ON THE GEOMETRY OF ALMOST HERMITIAN SYMMETRIC STRUCTURES

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ABSTRACT. The almost Hermitian symmetric structures include several important geometries, e.g. the conformal, projective, quaternionic or almost Grassmannian ones. The conformal case is known best and several efficient techniques have been worked out in the last 90 years. The present note provides links of the development presented in [CSS1, CSS2, CSS3, CSS4] to several other approaches and it suggests extensions of some techniques to all geometries in question.

Since the aim of this paper is to present explicit links between the development in [CSS1, CSS2, CSS3, CSS4] to other approaches established in the literature mostly for the special case of the conformal structures, we have first to review the main concepts and results. Then we extend some of the known techniques to all AHS structures, in particular we discuss the most classical method of the 'variation' of the Riemannian metric in the conformal class and the 'conformal calculus' due to Wünsch and Günther.

1. The AHS structures

The study of the conformal and projective structures on manifolds has a long history and the techniques leading to tensorial invariants and invariantly defined differential operators belonged always to the main aims. These two particular structures were known to allow a common development of the basic theory in the terms of the so called $|1|$-graded Lie algebras, see e.g. [Kob]. A general discussion on geometries associated to such Lie algebras was worked out in [Och]. The algebras in question are semisimple Lie algebras $\mathfrak{g}$ equipped with a grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

Let us fix such an algebra $\mathfrak{g}$, a connected Lie group G with the Lie algebra $\mathfrak{g}$, its subgroup B with the Lie algebra $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the normal subgroup B_1 corresponding to $\mathfrak{g}_1$ and the group $B_0 = B/B_1$ with the Lie algebra $\mathfrak{g}_0$. The associated geometric structures on manifolds M are reductions of the linear frame bundles P^1M to the structure group B_0, except the projective structures where $B_0 = GL(m,\mathbb{R})$ and we have to consider the reductions of P^2M to the structure group B. The flat models for such geometries are the Hermitian symmetric spaces G/B and we shall call such structures the *almost Hermitian symmetric structures*, following [Bas2], briefly the

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AHS structures. The main idea behind Ochiai's development was to construct a principal fiber bundle with structure group B, in a functorial way, and to discuss whether there are distinguished connections on this bundle which should then play the role of the Levi-Cività connections on the Riemannian linear frame bundles. It follows from the semisimplicity of $\mathfrak{g}$ that $\mathfrak{g}_1$ is the classical first prolongation of $\mathfrak{g}_0$ and the next prolongation is already trivial. Thus, the group B encodes the information about all derivatives of morphisms of the first order B_0-structures and the bundle P should be a 'higher order frame bundle' for the structures in question. Ochiai applied the classical prolongation theory of first order structures and he looked for the reduction of the second order frame bundle P^2M to the structure group B. Thus he had to preassume that his B_0-structures admit a connection without torsion, which led to a vaste simplification of the whole theory. He used the standard Lie algebra cohomology of the abelian $\mathfrak{g}_{-1}$ with coefficients in $\mathfrak{g}$ (the Spencer cohomology) to discuss the obstructions against the constructions of P and the so called Cartan connection on P, and to normalize the chosen Cartan connection.

In the study of the conformal and projective structures, the bundle P and the Cartan connection ω turn out to be unique. This fact was heavily used by Kobayashi in his study of the transformation groups. The first very explicit use of the existence of an analogy to the Riemannian connections appeared probably in [Bas1], however his arguments and even some formulations of the new results are very unprecise. Baston discusses the conformal structures only, but he remarks, that in fact the approach should be extendable to other AHS structures as well. He comes back to this point in two papers [Bas2, Bas3]. Since the torsion of the linear connections compatible with the structures is quite often the only obstruction against the local flatness, the Ochiai's assumption was too much restrictive. Baston developed a quite different approach to the canonical connection which mimics the local twistor transport in the conformal case. This means that he constructs an intrinsic connection on an auxiliary vector bundle with structure group G defined canonically by the B_0-structure in question. Thus he avoids the explicit construction of the principal B-bundle P and his connection is equivalent to the canonical one on the extension of P to the structure group G, whenever the Ochiai's P and ω exist.

The other possibility of extending the Ochiai's work to all AHS structures was to construct the canonical bundles PM without referring to the second order frame bundle P^2M. This was solved completely in [CSS2] and the key observation was that it was not the cohomology but simple linear algebra arguments (equivalent to the harmonicity requirements used by Ochiai in the torsion-free cases, but different in general) which led to the construction of P, equipped with a canonical form, via the standard classical prolongation of the B_0-structure. Moreover, the next prolongation leads to the same P (since the next prolongation of the algebra is trivial), but the next canonical form is just the required Cartan connection ω. This construction enables us to use ω to differentiate sections of bundles induced from representations of B in a very straightforward way (as roughly indicated already in [Bas1]). Of course, this is essentially possible in the Baston's approach as well, however we have first to embed our representation of B into a composition series of a representation of G and to control also the behavior of the other components. The obstructions against the local flatness are always certain components of the curvature of the canonical connection and one can get very explicit descriptions of them, see [Bas2] and [CS].

Another construction of the bundles P, which works even in more general cases than the AHS structures, uses the general technique of the principal prolongations of principal bundles, see [Slo].

2. The underlying connections of an AHS structure and natural operators

On each manifold with conformal Riemannian structure, the linear connections which preserve the metrics in the conformal class up to a multiple form an affine space modeled on one-forms (the difference of the Levi-Cività connections determined by two metrics in the class is expressed by the derivatives of the rescaling functions). This has a complete analogy in general as discussed in details in [Bas2].

