully determined by the couple $(\varphi_0, \bar\varphi)$ such that $\varphi(x,e) = (\varphi_0(x), \bar\varphi(x))$. Moreover, $\bar\varphi(x) = \bar\varphi(0).\bar\varphi(0)^{-1}.\bar\varphi(x)$ and so it is straightforward to verify that, as a Lie group,

$$W^1_mB_0 = (G^1_m \times B_0) \rtimes (\mathbb{R}^{m*} \otimes \mathfrak{b}_0)$$

where $\mathfrak{b}_0$ is the Lie algebra of B_0, $G^1_m = GL(m,\mathbb{R})$ is the first order jet group, and $\mathbb{R}^{m*} \otimes \mathfrak{b}_0$ is endowed with the structure of the vector group. The multiplication is then expressed by the formula

$$(A', a', Z').(A, a, Z) = (A' \circ A, a' \circ a, Ad(a^{-1}) \circ Z' \circ A + Z) \tag{1}$$

with $A, A' \in G^1_m$, $a, a' \in B_0$, $Z, Z' \in \mathbb{R}^{m*} \otimes \mathfrak{b}_0$.

Let us write P^1M for the linear frame bundle over M while J^1 will denote the functor of the first jet prolongations of fibered manifolds. Each local trivialization $\psi\colon \mathbb{R}^m \otimes B_0 \to P$ corresponds to the couple $(\psi_0, \tilde\psi)$ given by $\psi(x,e) = \tilde\psi \circ \psi_0$ with a section $\tilde\psi\colon M \to P$ and $\psi_0\colon \mathbb{R}^m \to M$. Thus we have the identification $W^1P = P^1M \times_M J^1P$ with the appropriate right action of $W^1_mB_0$ given by

$$(u, w).(A, a, Z) = (u \circ A, Tr(w, T\ell_a \circ Z \circ A^{-1} \circ u^{-1})) \tag{2}$$

with $u \in P^1M$ over $x \in M$, $w \in J^1P$ viewed as an element in $\operatorname{Hom}(T_xM, T_{\beta w}P)$, $(A, a, Z) \in W^1_mB_0$, $r\colon P \times B_0 \to P$ is the principal action, while ℓ_a means the left multiplication in B_0 by a.

1.2. The soldering form. Analogously to the definition of the soldering form on P^1M, we define the form $\theta = \theta_{-1} \oplus \theta_0 \in \Omega^1(W^1P, \mathbb{R}^m \oplus \mathfrak{b}_0)$ as follows. Given a tangent vector $\xi \in T_uW^1P$, we project it down by means of the projection $\beta\colon W^1P \to P$ onto the target of the jets, and we express $T\beta(\xi)$ in the base of $T_{\beta(u)}P$ induced by u.

The vector space $\mathbb{R}^m \oplus \mathfrak{b}_0 = T_{(0,e)}(\mathbb{R}^m \times B_0)$ can be considered as the standard fiber of the vector bundle of the right invariant vector fields on P, which is an associated bundle to W^1P corresponding to the action ℓ of $W^1_mB_0$ given by $\ell(j^1\psi(0,e))(j^1_0c) = j^1_0(r_{\bar\psi(0)^{-1}} \circ \psi \circ c)$.

Lemma. (1) *$\theta_{-1}(\xi) = 0$ if and only if ξ is vertical on $W^1 \to M$*

(2) *for each element $X + Y + Z$ from the Lie algebra $\mathfrak{g}^1_m + \mathfrak{b}_0 + (\mathbb{R}^{m*} \otimes \mathfrak{b}_0)$, $\theta(\zeta_{X+Y+Z}) = Y$.*

(3) *θ is a pseudotensorial form of type ℓ, i.e. $r^*_a(\theta_P) = \ell_{a^{-1}} \circ \theta_P$ for all $a \in W^1_mB_0$.*

Proof. The claims are verified by a straightforward computation, see e.g. [Kolář, Michor, Slovák, 93, p. 155]. □

