ON LOCAL FLATNESS OF MANIFOLDS WITH AHS–STRUCTURES

Andreas Čap, Jan Slovák

Abstract. The AHS–structures on manifolds are the simplest cases of the so called parabolic geometries which are modelled on homogeneous spaces corresponding to a parabolic subgroup in a semisimple Lie group. It covers the cases where the negative parts of the graded Lie algebras in question are abelian. In the series [Čap, Slovák, Souček, 94, 95], the authors developed a consistent frame bundle approach to the subject. Here we give explicite descriptions of the obstructions against the flatness of such structures based on the latter approach. In particular we recover the complex real-analytic results from [Baston] in the real smooth setting.

AMS Classification: 53C10, 53C05

1. Introduction

This note is an addendum to the series of papers [Čap, Slovák, Souček, 94, 95]. In the second paper of this series we have shown how to construct a canonical Cartan connection on a manifold with an almost Hermitian symmetric structure, and we observed that the classical theory of prolongations of G–structures implies that such a structure is locally isomorphic to the homogeneous flat model if and only if this canonical Cartan connection has zero curvature.

The curvature of the canonical Cartan connection naturally splits into three parts according to the $|1|$–grading of the Lie algebra under consideration. Using the known results about the Spencer cohomologies and the calculus for Cartan connections developed in [Čap, Slovák, Souček, 94], it is rather easy to analyze, which of these parts are true obstructions and which vanish automatically. Moreover, for each of the structures in question we can compute these obstructions explicitly in terms of any of the underlying linear connections belonging to the distinguished class.

We will use the notations of [Čap, Slovák, Souček, 94, 95], and citations starting with I or II refer to the corresponding items in parts I and II of this series.

2. $|1|$–graded Lie algebras and Spencer cohomology

2.1. We start with a semisimple real $|1|$–graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and consider the Spencer cohomology $H^*(\mathfrak{g}_{-1}, \mathfrak{g})$, which is just the Lie algebra cohomology

The paper is in final form and no version of it will be submitted elsewhere.
Second author supported by the grant Nr. 201/93/2125 of the GAČR.

of the abelian Lie algebra $\mathfrak{g}_{-1}$ with coefficients in the module $\mathfrak{g}$. The standard complex for computing this cohomology is given by $C^k := \Lambda^k \mathfrak{g}_{-1}{}^* \otimes \mathfrak{g}$ with the differential $\partial : C^k \to C^{k+1}$ defined by

$$\partial\varphi(X_0, \dots, X_k) := \sum_{i=0}^{k} (-1)^i [X_i, \varphi(X_0, \dots, \widehat{X_i}, \dots, X_k)],$$

where we view C^k as the space of k–linear maps $\mathfrak{g}_{-1}{}^k \to \mathfrak{g}$.

Now the grading of $\mathfrak{g}$ clearly induces a grading on each of the spaces C^k, by putting $C^{k,\ell} := \Lambda^k \mathfrak{g}_{-1}{}^* \otimes \mathfrak{g}_\ell$ for $\ell = -1, 0, 1$. Obviously the differential ∂ satisfies $\partial(C^{k,\ell}) \subset C^{k+1,\ell-1}$. Thus we get an induced grading on the cohomology $H^*(\mathfrak{g}_{-1}, \mathfrak{g}) = \oplus_\ell H^{*,\ell}(\mathfrak{g}_{-1}, \mathfrak{g})$.

Note that $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$, the adjoint action makes $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ into $\mathfrak{g}_0$–modules, and the differential ∂ is actually a homomorphism of $\mathfrak{g}_0$–modules for the induced structures. In particular, this implies that the cohomology spaces are $\mathfrak{g}_0$–modules.

2.2. It is well known (see [Ochiai, lemma 3.3]) that for a semisimple $|1|$–graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, the Cartan Killing form induces an isomorphism $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}{}^*$ of $\mathfrak{g}_0$–modules. Now let $\{\xi_i\}$ be a basis of $\mathfrak{g}_{-1}$ and let $\{\zeta_i\}$ be the dual basis of $\mathfrak{g}_1$. Following [Kostant], we define an operator $\partial^* : C^{k,\ell} \to C^{k-1,\ell+1}$ by putting

$$\partial^*\varphi(X_1, \dots, X_{k-1}) := \sum_i [\zeta_i, \varphi(\xi_i, X_1, \dots X_{k-1})]$$

It can be shown, see [Ochiai, proposition 4.1], that the operator ∂^* is the adjoint of ∂ with respect to a certain inner product on the complex C^*. In particular, this implies that the kernel of ∂^* and the image of ∂ are complementary subspaces.