The way how one-forms and torsions appear is very transparent in the frame bundle approach: Whenever $P \to M$ is a principal B-bundle, then there is also the principal B_0-bundle $P_0 := P/B_1$ over M, the space of all global B_0-equivariant sections $\sigma\colon P_0 \to P$ is non-empty and it is an affine space modeled on the space of all one-forms on M. Moreover, if P is equipped with a Cartan connection ω compatible with the soldering form on P_0, then for each such σ, the pullback of the $\mathfrak{g}_0$-component of ω is a linear connection on M and the pullback of the $\mathfrak{g}_{-1}$-component is the torsion of this connection. In particular, all such connections have the same torsion, see [CSS1, Lemma 3.6] for the details.

Thus a choice of an equivariant section $P_0 \to P$ is equivalent to the choice of a connection from the distinguished class for all AHS structures, generalizing in this way the well known correspondence for the frame bundles $P^2M \to P^1M \to M$.

In general, it is not evident what should be a good definition for the 'invariant operators'. The general notion of the natural operators as those differential operators on sections of bundles induced by representations of B which commute with the induced actions of morphisms of the geometric structures in question is not satisfactory since there are no morphisms except identities (even locally) on 'nearly all' manifolds with the AHS structures. Thus the standard naturality approach covers just the locally flat AHS-structures, cf. [KMS]. The standard trick used in the discussion of the Riemannian manifolds, namely to include the structure itself as another argument for the operators and to apply the naturality with respect to all locally invertible smooth mappings excludes many representations of B and so we loose most of the operators.

Another possibility is to deal with the class consisting of those differential operators expressed exclusively by means of a connection from the distinguished class and independent on its choice. In particular, this is the setting generally accepted in the conformal case, where most authors work with polynomial expressions in the covariant derivatives and curvatures with respect to any metric from the conformal class.

3. Admissible Cartan connections and the invariant differential

Let us fix a principal fiber bundle $P \to M$ with structure group $B \subset G$ and denote by ζ_X the fundamental vector field corresponding to $X \in \mathfrak{b}$. A $\mathfrak{g}$-valued one form $\omega \in \Omega^1(P, \mathfrak{g})$ with the properties

(1) $\omega(\zeta_X) = X$ for all $X \in \mathfrak{b}$
(2) $(r^b)^*\omega = \operatorname{Ad}(b^{-1}) \circ \omega$ for all $b \in B$
(3) $\omega|_{T_uP} \colon T_uP \to \mathfrak{g}$ is a bijection for all $u \in P$

is called a *Cartan connection.* Thus the Cartan connections are absolute parallelizms with the proper invariance properties.

In particular, for each vector $X \in \mathfrak{g}$ there is the vector field $\omega^{-1}(X)$ on P. The standard differentiation of smooth functions on P in direction of these parallel vector fields can be viewed as a mapping $C^\infty(P, V) \to C^\infty(P, \mathfrak{g}^* \otimes V)$. If V is a representation space of B and if we differentiate a section $s\colon P \to V$ of the induced associated bundle (i.e. s obeys the proper equivariance properties), then the differentiation with $X \in \mathfrak{g}_0 \oplus \mathfrak{g}_1$ expresses the equivariance of s, while we obtain an analogy of the covariant derivative in the direction of a vector field for $X \in \mathfrak{g}_{-1}$. In the particular case of an affine connection on the linear frame bundle (which is an example of a Cartan connection) we recover the standard covariant derivative. Thus we use the notation $\nabla^\omega s(X)$ or $\nabla^\omega_X s$ for such derivatives. By the definition, this procedure can be iterated and we obtain the *kth invariant differential*

$$(\nabla^\omega)^k \colon C^\infty(P, V) \to C^\infty(P, \otimes^k \mathfrak{g}^*_{-1} \otimes V).$$

The curvature $K \in \Omega^2(P, \mathfrak{g})$ of a Cartan connection ω is defined by the structure equation $d\omega = -\frac{1}{2}[\omega, \omega] + K$. A direct computation using property (2) of Cartan connections shows that the curvature is always a horizontal 2-form, i.e. it vanishes if one of the vectors is vertical. Thus it is fully described by the function $\kappa \in C^\infty(P, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}^*_{-1} \otimes \mathfrak{g})$, $\kappa(u)(X, Y) = K(\omega^{-1}(X), \omega^{-1}(Y))(u)$. In particular for $X, Y \in \mathfrak{g}_{-1}$, we obtain $\kappa(u)(X, Y) = -\omega(u)([\omega^{-1}(X), \omega^{-1}(Y)])$, so the components of κ with values in $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ are the obstructions against integrability of the horizontal distribution defined by ω.

The defining bundle $P \to M$ of an AHS structure comes always equipped with the canonical form $(\theta_{-1} \oplus \theta_0) \in \Omega^1(P, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$. The component θ_{-1} is always the pullback $p^*(\theta)$ of the canonical soldering form on P_0 with respect to the standard projection $p\colon P \to P_0$. This canonical form has exactly the equivariance properties of the first two components of a Cartan connection on P. The Cartan connections $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$ with ω_1 arbitrary are called the *admissible Cartan connections.* In particular, each linear connection $\gamma = \sigma^*(\theta_0)$ in the distinguished class on P_0 induces a unique admissible Cartan connection $\tilde\gamma = \theta_{-1} \oplus \theta_0 \oplus \omega_1$ on P with ω_1 vanishing on $T\sigma(TP_0) \subset TP$. The pullback $\sigma^*\kappa_{-1}$ of the $\mathfrak{g}_{-1}$-part of the curvature of $\tilde\gamma$ is the torsion of γ, while $\sigma^*\kappa_0$ is the curvature of γ (in fact $\theta \oplus \gamma$ is a Cartan connection on P_0 σ-related to $\tilde\gamma$).