2. The B-structures

2.1. Let us fix several assumptions for the sequel. We shall consider a Lie algebra with grading $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, a connected Lie group B with the algebra $\mathfrak{b}$ and its subgroup

$B_1 \subset B$ corresponding to the Lie algebra $\mathfrak{g}_1$. Then B_1 is a normal subgroup and the grading determines the semidirect product structure $B = B_0 \rtimes B_1$ where $B_0 = B/B_1$ is the subgroup corresponding to $\mathfrak{g}_0$. In the sequel, we shall moreover assume that $\mathfrak{g}_0 \subset \mathfrak{gl}(m, \mathbb{R})$, $B_0 \cap B_1 = \{e\}$, and B_1 is the first prolongation of B_0 in the sense in the classical theory of G-structures, we recall the definition below. In particular, if we start with $\mathfrak{g}_0 \subset \mathfrak{gl}(m, \mathbb{R})$ and consider a covering of the corresponding Lie subgroup in $GL(m, \mathbb{R})$, then the first prolongation of $\mathfrak{g}_0$ leads to the semidirect product $B_0 \rtimes B_1$ which is a covering of a subgroup in G^2_m and which satisfies all the required properties.

All the almost Hermitian symmetric structures as considered in [Baston, 91], except the almost Grassmanian case with parameters $p = 1$, q arbitrary, i.e. except the projective structures, satisfy our requirements, cf. [Ochiai, 70], , [Cap, Slovák, Souček].

In the projective case, we take $B_1 = \mathbb{R}^{m*}$ while the first prolongation of $B_0 = GL(m, \mathbb{R})$ leads to the whole second order jet group $G^2 m$.

Let $P_0 \to P^1 M$ be a reduction to the structure group B_0, i.e. an arbitrary morphism of principal fiber bundles over the identity on the base manifold, such that the corresponding Lie group morphism is injective on the Lie algebra level. This means we involve the coverings of principal subbundles in $P^1 M$. For example, the spin structures are involved in the case of the conformal Riemannian structures. Further, let us consider a linear connection γ on P_0, viewed as a B_0 equivariant section $\gamma\colon P_0 \to J^1 P_0$. The choice of the connection defines the mapping

$$P_0 \to P_0 \times_M P_0 \to P_0 \times_M J^1 P_0 \to P^1 M \times_M J^1 P = W^1 P$$
$$u \mapsto (u, u) \mapsto (u, \gamma(u))$$

Let us denote it by $\bar{\gamma}\colon P_0 \to W^1 P_0$.

According to 1.1.(1), there is the subgroup $B_0 \rtimes (\mathbb{R}^{m*} \otimes \mathfrak{g}_0)$ in $W^1_m B_0$ consisting of the elements of the form (a, a, Z). Let us notice that the (right) action of $(a, a, 0) \in B_0$ on (e, e, Z) is exactly the standard tensorial action of a^{-1} on Z.

Furthermore, there is the B_0-invariant subgroup (vector subspace)

$$B_1 \subset \mathbb{R}^{m*} \otimes \mathfrak{g}_0 \subset \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^m$$

given by the symmetrization and the corresponding subgroup $B_0 \rtimes B_1 \in W^1 P_0$. The subgroup B_1 is the classical first prolongation of B_0.

In the next Lemma, we shall construct the distinguished first prolongation $P_{[\gamma]}$ of P_0 depending on our choice of γ. As a fiber bundle, $P_{[\gamma]}$ will be always isomorphic to $P_0 \times B_1$, and there will be the structure of the principal fiber bundle with structure group B there.

2.2. Lemma. *For each choice of γ we have*

(1) *The mapping $\bar{\gamma}\colon P_0 \to W^1 P_0$ is a reduction of $W^1 P_0$ to the structure group B_0 with the corresponding group morphism $a \mapsto (a, a, 0)$.*

(2) *There is a unique principal right action of B on $P_0 \times B_1$ such that the mapping*

$$\hat{\gamma}\colon P_0 \times B_1 \to W^1 P_0, \quad (u, b) \mapsto \bar{\gamma}(u).b$$

is a morphism of the principal fiber bundles.