2.3. The construction of the adjoint operator ∂^* and the resulting Hodge theory for the Spencer cohomology is a crucial step in the computation of this cohomology, which was first done in [Kostant], see also [Ochiai, section 5]. What we will need in the sequel is just which components of the second cohomology are nontrivial. This is determined in [Baston] in the complex case, and by [Ochiai, lemma 2.4] the result is the same in the real case.

3. AHS–structures and the canonical Cartan connection

3.1. Let $\mathfrak{g}$ be a $|1|$–graded Lie algebra as above, and let G be a connected Lie group with Lie algebra $\mathfrak{g}$. By B, B_0 and B_1 we denote the Lie subgroups of G corresponding to $\mathfrak{b} := \mathfrak{g}_0 \oplus \mathfrak{g}_1$, $\mathfrak{g}_0$ and $\mathfrak{g}_1$, respectively.

Now let $P_0 \to M$ be a first order B_0–structure on a smooth manifold M which has the same dimension as $\mathfrak{g}_{-1}$. In [Čap, Slovák, Souček, 95] we have shown how to construct from this a principal bundle $P \to M$ with structure group B, which is called the first prolongation of $P_0 \to M$. Moreover the projection $P \to M$ factors over P_0 and $P \to P_0$ is a principal B_1–bundle. Note that in the case of projective structure this prolongation cannot be constructed from the first order part (which actually contains no information) but it has to be chosen as an ingredient of the structure.

3.2. Recall the definition of a Cartan connection on $P \to M$. This is a $\mathfrak{g}$–valued one form $\omega \in \Omega^1(P, \mathfrak{g})$ such that

(1) $\omega(\zeta_X) = X$ for all $X \in \mathfrak{b}$, where ζ_X denotes the fundamental vector field corresponding to X.
(2) $(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in B$, where r^b denotes the right principal action with b and Ad denotes the adjoint action.
(3) $\omega|_{T_uP} : T_uP \to \mathfrak{g}$ is a bijection for all $u \in P$.

The curvature $K \in \Omega^2(P, \mathfrak{g})$ of such a Cartan connection is defined by the structure equation $d\omega = -\frac{1}{2}[\omega, \omega] + K$. In I.2.1 we have shown that the curvature is completely described by the function $\kappa : P \to \mathfrak{g}_{-1}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}$, which is defined by $\kappa(u)(X, Y) = K(\omega^{-1}(X), \omega^{-1}(Y))(u)$.

3.3. In the second section of [Čap, Slovák, Souček, 95] it is shown that for all structures but the one dimensional projective ones, there is a unique Cartan connection ω on the first prolongation $P \to M$, such that $\partial^*(\kappa_{-1}(u)) = 0$ and $\partial^*(\kappa_0(u)) = 0$ for all $u \in P$, where we split $\kappa = \kappa_{-1} + \kappa_0 + \kappa_1$ according to the grading of $\mathfrak{g}$. This is called the canonical Cartan connection. Since $\partial^*(\kappa_1(u))$ is automatically zero (the relevant ∂^* has values in the zero space), the canonical Cartan connection is characterized by the fact that $\partial^* \circ \kappa = 0$.

For any group G as above, there is a canonical flat model of the corresponding structure. This is the homogeneous space G/B, and the bundle $G \to G/B$ is the first prolongation of the flat structure. In this case, the canonical Cartan connection is the Maurer–Cartan form, and the Maurer–Cartan equation says that this has zero curvature. Moreover, an AHS–manifold is locally flat, i.e. locally isomorphic (as a B_0–structure) to the flat model if and only if its canonical Cartan connection has zero curvature, see II.2.4.

3.4. Next recall from I.2.4 the Bianchi identity for the curvature of any Cartan connection:

$$\sum_{\text{cycl}} ([\kappa(X, Y), Z] - \kappa(\kappa_{-1}(X, Y), Z) - \nabla^\omega_Z \kappa(X, Y)) = 0,$$

where $\sum_{\text{cycl}}$ denotes the sum over all cyclic permutations and $X, Y, Z \in \mathfrak{g}_{-1}$. Now the first term in this equation can be rewritten as

$$-[Z, \kappa(X, Y)] + [Y, \kappa(X, Z)] - [X, \kappa(Y, Z)] = -(\partial\kappa)(X, Y, Z).$$

Splitting the resulting equation for $\partial\kappa$ according to the grading of $\mathfrak{g}$ we arrive at the following four equations (recall that $\partial\kappa_\ell$ has values in $\mathfrak{g}_{\ell-1}$):

$$(\partial\kappa_{-1})(X, Y, Z) = 0 \tag{1}$$

$$(\partial\kappa_0)(X, Y, Z) = -\sum_{\text{cycl}} \big(\kappa_{-1}(\kappa_{-1}(X, Y), Z) + \nabla^\omega_Z \kappa_{-1}(X, Y)\big) \tag{2}$$

$$(\partial\kappa_1)(X, Y, Z) = -\sum_{\text{cycl}} \big(\kappa_0(\kappa_{-1}(X, Y), Z) + \nabla^\omega_Z \kappa_0(X, Y)\big) \tag{3}$$

$$0 = -\sum_{\text{cycl}} \big(\kappa_1(\kappa_{-1}(X, Y), Z) + \nabla^\omega_Z \kappa_1(X, Y)\big) \tag{4}$$

Here ∇^ω denotes the absolutely invariant differentiation introduced in I.2.3.