Since any two admissible Cartan connections ω, $\bar\omega$ on P differ only in the $\mathfrak{g}_1$-component, there must always be a function $\Gamma \in C^\infty(P, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1)$ such that $\bar\omega = \omega - \Gamma \circ \theta_{-1}$. Since Γ turns out to be a pullback of a tensor on P_0, we call it the *deformation tensor* (deforming ω into $\bar\omega$).

It is quite easy to express the relations of the differentials and curvatures of $\bar\omega$

and ω evaluated in elements in $\mathfrak{g}_{-1}$, see [CSS1, Lemma 3.10] for details:

$$\bar{\omega}^{-1}(X)(u) = \omega^{-1}(X)(u) + \zeta_{\Gamma(u).X}(u) \tag{4}$$

$$(\bar{\kappa}_{-1} - \kappa_{-1})(u)(X,Y) = 0 \tag{5}$$

$$(\bar{\kappa}_0 - \kappa_0)(u)(X,Y) = -[X,\Gamma(u).Y] - [\Gamma(u).X,Y] \tag{6}$$

$$(\bar{\kappa}_1 - \kappa_1)(u)(X,Y) = \nabla^\omega_Y\Gamma(u).X - \nabla^\omega_X\Gamma(u).Y - \Gamma(u)(\kappa_{-1}(X,Y)) \tag{7}$$

In particular, the formulae (5)–(7) enable us to compute the explicit values of Γ deforming a given distinguished connection $\gamma = \sigma^*(\theta_0)$ into the canonical Cartan connection (which is normalized by a vanishing trace condition on the curvature). These computations are explicitly done in [CSS2]. For the conformal structures we obtain exactly the well known tensor Γ_{ij} which is expressed in terms of the Ricci curvature of γ by

$$\Gamma_{ij} = \frac{-1}{m-2}\left(R_{ij} - \frac{\delta_{ij}}{2(m-1)}R\right) \tag{8}$$

and which was used by many authors 'because of its nice transformation properties'.

Let us fix, for a moment, an admissible Cartan connection ω on P and a linear connection $\gamma = \sigma^*(\theta_0)$ on P_0. For each B_0 representation space V corresponding to a $\mathfrak{g}_0$-representation $\lambda\colon \mathfrak{g}_0 \to \mathfrak{gl}(V)$ we can easily compare the iterated covariant differential of a section $s \in C^\infty(P_0,V)$ and the iterated invariant differential of the expression $p^*s \in C^\infty(P,V)$ of the same section, in the terms of the tensor Γ deforming $\tilde{\gamma}$ into ω. The first derivative evaluated at $u \in P$ and $X \in \mathfrak{g}_{-1}$ is

$$\nabla^\omega(p^*s)(u)(X) = \nabla^\gamma s(p(u))(X) + \lambda([X,\tau(u)])(s(p(u))) \tag{9}$$

where $\tau\colon P \to \mathfrak{g}_1$ is determined by the section $\sigma\colon P_0 \to P$ and is given by the formula $u = \sigma(p(u)).\exp(\tau(u))$.

By induction, we can iterate such comparison for higher order derivatives. The main technical point is the Lemma 4.4 in [CSS1]:

Lemma 1. *Let V and λ be as before and let $f\colon P \to V$ be a mapping defined by*

$$f(u) = \tilde{f}(p(u))(\tau(u),\dots,\tau(u)),$$

where $\tilde{f}\colon P_0 \to \otimes^k\mathfrak{g}_1^* \otimes V$ is $\mathfrak{g}_0$-equivariant with respect to the canonical action $\tilde{\lambda}$ on the tensor product. Then

$$\begin{aligned}\nabla^\omega_Y f(u) = {}& \lambda([Y,\tau(u)])(f(u)) - \\ & \tfrac{1}{2}\sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,[\tau(u),[\tau(u),Y]],\dots,\tau(u)) + \\ & (p^*(\nabla^\gamma_Y\tilde{f}))(u)(\tau(u),\dots,\tau(u)) + \\ & \sum_{i=1}^k (p^*\tilde{f})(u)(\tau(u),\dots,\Gamma(u).Y,\dots,\tau(u)).\end{aligned}$$

Moreover, all the terms in the above expression for $\nabla^\omega f \colon P \to \mathfrak{g}_{-1}^* \otimes V_\lambda$ satisfy the assumptions of this lemma with the corresponding canonical representation on $\otimes^t \mathfrak{g}_1^* \otimes \mathfrak{g}_{-1}^* \otimes V_\lambda$, where t is the number of τ's entering the term in question.

For example, a direct application of this lemma yields the full expansion of the second invariant differential:

$$\begin{aligned}
&(\nabla^\omega)^2(p^*s)(u)(X,Y) = p^*((\nabla^\gamma)^2 s)(u)(X,Y) + \lambda([X,\Gamma(u).Y])(p^*s(u)) + \\
&\lambda([X,\tau(u)])(p^*(\nabla^\gamma_Y s))(u) + (\lambda^{(1)}([Y,\tau(u)])(p^*(\nabla^\gamma)s)(u))(X) + \\
&\lambda^{(1)}([Y,\tau(u)])(\lambda([_,\tau(u)])(p^*s)(u))(X) - \tfrac{1}{2}\lambda([X,[\tau(u),[\tau(u),Y]]])(p^*s)(u)
\end{aligned} \tag{10}$$

where $\lambda^{(k)}$ means the induced representation on the tensor product $\otimes^k \mathfrak{g}_{-1}^* \otimes V$.