(3) *The image $\hat{\gamma}(P_0 \times B_1) =: P_{[\gamma]} \subset W^1_m P_0$ is a principal subbundle in $W^1 P_0$ which is a reduction of $W^1 P_0$ to the structure group $B = B_0 \rtimes B_1$.*

Proof. The connection $\gamma\colon P_0 \to J^1P_0$ is a B_0-equivariant mapping, with respect to the right action of B_0 on P_0 given by pointwise multiplication. Thus in order to verify the first claim we have only to check that the action of an element $(a,a,0) \in B_0 \subset W^1_m B_0$ on the frame $(u,w) \in P_0 \times_M J^1P_0$ with w over u, coincides with the action of a on w in J^1P_0. But this is visible easily from 1.1.(2) with $Z = 0$.

The mapping $\bar\gamma$ is injective by the definition. If $\bar\gamma(u_1).b_1 = \bar\gamma(u_2).b_2$ with u_1, u_2 over the same point in M and $b_1, b_2 \in B_1$, then $u_2 = u_1.b_0$ and we obtain $\bar\gamma(u_1).b_0.b_2 = \bar\gamma(u_1).b_1$. Since $B_0 \cap B_1 = \{e\}$, the latter implies $b_0 = b_1 = b_2 = e$, and we have verified that $\hat\gamma$ is injective. Moreover, B_1 is a normal subgroup and so for arbitrary $u \in P_0$, $b_0 \in B_0$, $b \in B_1$, we obtain

$$\hat\gamma(u,b).b_0 = \bar\gamma(u).bb_0 = \bar\gamma(u).b_0b_0^{-1}bb_0 = \bar\gamma(u.b_0).b_0^{-1}bb_0 = \hat\gamma(u.b_0, b_0^{-1}bb_0).$$

This defines the structure of a principal fiber bundle on $P_0 \times B_1$ and $\hat\gamma$ is a morphism by the definition. The corresponding Lie group homomorphism is the inclusion $B_0 \rtimes B_1 \hookrightarrow B_0 \rtimes (\mathbb{R}^{m*} \otimes \mathfrak{g}_0) \hookrightarrow W^1_m B_0$. Since $\hat\gamma$ is over the identity on the base manifold, we have proved claims (2) and (3). □

2.3. Lemma. *The pullback $\theta_{P_{[\gamma]}}$ of the canonical soldering form on W^1P to the reduction $P_{[\gamma]}$ satisfies*

(1) *$\theta_{P_{[\gamma]}}$ is strictly horizontal over M*
(2) *$\theta_{P_{[\gamma]}}(\zeta_{Y+Z}(u)) = Y$ for all $(Y+Z) \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$*
(3) *$\theta_{P_{[\gamma]}} \in C^\infty(P_{[\gamma]}, \mathbb{R}^m \oplus \mathfrak{g}_0)$ is a pseudo-tensorial form of type ℓ, where ℓ is the restriction of the action from 1.2 to the subgroup $B_0 \rtimes B_1 \subset W^1_m B_0$.*

Proof. All the properties follow immediately from the properties of the soldering form on W^1P_0, see 1.2. □

We call $\theta_{P_{[\gamma]}}$ the *soldering form* of $P_{[\gamma]}$ and the principal fiber bundle $P_{[\gamma]} \to M$ together with the soldering form is called the B-structure on M (induced by the choice of γ). As mentioned in 1.2, ℓ is the action on the standard fiber of the space of right invariant vector fields on $P_0 \to M$.