Using these formulae it is now fairly easy to discuss, which parts of the curvature of the canonical Cartan connection are actual obstructions to local flatness and which vanish automatically as follows: First we see that for the component κ_{-1} we have $\partial\kappa_{-1} = 0$, so κ_{-1} is a Spencer–cocycle in $C^{2,-1}$. On the other hand, $\partial^*\kappa_{-1} = 0$ and the kernel of ∂^* is complementary to the image of ∂. Thus we see that κ_{-1} vanishes automatically if $H^{2,-1}(\mathfrak{g}_{-1}, \mathfrak{g}) = 0$, and is a true obstruction otherwise.

Next, let us assume that $\kappa_{-1} = 0$. Then, according to equation (2), this implies that κ_0 is a cocycle, so as before we conclude that this vanishes automatically if $H^{2,0}(\mathfrak{g}_{-1}, \mathfrak{g}) = 0$ and is a true obstruction otherwise.

Finally, if both κ_{-1} and κ_0 vanish then the equation (3) shows that κ_1 is a cocycle, so this vanishes automatically if $H^{2,1}(\mathfrak{g}_{-1}, \mathfrak{g}) = 0$ and is a true obstruction otherwise.

3.5. We shall give explicit expressions for the obstruction terms in terms of any of the connections from the so called distinguished class of connections on $P_0 \to M$. The connections in this class are in bijective correspondence with the space of all B_0–equivariant sections of $P \to P_0$ and they can be parametrized by exterior 1–forms on M. This bijection is given by mapping a section σ to the connection with connection form $\sigma^*\omega_0 \in \Omega^1(P_0, \mathfrak{g}_0)$, where ω_0 is the $\mathfrak{g}_0$–component of the canonical Cartan connection on $P \to M$, see I.3.6 and II.1.7.

There is also an alternative description of the distinguished class of connections in all cases except the projective structures: Recall that since $P_0 \to M$ is a first order B_0–structure one can assign a torsion to any principal connection on this bundle, which can be viewed as a smooth function $P_0 \to {\mathfrak{g}_{-1}}^* \wedge \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$, see II.1.7. The distinguished connections are then precisely those for which the torsion in each point is ∂^*–closed. Moreover, there is only one possible torsion function in each case, so all connections in the distinguished class have the same torsion (in fact this torsion function is the structure function of the B_0–structure $P_0 \to M$), and the pullback of this function to P is precisely the component κ_{-1} of the curvature of the canonical Cartan connection. For example, in the case of the conformal pseudo–Riemannian structures the distinguished connections are exactly the Levi-Cività connections of the metrics from the conformal class.

On the other hand, having given a connection γ from the distinguished class and the corresponding section $\sigma : P_0 \to P$, we can form the induced Cartan connection $\tilde\gamma$ on $P \to M$, see I.3.7. The pullback of the curvature of this induced Cartan connection to P_0 is just the curvature and torsion of γ, see I.3.8.

3.6. From the above discussion it is clear that the first obstruction to local flatness (corresponding to κ_{-1}) is the existence of a torsion free principal connection on $P_0 \to M$.

Now let us assume that this first obstruction vanishes, take a torsion free connection γ on $P_0 \to M$ corresponding to a section σ: $P_0 \to P$ and let $\tilde\gamma$ be the induced Cartan connection on $P \to M$. The difference between this induced Cartan connection and the canonical one is described by the deformation tensor $\Gamma \in C^\infty(P, {\mathfrak{g}_{-1}}^* \otimes \mathfrak{g}_1)$ which is always a pullback of a tensor on P_0, see I.3.9. Formulae (4), (5) and (6) of I.3.10 give an explicit description of the effect of the deformation tensor on the curvatures

($\bar{\kappa}$ is the 'deformed' curvature):

$$(\kappa_{-1} - \bar{\kappa}_{-1})(u)(X, Y) = 0 \tag{5}$$

$$(\kappa_0 - \bar{\kappa}_0)(u)(X, Y) = [X, \Gamma(u).Y] + [\Gamma(u).X, Y] \tag{6}$$

$$(\kappa_1 - \bar{\kappa}_1)(u)(X, Y) = \nabla^{\omega}_X \Gamma(u).Y - \nabla^{\omega}_Y \Gamma(u).X + \Gamma(u)(\kappa_{-1}(X, Y)) \tag{7}$$