4. The jets and the obstruction method

Let us consider a representation λ of B on V and write $E_\lambda \to M$ for the associated bundles to the defining principal B-bundles $P \to M$ of the AHS structure. It is remarkable that there is no natural identification of higher order jet prolongations $J^k E_\lambda$, $k > 1$, with the induced associated bundles from a representation constructed in the homogeneous case. However, there are natural semi-holonomic jet prolongations of each bundle $E_\lambda \to M$ which are closely related to the invariant differentials with respect to Cartan connections, first discussed in [CSS1].

For any fixed Cartan connection ω on P, the mapping $C^\infty(P,V)^B$, $s \mapsto (s, \nabla^\omega s)$, identifies the first jet prolongation $J^1 E_\lambda$ with the associated bundle induced by a B-representation (the latter representation is understood easily in the flat homogeneous case $G \to G/B$ with the standard Maurer–Cartan form ω — it corresponds to the standard functorial jet prolongation). Since the above identification of one-jets works for all B-representations, we can consider these identifications for the iterated applications of the functor J^1. Moreover, there are always subbundles of the so called semi-holonomic jets $\bar{J}^k E_\lambda \subset (J^1 \dots J^1) E_\lambda$ and the mappings $s \mapsto (s, \nabla^\omega s, \dots, (\nabla^\omega)^k s)$ are sections of these bundles. We write also $\bar{J}^k V$ for the corresponding B-modules. See [CSS1, Section 5] for the details on the algebraic construction of the semi-holonomic jet prolongation $\bar{J}^k V$ of the representation space V and the relation to the invariant derivative.

For another B-representation W inducing the associated bundles E_μ, each B-module homomorphism $\Phi \colon \bar{J}^k V \to W$ yields the zero order operator $\bar{J} E_\lambda \to E_\mu$ and its composition with the standard inclusion $J^k E \subset \bar{J}^k E$ provides us then with a natural differential operator. The composition of such a homomorphism with the iterated invariant differential viewed as an operator with values in the semi-holonomic jets yields always a natural operator which transforms sections into sections. On the other hand, whenever we find such a B–homomorphism Φ, then the canonical Cartan connection ω yields a natural operator.

In particular, the problem gets easier if λ and μ are irreducible B-representations. Then there is a bijective correspondence between the B-module homomorphisms on the jets and the B_0-homomorphisms

$$\Phi \colon \bar{J}^k V \supset \otimes^k \mathfrak{g}_{-1}^* \otimes V_\lambda \to V_\mu$$

vanishing on the image of the B_1-action on $\bar{J}^k V$.

Unfortunately, not all natural operators arise in such a simple algebraic way. The best known example is the conformally invariant extension of the flat second power of the Laplacian on four-dimensional conformal Riemannian manifolds, see [ESi] and [ESl]. The existence of such operators is related to the fact, that the values of iterated invariant differentials do not fill the whole space of non-holonomic jets and one should try to refine the jets to submodules with more symmetries. The background for doing that is to be found in [CSS1], but a general 'algebraic' theory of such exceptional operators is not available yet. One could believe that such examples are rather rare in the even dimensional conformal geometry and we do not know any in the other geometries.

On the other hand, the calculus of the comparison of the invariant differentials with the covariant ones suggests an obvious method for searching for invariants which seems to be discussed first in [CSS1, CSS2]. As before, let us consider B_0-representation spaces V, W corresponding to the representations λ and μ of $\mathfrak{g}_0$, and write $E_\lambda \to M$ and $E_\mu \to M$ for the induced associated bundles.

Everything constructed by means of the canonical Cartan connection ω on $P \to P_0 \to M$ should lead to natural operators (in any of the definitions), but the invariant derivatives of the sections of the natural bundles E_λ are no more sections in general, i.e. our procedure of taking the invariant derivatives is not 'covariant'. This is equivalent to the statement that the resulting expression is not $\mathfrak{g}_1$-invariant, in general. On the other hand, once we expand the invariant differentials in terms of the covariant derivatives with respect to an underlying connections γ, we obtain an expression built of covariant terms, except the τ entries which concentrate the failure of being $\mathfrak{g}_1$-invariant. Thus the idea is to consider operators of the general form

$$D = \sum_{\ell=0}^{k} A_\ell \circ (\nabla^\omega)^\ell \tag{11}$$

where the 'coefficients' A_ℓ are operators of order zero (which have to be chosen carefully from a suitable class), to expand the iterated differentials in the terms of the underlying connections and the τ's (and the same with the coefficients if necessary), and to discuss under which conditions all the terms involving τ's vanish in the expansion of D. For the sake of simplicity, let us consider now that all A_ℓ are B_0-homomorphisms with values in W, as above (we shall come back to a more general case in Section 6 below). After the expansion, the operator D splits into a sum of terms D_j, j-linear in τ and depending on the choice of the underlying connection γ

$$Ds(u) = D_0 s(\gamma, \Gamma)s(u) + D_1(\gamma, \Gamma, \tau)s(u) + \dots. \tag{12}$$

We call D_0 the *covariant part* of D, while D_j is called the *obstruction part* of order j. An important observation is that if the term D_1 vanishes identically for all γ and τ, then the other D_j, $j > 1$, vanish as well. See [CSS1, Section 4] for details.