2.4. Proposition. *Let $P_0 \to M$ be a reduction of the linear frame bundle to B_0, let $P_{[\gamma]} \to M$ be the reduction of W^1P_0 to B given by a connection γ on P_0 and let us write $\theta_{P_{[\gamma]}} = \theta_{-1} \oplus \theta_0$ for the corresponding soldering form on $P_{[\gamma]}$.*

(1) *For each equivariant section $\sigma\colon P_0 \to P_{[\gamma]}$ we have $\sigma^*(\theta_{-1}) = \theta_{P_0}$, the standard soldering form of the first order B_0 structure.*
(2) *There is a unique B_0-equivariant section $\sigma\colon P_0 \to P_{[\gamma]}$ such that the connection form of γ is $\sigma^*(\theta_0)$.*
(3) *There is a bijective correspondence between the space of linear connections η_σ on P_0 with the same torsion as γ and the space of B_0-equivariant sections σ of $P_{[\gamma]} \to P_0$. In this correspondence, the connection form of the connection η_σ is $\sigma^*\theta_0$.*
(4) *A connection η on P_0 defines the same reduction $P_{[\eta]} = P_{[\gamma]}$ as γ if and only if the torsions of η and γ coincide. All such connections are sections of an affine subbundle modeled over the vector bundle of one-forms on M.*

Proof. Since the standard soldering form θ_{P_0} is the pullback of the soldering form on P^1M by the reduction $P_0 \to P^1M$, the first statement follows directly from the construction of $P_{[\gamma]}$ and the commutativity of the diagram (cf. [Kolář, Michor, Slovák, 93, p. 156])

$$\begin{array}{ccc} TW^1P_0 & \xrightarrow{\ \theta\ } & \mathbb{R}^m \oplus \mathfrak{g}_0 \\ {\scriptstyle T\mathrm{pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pr}_1} \\ TP^1M & \xrightarrow{\ \theta_{P^1M}\ } & \mathbb{R}^m \end{array}$$

Let us view now the affine bundle $J^1P_0 \to P_0$ as the subspace

$$P_0 \times_{P_0} J^1P_0 \subset P_0 \times_M J^1P_0 \subset W^1P_0.$$

The corresponding modeling vector bundle has the standard fiber $\mathbb{R}^{m*} \otimes \mathfrak{g}_0$. Let η be a linear connection on P_0, viewed as the B_0-equivariant section $\eta\colon P_0 \to J^1P_0$. Its connection form is defined by means of the vertical projection on TP_0 determined by η and the fundamental vector field mapping. On the other hand, the canonical soldering form on W^1P_0 is defined by means of the tangent map to the target jet projection $\beta\colon W^1P_0 = P^1M \times_M J^1P_0 \to P_0$ and the decomposition of W^1P_0 is given by the decomposition of a local trivialization $\psi\colon \mathbb{R}^m \times B_0 \to P_0$ into the local chart $\psi_0\colon \mathbb{R}^m \to M$ and the section $x \mapsto \psi(\psi_0^{-1}(x), e)$. Hence it follows that the connection form of η is the pullback of the $\mathfrak{g}_0$-part of the canonical soldering form θ on W^1P_0 by the induced mapping $\bar{\eta}\colon P_0 \to W^1P_0$.

For each B_0-equivariant section $\sigma\colon P_0 \to P_0 \times B_1 \simeq P_{[\gamma]}$, $\sigma(u) = (u, \sigma_1(u))$ the composition $\hat{\gamma} \circ \sigma$ satisfies

$$\hat{\gamma} \circ \sigma\colon u.b_0 \mapsto (u.b_0, b_0^{-1}\sigma_1(u)b_0) \mapsto (u.b_0, \gamma(u).\sigma_1(u)b_0)$$

and so it is a reduction $P_0 \to W^1P_0 = P^1M \times_M J^1P_0$ to the structure group B_0 which has the form $u \mapsto (u, \eta(u))$ for a connection $\eta\colon P_0 \to J^1P_0$. Now, if η is a connection on P_0 with the values of $\bar{\eta}$ in the image of $\hat{\gamma}$, it means exactly that η is given by a right invariant section of $P_{[\gamma]}$. By the preceding part of the proof, the connection form of η is the pullback of the $\mathfrak{g}_0$-part of the soldering form on W_1P_0 and thus the pullback of the $\mathfrak{g}_0$-part of the induced soldering form on $P_{[\gamma]}$.

By the construction, two different sections $P_0 \to P_{[\gamma]}$ give rise to two different connections (consequently with two different connection forms). In particular, the only section corresponding to the initial connection γ is the section $u \mapsto \hat{\gamma}(u, e)$ which proves (2).