Thus, D yields a natural operator if and only if the whole obstruction part of D vanishes. In the special case of (11) where D involves just one power $(\nabla^\omega)^k$, there is the part of the operator D_1 in the expansion, all terms of which contain $(k-1)$st derivatives of s. The algebraic vanishing of this part is the obstruction against the fact that $A_k(\nabla^\omega)^k$ comes from a B-module homomorphism in the way indicated above. At the same time, this is the most simple part of the expansion (12) and it can be written down explicitly without any recurrence procedure, see [CSS1, Section 5]. We call this the *algebraic obstruction term.* The full discussion of the obstruction terms in (12) yields also operators like the second power of the Laplacian mentioned above, however we have to use the special symmetries of the invariant differentials in order to combine the (algebraically different) obstruction terms together.

The direct approach to the obstruction method using the finite-dimensional representation theory has been worked out in the forthcoming paper [CSS3]. This is very efficient for the couples of representations λ and μ such that μ appears as a component in $\bar{J}^k V$ without multiplicities and it yields the existence and closed formulae of a large class of operators of all orders in these cases. For operators of low orders (at least the first and second order operators) we can also obtain full classification lists in this way.

The method also does not restrict to the B_0-representations, on the contrary, for a general representation λ of B we can consider its composition series. Then the equivariance properties of the sections can be written down in a form which satisfies the assumptions of Lemma 1 and so the same recursion procedure for the expansions applies. In particular, one can involve the curvature of the canonical connection ω in this way and there should be an explicit link to the invariance theory developed in [BEGr]. However, these ideas have not been worked out in details yet.

On the other hand, the obstruction method doesn't seem to be efficient for non-existence proofs, in general.

5. The translation procedure

It is well known that on manifolds with the locally flat AHS structures, there is the bijective correspondence between the natural linear differential operators and the homomorphisms of the generalized Verma modules (which are the dual spaces to the infinite jets of the bundles in question). These homomorphisms have been classified by means of the so called Jantzen-Zuckerman translation principle, see [BC]. The main point is, that the description of the natural operators is done explicitly for a few concrete examples (the exterior differential, the Dirac operator and low powers of Laplacian in the conformal geometries) and then these partial results are 'exported' to the whole category by a class of nice functors. At the first look, this procedure fails completely for the general curved spaces because of the lack of the canonical associated bundle structure on the jet prolongations, cf. the discussion above.

But still there are several possibilities to try but, of course, we can hardly hope to cover operators which do not arise from $\mathfrak{b}$-module homomorphisms in this way. In particular, we can switch back to the jet level (instead of their duals) and try to understand better the flat case in this setting, i.e. to understand the differential operators which realize the translations. The first attempt to do this can be found in

[ERi] in the conformal case and we refer to this approach as the *Eastwood's curved translation principle*. These ideas were further extended and reformulated in [Bas1] and another related approach using in fact the canonical Cartan connection on an auxiliary vector bundle is worked out in [BEGo]. A very intricate extension of this approach in the general setting of the AHS structures is published in [Bas3]. (However, the author of the present paper has not met anybody who understands the arguments there in detail.) The original ideas of the Eastwood's approach are worked out in full generality in the framework of the algebraical structure of the semi-holonomic jets in [CSS4].

Another 'translation' idea can be found in [Gov], see the survey paper [BE] on conformal invariants for further details.

The other possibility to extend the translations to the curved case is to stick to the duals of jets, but to replace the standard jets by the non-holonomic ones and to study their algebraical structure in a similar way to that of the generalized Verma modules. The first steps in this direction are done in [ESl].

The construction of the operators essential for the translations is usually related to the various generalizations of the local twistor transport known from the conformal geometry. In fact we have to extend the canonical Cartan connection on P to its extension $\tilde{P}$ to the structure group G. We get a classical principal connection $\tilde{\omega}$ in this way and we can use its covariant derivative on all bundles induced from a representation of the whole group G. The formula is very simple:

$$\nabla^{\tilde{\omega}} s(u)(X) = \nabla^{\omega} s(u)(X) + \tilde{\zeta}_X . s(u) = \nabla^{\omega} s(u) - \lambda(X) \circ s(u)$$

for all $u \in P \subset \tilde{P}$, $X \in \mathfrak{g}_{-1}$. This approach is also a very efficient tool for construction of invariant operators, see e.g. [BE], [BEGo], [Bas3].

6. The variation of the underlying connections

Consider now two B_0-representation spaces V, W with B_0-representations λ and μ. The classical method how to construct the 'invariant operators' is to consider a general (usually polynomial) expression in the covariant derivatives and curvatures of one of the connections in the distinguished class with unknown coefficients and to discuss under which conditions the resulting operator does not change if the chosen connection is replaced by another one. This simple idea turns into a quite powerful method once we observe, that it is enough to compute the change up to the first derivatives only. Moreover, we can identify P_0 with the trivial bundle over its factor P_0' by the action of the one-dimensional center of B_0, with the fiber $\mathbb{R}$. Then the representations λ, μ correspond to representations λ' and μ' of the semisimple part of B_0 together with the given scalar actions w_λ, w_μ of the center. The choice of the underlying connection γ is then equivalent to the choice of the linear connection γ on P_0' and the sections of E_λ and E_μ can be then considered as sections of the bundles $E_{\lambda'}$ and $E_{\mu'}$ which rescale upon the change of γ according to the weights w_λ and w_μ. This is the classical point of view in the conformal geometry.

There are many deep papers on conformally invariant objects using various modifications of this approach, see e.g. [Bra1, Bra2, Bra3], [Feg], [Jak], [Ørs], [Wün].

Let us now discuss how this approach fits into our scheme. Obviously it is natural to vary the chosen equivariant section σ and to study the effect of this change on the distinguished linear connections. It turns out that if we use the calculus for the admissible Cartan connections to compare the two induced connections $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ corresponding to equivariant sections σ_1, σ_2, we get a complete analogy of the transformation rules describing the rescaling of a Riemannian metric in the conformal geometry. In the sequel, the $|1|$-graded algebra $\mathfrak{g}$ will be fixed, but arbitrary, ω will be the canonical Cartan connection on the bundle in question and Γ will be the standard deformation tensor transforming $\tilde{\gamma}$ into ω for an underlying connection γ.

Theorem 1. *Let σ_1 and σ_2 be equivariant sections $P_0 \to P$.*

(i) *For all $u \in P_0$, $\sigma_2(u) = \sigma_1(u).\exp(\Upsilon(u))$ where $\Upsilon\colon P_0 \to \mathfrak{g}_1$ is the frame form of a differential one-form which describes the difference of the corresponding connections γ_1, γ_2.*

(ii) *For each section $s \in C^\infty(P_0, V)^{B_0}$, the covariant derivatives with respect to the induced underlying connections γ_1 and γ_2 satisfy for all $X \in \mathfrak{g}_{-1}$*

$$\nabla^{\gamma_2} s(u)(X) = \nabla^{\gamma_1} s(u)(X) + \lambda([X, \Upsilon(u)]) \circ s(u).$$

(iii) *The standard deformation tensors Γ_1, Γ_2 determined by the connections γ_1 and γ_2 satisfy for all $X \in \mathfrak{g}_{-1}$*

$$\Gamma_2(u)(X) = \Gamma_1(u)(X) - \nabla^{\gamma_1}\Upsilon(u)(X) - \tfrac{1}{2}[\Upsilon(u), [\Upsilon(u), X]].$$

Proof. The first assertion is in fact the defining formula for Υ, see [CSS1, Lemma 3.6] for the proof that this really gives a one-form.

To see the next equality we have to observe, that the induced admissible connections $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ differ only in the $\mathfrak{g}_1$-component. Thus $\nabla^{\tilde{\gamma}_1}(p^*s) = \nabla^{\tilde{\gamma}_2}(p^*s)$. But according to (9),

$$\begin{aligned}
\nabla^{\tilde{\gamma}_2}(p^*s)(\sigma_2(u))(X) &= \nabla^{\gamma_2} s(u)(X) \\
\nabla^{\tilde{\gamma}_1}(p^*s)(\sigma_2(u))(X) &= \nabla^{\gamma_1} s(u)(X) + \lambda([X, \Upsilon(u)]).
\end{aligned}$$

The computation of the difference $\delta\Gamma = \Gamma_2 - \Gamma_1$ of the deformation tensors is a little bit more tricky. In view of (4) we have just to compare the $\mathfrak{g}_1$ components of the horizontal fields $\tilde{\gamma}_1^{-1}(X)$ and $\tilde{\gamma}_2^{-1}(X)$ corresponding to $X \in \mathfrak{g}_{-1}$ in an arbitrary point in the fiber of $P \to P_0$. Let us choose the point $\sigma_2(u)$. The already known difference between γ_1 and γ_2 implies the relation between the horizontal lifts of $X \in \mathfrak{g}_{-1}$

$$T_u P_0 \ni \gamma_2^{-1}(X)(u) = \gamma_1^{-1}(X)(u) - \zeta_{[X, \Upsilon(u)]}(u).$$

By the definition of the induced admissible Cartan connections, $\tilde{\gamma}_2^{-1}(X)(\sigma_2(u)) = T\sigma_2(\gamma_2^{-1}(X)(u))$ and $\sigma_2(u) = \sigma_1(u).\exp(\Upsilon(u))$. Thus we can compute using the equivariance of the parallel vector fields $\tilde{\gamma}_1^{-1}(X)$ and the standard formula for $\operatorname{Ad}(\exp Z).X$, see [CSS1, Section 3] for similar computations. We shall denote by r the right principal action.

$$\begin{aligned}
\tilde{\gamma}_2^{-1}(X)(\sigma_2(u)) &= Tr \circ (\sigma_1, \exp \Upsilon)(\gamma_1^{-1}(X)(u) - \zeta_{[X,\Upsilon(u)]}(u)) \\
&= \zeta_{T\Upsilon\circ(\gamma_1^{-1}(X))(u)}(\sigma_2(u)) + \zeta_{[[X,\Upsilon(u)],\Upsilon(u)]}(\sigma_2(u)) + \\
&\qquad T(r^{\exp \Upsilon(u)}) \circ (\tilde{\gamma}_1^{-1}(X) - \zeta_{[X,\Upsilon(u)]})(\sigma_1(u)) \\
&= \zeta_{T\Upsilon\circ(\gamma_1^{-1}(X))(u)}(\sigma_2(u)) + \zeta_{[[X,\Upsilon(u)],\Upsilon(u)]}(\sigma_2(u)) + \\
&\qquad \tilde{\gamma}_1^{-1}(X + [X,\Upsilon(u)] + \tfrac{1}{2}[\Upsilon(u),[\Upsilon(u),X]])(\sigma_2(u)) - \\
&\qquad \zeta_{([X,\Upsilon(u)]+[\Upsilon(u),[\Upsilon(u),X]])}(\sigma_2(u)) \\
&= \tilde{\gamma}_1^{-1}(X)(\sigma_2(u)) + \zeta_{(T\Upsilon\circ(\gamma_1^{-1}(X))+\frac{1}{2}[\Upsilon(u),[\Upsilon(u),X]])}(\sigma_2(u)).
\end{aligned}$$