Now also (3) follows easily. Indeed, for each connection $\eta\colon P_0 \to J^1P_0 \simeq P_0 \times_{P_0} J^1P_0$ there is the difference of γ and η lying in the modeling vector bundle to the affine bundle J^1P_0. The mapping η admits the expression as a composition of $\hat{\gamma}$ with a B_0-equivariant section $P_0 \to P$ if and only if the image of the mapping η lies in the image of $\hat{\gamma}$. By the definition of B_1, the latter is equivalent to the vanishing of the antisymmetric part of the difference, i.e. the torsions of γ and η have to coincide.

By the above arguments, two connections γ and η give rise to the same subbundle in W^1P_0 if an only if the difference of the corresponding sections lies in B_1. By the previous part, this is equivalent to the equality of their torsions. Moreover, their difference is a mapping in $C^\infty(P_0, \mathfrak{g}_1)$, equivariant with respect to the Ad action of B_0 on $\mathfrak{g}_1$. But this action is the restriction of the standard tensorial action on $\mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^m$ to $\mathfrak{g}_1$. □

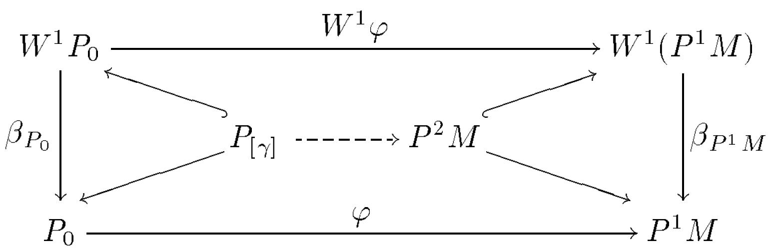
2.5. The linear connections on P_0 which are pullbacks of the soldering form on $P_{[\gamma]}$ are called the *distinguished linear connections on* $P_{[\gamma]}$. We have seen that once we choose one of them, all the other ones correspond to a choice of a one-form on M.

There is the structure equation for the soldering form $\theta_{-1} \oplus \theta_0$ on $P_{[\gamma]}$

$$d\theta_{-1} = -[\theta_0, \theta_{-1}] + T$$

where the two form T on $P_{[\gamma]}$ is called the *torsion* of the B-structure $P_{[\gamma]}$. A direct evaluation shows that the torsion T is always horizontal over M and thus T is determined by an equivariant function $t \in C^\infty(P_{[\gamma]}, \mathbb{R}^{m*} \wedge \mathbb{R}^{m*} \otimes \mathbb{R}^{m*})$, $T(\xi, \eta) = t(\theta_{-1}(\xi), \theta_{-1}(\eta))$.

2.6. For each manifold M, there is the canonical reduction $P^2M \hookrightarrow W^1(P^1M)$ of the principal prolongation of the linear frame bundle to the structure group G^2_m, the second order jet group. Indeed, if $\psi\colon \mathbb{R}^m \to M$ is a local chart, then $P^1\psi\colon \mathbb{R}^m \times G^1_m = P^1\mathbb{R}^m \to P^1M$ is a local trivialization and $j^1(P^1\psi)(0, e) \in W^1(P^1M)$ depends on $j^2\psi(0)$ only, and in this way we get a mapping $\nu_M\colon P^2M \to W^1(P^1M)$. The corresponding homomorphism of Lie groups is obtained just by the restriction of $\nu_{\mathbb{R}^m}$ to the fiber over zero. For more details and higher order analogies see [Kolář, Michor, Slovák, 93, p. 153]. In general, for each reduction $\varphi\colon P_0 \to P^1M$ with a given linear connection γ on P_0 there is the following diagram of reductions where the dashed one may, but need not, exist



Proposition. *Let $\varphi\colon P_0 \to P^1M$ be a reduction to B_0 and $P_{[\gamma]} \to M$ be the B-structure determined by the linear connection γ on P_0. Then*