Now, the canonical Cartan connection ω satisfies

$$\omega^{-1}(X) = \tilde{\gamma}_1^{-1}(X) + \zeta_{\Gamma_1.X} = \tilde{\gamma}_2^{-1}(X) + \zeta_{\Gamma_2.X}$$

and the above computation yields the negative of the required difference $\Gamma_2 - \Gamma_1$. $\square$

The remarkable fact is, that exactly as in the conformal geometries, the transformation of the tensor Γ involves the negative of the first covariant derivative of Υ and the rest is bilinear in Υ. It was exactly this property which led to classical procedures for the constructions of invariant operators. Basically, since one has to compute the deformation up to the terms linear in Υ only, we can eliminate the higher derivatives of Υ in the transformation rules for the constructed operator by adding terms with the tensor Γ. But this is exactly what our expansion of the invariant differentials automatically does, cf. the last line in the formula in Lemma 1, so that the covariant parts of the iterated invariant derivatives may be considered as good approximations of invariant operators. The next Lemma clarifies further this observation.

Lemma 2. *Let us consider two underlying connections γ_1, γ_2 and the expansion (12) of the invariant diferential $(\nabla^\omega)^k$. Then for each section s of E_λ and each point $u \in P_0$, the difference of the covariant parts of the expansion equals*

$$D_0(\Gamma_2, \gamma_2)s(u) - D_0(\Gamma_1, \gamma_1)s(u) = D_1(\Gamma_1, \gamma_1, \Upsilon)s(u) + \cdots + D_k(\Gamma_1, \gamma_1, \Upsilon)s(u).$$

In particular, the covariant parts of our invariant differentials involve no derivatives of Υ in their variations under the change of the underlying connections.

Proof. Let σ_1 and σ_2 be the equivariant sections of $P \to P_0$ corresponding to γ_1 and γ_2. Expanding the iterated differential with respect to the Cartan connection ω in the point $\sigma_2(u)$ in terms of γ_1 and γ_2, we obtain

$$\begin{aligned}
(\nabla^\omega)^k p^* s(\sigma_2(u) &= D_0(\Gamma_2, \gamma_2)s(u) + 0 \\
&= D_0(\Gamma_1, \gamma_1)s(u) + D_1(\Gamma_1, \gamma_1, \Upsilon)s(u) + \cdots + D_k(\Gamma_1, \gamma_1, \Upsilon)s(u)
\end{aligned}$$

see Theorem 1 for the relation of the τ's from Section 3 and our Υ's. This yields just the required difference and all the obstruction terms involve no derivatives of the τ's by the recurrence procedure. $\square$

Lemma 3. *Let E_λ and E_μ be as above and consider an operator*

$$D(\gamma) = \sum_{\ell=0}^{k} A_\ell(\gamma) \circ (\nabla^\omega)^\ell$$

depending on a choice of the underlying connection γ. Moreover, let us assume that the coefficients $A_\ell \in C^\infty(P, \mathrm{Hom}(\otimes^\ell \mathfrak{g}_{-1}^* \otimes V_\lambda, V_\mu))$ are polynomial expressions in the curvature and covariant derivatives of γ and τ's. If the variation of the covariant part of $D(\gamma)$ in its expansion (12) under the change of the underlying connection γ involves no derivatives of Υ, then also the variations of all the coefficients $A_\ell(\gamma)$ do not involve the derivatives of Υ.

Proof. The coeficient $A_k(\gamma)$ appears at the iterated covariant differential $(\nabla^\gamma)^k$ and there is no other term of this order. Thus the variation of $A_k(\gamma)$ must not involve any derivative of Υ. Thus, variation of the covariant part of $A_k(\gamma)(\nabla^\omega)^k$ does not involve any derivative of Υ, according to Lemma 2. Now the covariant part of the expansion of the difference $D - A_k(\gamma)(\nabla^\omega)^k$ has the same property and so the variation of $A_{k-1}(\gamma)$ must not involve any derivative of Υ. By induction, the Lemma is proved. $\square$

The curvature $\kappa_{-1} \oplus \kappa_0 \oplus \kappa_1 \in C^\infty(P, \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g})$ satisfies all assumptions of Lemma 1 and according to the recurrence procedure for the expansion from Section 3 all iterated invariant derivatives of the individual parts of the curvature can be used for building the coefficients A_ℓ. The next proposition shows that it is enough to allow such coefficients in order to obtain all invariant operators which are available by means of the variation of the underlying connection. Moreover, we give strongly restrictive properties of the possible coefficients A_ℓ.

Theorem 2. *Let F be a polynomial expression in terms of the covariant derivatives and curvatures of the underlying connections on P_0. If the values of F on sections of associated bundles E_λ induced from a B_0 representation λ do not depend on the particular choices of the linear connection, then there are coefficients $A_\ell \in C^\infty(P, \mathrm{Hom}(\otimes^\ell \mathfrak{g}_{-1}^* \otimes V_\lambda, V_\mu))$ expressed by means of the curvature of the canonical Cartan connection ω and its invariant iterated differentials, $\ell = 1, \dots, k$, such that the covariant part $D_0(\gamma, \Gamma)s(u)$ of the expansion of the operator $D = \sum_{\ell=0}^{k} A_\ell \circ (\nabla^\omega)^\ell$ in the terms of the underlying connections coincides with F while the obstruction terms vanish identically. In particular, the top degree coefficient A_k must then be an absolutely invariant tensor.*

Proof. Let us assume first that we are given an expression $\sum_{\ell=0}^{k} A_\ell \circ (\nabla^\omega)^\ell$ for which the covariant part $D_0(\Gamma, \gamma)s$ of the expansion does not depend on the choice of the connection γ. Let us fix such a connection γ_1 and compare the expansions for this one and another connection γ_2. Let us write σ_1 and σ_2 for the corresponding equivariant sections $P_0 \to P$. For all $u \in P_0$ we have

$$\begin{aligned}\sum_{\ell=0}^{k} A_\ell \circ (\nabla^\omega)^\ell s(\sigma_2(u)) &= D_0(\Gamma_1, \gamma_1)s(u) + D_1(\Gamma_1, \gamma_1, \Upsilon)s(u) + \dots \\ &= D_0(\Gamma_2, \gamma_2)s(u) + 0\end{aligned}$$

But since both the operator D and its covariant part D_0 are independent of the choice of γ, all the obstruction terms in the expansion on the first line must vanish. For fixed γ_1 we can achieve each value of Υ by a proper choice of γ_2, so all the obstructions vanish identically for the connection γ_1. But γ_1 was arbitrary.

So the Theorem will be proved, once we verify that each expression F independent of the choice of γ is available among the covariant parts of the expansions of our operators D.

By induction, the covariant part of the expansion of the difference $(\nabla^\omega)^k(p^*s) - p^*(\nabla^\gamma)^k s$ is an operator of order at most $k-2$. Thus all expressions F built of the iterated covariant differentials $(\nabla^\gamma)^\ell$ and the covariant parts of the curvatures κ_{-1}, κ_0, κ_1 of ω and their covariant derivatives are available among the covariant parts of our operators D. Now we have to recall the explicit formulae (5)–(7) for the deformations of the curvature of the admissible Cartan connections. An obvious consequence is that κ_{-1} is the pullback of the common torsion t_γ of all underlying connections, the covariant part of κ_0 is the trace-free part of the curvature of γ, i.e. the Weyl curvature (its explicite formula $C(X,Y) = R(X,Y) - [X, \Gamma.Y] - [\Gamma.X, Y]$ is visible from (6), see the discussion in Section 3 and [CSS1]), while the covariant part of the $\mathfrak{g}_1$-part is $2\,\mathrm{Alt}(\nabla^\gamma\Gamma) - \Gamma\circ t_\gamma$, a generalization of the so called Cotton-York tensor well known from the conformal Riemannian geometry. The covariant parts of the iterated invariant differentials of the curvature yield then the covariant derivatives of these expressions. Moreover, each antisymmetrization in the arguments of the ℓ's iterated differential yields an expression of order $\ell - 2$ involving another curvature term. Thus we can restric ourselves to the symmetrized invariant differentials of the three parts of the curvature only.

Assume for a moment that also the deformation tensor Γ itself and the full symmetrizations $\mathrm{Sym}(\nabla^\gamma)^\ell\Gamma$ may appear in the coefficients (more explicitly, the pullbacks of these quantities). Since the curvature of γ can be always recovered from its trace–free part and the deformation tensor Γ, and also the antisymmetrization of $\nabla^\gamma\Gamma$ is available from κ_1 once we admit the tensor Γ, the above induction argument shows, that all differential operators built polynomially of ∇^γ and the curvature of γ are available now among the covariant parts of the expansions of D. So let us fix a choice of A_ℓ leading to the given expression F. Now, according to Lemma 3, the variation of an arbitrary coefficient A_ℓ under the change of the connection γ cannot involve derivatives of Υ.

The torsion is covariant and has variation zero. The trace-free part of γ has the same property. The variation of the alternation of the first derivative of Γ could be read of formula in (iii) of Theorem 1. However, it turns out to be much easier to follow the same argument as in Lemma 2, i.e. to express $\kappa_1(\sigma_2(u))(X,Y)$ first in terms of γ_2 and then in terms of γ_1. The comparison then yields the required variation which obviously does not involve any derivative of Υ. Moreover, we have seen that all three parts of the curvature satisfy the assumptions of Lemma 1, see [CSS1, Lemmas 3.8 and 3.10] for more details. Thus also the covariant parts of all iterated invariant derivatives of them have the same property.

On the contrary, an easy induction shows that the variation of the symmetrized covariant derivatives $\mathrm{Sym}((\nabla^\gamma)^\ell\Gamma)$ involves the term $-(\nabla^\gamma)^{\ell+1}\Upsilon$ and there are no other linear terms in Υ there, see (iii) of Theorem 1. Since there are no derivatives of Υ among the derivatives of κ, the whole contribution of the symmetrized derivatives of Γ vanishes whenever these terms cancel.

Finally, if the linear obstruction term in the expansion of the top degree coefficient A_k is non–zero, then this term appears as a coefficient at $(\nabla^\gamma)^k$ and there is no other term of degree k in ∇^γ which could cancel with this one. Thus the linear obstruction term of A_k vanishes identically. $\square$

This theorem shows that the obstruction technique recovers exactly the 'conformal calculus' developed by Wünsch and Günther for conformal Riemannian manifolds of dimensions $m \geq 4$. This calculus was probably the finest version of the method of variations of the Riemannian metric in the conformal class. In our approach, we have obtained a straightforward extension of this calculus to all structures in question (in particular for the three–dimensional Riemannian conformal geometries).

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