(1) *The torsion of the soldering form on $P_{[\gamma]}$ vanishes if and only if the torsion of γ is zero.*
(2) *If there is a reduction $P_{[\gamma]} \to P^2M$ to the structure group B over φ, then there is a connection γ with vanishing torsion on P_0.*
(3) *If the torsion of γ is zero, then there is a reduction $\nu\colon P_{[\gamma]} \to P^2M$ making the above diagram commutative such that the distinguished soldering form on P_γ is the pullback of that one on P^2M.*

Proof. In view of 2.4.(1) and 2.4.(3), the pullback of the structure equation for the torsion T of the soldering form $\theta_{P_{[\gamma]}}$ with respect to a B_0-equivariant section $\sigma\colon P_0 \to P_{[\gamma]}$

is just the standard structure equation for the torsion of the corresponding connection $\sigma^*\theta_0$. This proves (1).

All the preceding constructions can be applied to the connected component of the identity of the whole second order jet group $G^2_m = G^1_m \rtimes B_1$ where $B_1 = \mathbb{R}^{m*} \otimes \mathfrak{gl}(m)$. If we choose any linear connection η on P^1M without torsion, then the corresponding reduction of W^1P^1M to G^2_m will be (up to the identification of P^2M with the corresponding subspace in W^1P^1M) exactly the above mentioned standard reduction $P^2M \hookrightarrow W^1P^1M$. In particular, we get the standard second order soldering form $\theta^{(2)}$ on P^2M with vanishing torsion, cf. [Kobayashi, 72] or [Kolář, Michor, Slovák, 93, p. 156].

Let us now assume that $\nu\colon P_{[\gamma]} \to P^2M$ is a reduction of the second order frame bundle to the structure group $B = B_0 \rtimes B_1$. Since the torsion of the soldering form $\theta^{(2)}$ vanishes, $\nu^*\theta^{(2)} = \theta_{-1} \oplus \theta_0$ is a soldering form with vanishing torsion on $P_{[\gamma]}$. But then for each local section $\sigma\colon P_0 \to P$, the pullback $\sigma^*\theta_0$ is a linear connection without torsion on P_0. Now we can use a partition of the unity to glue such local connections with vanishing torsions to a global one.

On the other hand, if we have started with a linear connection γ on P_0 with vanishing torsion then, by the construction, the values of $\hat{\gamma}$ are in the image of P^2M under the above reduction and so we can define uniquely the dashed arrow in the diagram. Since the reduction of W^1P^1M to P^2M can be achieved by means of the same connection, considered as a connection on the whole P^1M, this implies all the required properties.

3. Harmonic prolongations of almost Hermitian structures

As already mentioned, our initial motivation comes from the study of the almost Hermitian symmetric structures. Let us now specify the results of the previous section in these special cases. So let us assume that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a grading of the Lie algebra of a semisimple Lie group G, $B = B_0 \rtimes B_1$ be the Lie group corresponding to $\mathfrak{g}_0 \oplus \mathfrak{g}_1$, and let M be a manifold endowed with a reduction $P_0 \to P^1M$ to the structure group B_0, $\dim M = \dim \mathfrak{g}_{-1}$. Then also all requirements put on B in the previous section are fulfilled, except in the case of the projective structures, where the first prolongation of $\mathfrak{g}_0$ leads to the whole second order jet group G^2_m, but the structures are reductions of the second order frame bundles P^2M by definition, see e.g. [Ochiai, 70] or [Baston, 91].

3.1. The torsion of a connection γ on P_0 is a section of the modeling vector bundle of the affine bundle J^1P_0. If we consider the torsions as the (vector valued) functions on P_0, i.e. we deal with the frame forms of them, then their values are in $(\mathbb{R}^{m*} \otimes \mathfrak{g}_0) \cap ((\mathbb{R}^{m*} \wedge \mathbb{R}^{m*}) \otimes \mathbb{R}^m)$. According to the general theory, the difference of the torsions of two connections on P_0 lies always in the image of the so called Spencer operator $\partial\colon \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_0 \to (\mathfrak{g}^*_{-1} \wedge \mathfrak{g}^*_{-1}) \otimes \mathfrak{g}_{-1}$ (defined by the alternation) and each value in $\operatorname{Im}\partial$ can be achieved. Thus there is always a connection γ on P_0 such that the $\operatorname{Im}\partial$-part of the torsion vanishes.

The Spencer operator is the special case of the differential in the bigraded Lie algebra cohomology of the abelian algebra $\mathfrak{g}_{-1}$ with values in $\mathfrak{g}$. The cochains are $C^{p,q} = \Lambda^p\mathfrak{g}^*_{-1} \otimes \mathfrak{g}_q$, $\partial\colon C^{p,q} \to C^{p+1,q-1}$. The semisimplicity of $\mathfrak{g}$ and its grading enables us to exchange the roles of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ in the cohomology theory and this gives rise to the adjoint operator ∂^* to ∂ and to the corresponding Hodge theory with the Laplacian $\Box = \partial\partial^* + \partial^*\partial$. Then each $C^{p,q}$ decomposes into $\mathrm{Im}\partial \oplus \mathrm{Ker}\partial^* = \mathrm{Im}\partial \oplus \mathrm{Ker}\Box \oplus \mathrm{Im}\partial^*$ as the $\mathfrak{g}_0$-module, see [Baston, 91] or [Kostant, 61] for further details.

In particular, the torsions have values in $C^{2,-1}$. On this space, the kernel of the Laplacian coincides with the kernel of ∂^* and $C^{2,-1}$ decomposes as $\mathrm{Im}\,\partial \oplus \mathrm{Ker}\,\Box$ as a $\mathfrak{g}_0$-module.

3.2. Definition. Let P_0 be a reduction of P^1M to the structure group B_0. A linear connection γ on P_0 is called a *harmonic connection* if its torsion has values in the subspace $\mathrm{Ker}\,\partial^* = \mathrm{Ker}\,\Box \subset (\mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1}$.

It turns out that for each manifold with an almost Hermitian symmetric structure the ∂^*-closedness condition on the torsion determines uniquely the prolongation:

3.3. Proposition. *For each reduction $P_0 \to M$ of P^1M to the structure group B_0 defining an almost Hermitian symmetric structure on M, there is a unique B-structure $P_{[\gamma]}$ on M over P_0 which is a reduction of W^1P_0 to B and all the distinguished linear connections on P_0 are harmonic. In particular, if P_0 admits a connection with vanishing torsion, then the harmonic connections on P_0 are just the connections without torsion on P_0 and $P_{[\gamma]}$ is a reduction of the second order frame bundle P^2M.*

Proof. Let us fix a linear connection η on P_0. Its torsion decomposes into the $\mathrm{Im}\,\partial$-part and the harmonic part. By the general theory, each other linear connection on P_0 can differ only in the $\mathrm{Im}\,\partial$-part of the torsion and we can always find a connection γ with the vanishing $\mathrm{Im}\partial$-part of the torsion. Thus, there are always harmonic connections on P_0 and all torsions of the harmonic linear connections on P_0 are equal. But then each of them defines the same B-structure $P_{[\gamma]}$, see 2.4. The rest follows from 2.6. □

3.4. Remark. Whenever we fix a complement of $\mathrm{Im}\,\partial$ in $C^{2,-1}$, the previous construction yields a canonical prolongation of any almost Hermitian symmetric structure P_0 with the torsion lying in the prescribed complement. However, our choice of the harmonic torsions leads to a more transparent structure of a B-structure and to the existence of a canonical Cartan connection on P.

A more explicit and direct construction of the prolongation in this case and the explicit construction of the canonical Cartan connections is worked out in the second part of the forthcoming series [Čap, Slovák, Souček]. In fact, the Cartan connections are absolute parallelisms with certain additional properties and the idea behind their construction is to apply the construction of the canonical prolongation once more to the B-structure P. Since the second prolongation of the group B_0 is already trivial, the new soldering form turns out to be such an absolute parallelism. It seems, that this idea should also lead to an iterative procedure in our more general setting.

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