Moreover, in the torsion free case the equivariance properties of the curvature components derived in I.3.8 are

$$\kappa_0(u)(X, Y) = \kappa_0(\sigma(p(u)))(X, Y) \tag{8}$$

$$\kappa_1(u)(X, Y) = [\kappa_0(\sigma(p(u)))(X, Y), \tau(u)] \tag{9}$$

where $\tau(u) \in \mathfrak{g}_1$ is qiven by the equality $u = \sigma(p(u)).\exp(\tau(u))$. Thus if the torsion of the canonical Cartan connection is zero, then it suffices to compute κ_0 on $\sigma(P_0) \subset P$, where we already know that the curvature of $\tilde{\gamma}$ is just given by the curvature and torsion of γ. Furthermore, by the construction of the canonical Cartan connection as that one with ∂^*–closed curvature, the achieved $\mathfrak{g}_0$–part κ_0 coincides on $\sigma(P_0)$ exactly with the trace–free part of the curvature of the underlying connection γ, the so called Weyl curvature tensor.

4. Obstructions against local flatness

In section 3 of [Čap, Slovák, Souček, 95] we have computed explicitly the deformation tensor Γ giving the canonical Cartan connection for several real forms of the main complex series of the AHS–structures, in terms of the Ricci curvature tensor of a chosen distinguished connection γ. In this section we derive the explicit results on the obstructions for these individual structures. The splitting into the various cases is dictated by the different second cohomology groups. On the other hand, as mentioned above the vanishing of the second cohomologies is independent of our choice of the real forms, thus the discussion below aplies to all of them, for a classification list see [Ochiai, section 7]. In particular, the obstruction coming from κ_{-1} is always the torsion of the underlying linear connections, while that one corresponding to κ_0 is the Weyl curvature (the trace free part) of them. Let us notice, that all the found obstruction tensors are invariants of the structures in question.

Let us first start with the $\mathfrak{sl}(p+q)$ series, the corresponding structures are called almost Grassmannian (the flat models are the Grassmannian manifolds). We do not discuss the case of one–dimensional projective structures, since there is no canonical Cartan connection in this case. As stated before we take the results on the second cohomology from [Baston, table 2].

4.1. Two–dimensional projective structures. This is the special case $p = 1$, $q = 2$ of an almost Grassmannian structure, see I.3.3, so $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$. In this case the cohomologies $H^{2,-1}(\mathfrak{g}_{-1}, \mathfrak{g})$ and $H^{2,0}(\mathfrak{g}_{-1}, \mathfrak{g})$ are trivial, while $H^{2,1}(\mathfrak{g}_{-1}, \mathfrak{g})$ is nonzero. Thus in this case, there always is a torsion free connection γ in the distinguished class on $P_0 \to M$ (in fact P_0 is the whole first order frame bundle P^1M), and the only obstruction comes from κ_1. Writing $\omega = \tilde{\gamma} - \Gamma \circ \theta_{-1}$ for the canonical Cartan connection, we get from II.3.9 the formula $\Gamma_{jk} = R^l_{jlk} + R^l_{ljk}$ for the uniquely defined deformation tensor, where R^i_{jkl} means the curvature tensor of γ. By (9) the $\mathfrak{g}_1$–component of the curvature of $\tilde{\gamma}$ is trivial on $\sigma(P_0)$, so that (7) gives the curvature component $\bar{\kappa}_1$ of the canonical Cartan connection:

$$\bar{\kappa}_1(u)(X,Y) = \nabla^{\tilde{\gamma}}_Y\Gamma(u).X - \nabla^{\tilde{\gamma}}_X\Gamma(u).Y. \tag{10}$$

Considering Γ as a a tensor on P_0 (which we actually already did above) we see from I.3.8.(1) that we may replace the invariant differentials $\nabla^{\tilde{\gamma}}$ (still on $\sigma(P_0)$) by covariant derivatives with respect to γ. Using the Bianchi identity for principal connections and the Ricci tensor $R_{jk} = R^l_{jlk}$ of γ yields $R^l_{ljk} = R_{jk} - R_{kj}$, and so we obtain $\Gamma_{jk} = 2R_{jk} - R_{kj}$. Thus, the coordinate expression for the only obstruction against the flatness of a two–dimensional projective structure is the tensor

$$t_{jkl} = 2R_{jk;l} - R_{kj;l} + 2R_{jl;k} - R_{lj;k}. \tag{11}$$

Notice that if the choosen connection happens to be a Riemannian one, than the tensor t_{jkl} is the symmetrization of the first covariant differential of the Ricci curvature.

4.2. Higher dimensional projective structures. Now we deal with the cases $p = 1$, $q > 2$ of almost Grassmannian structures, see I.3.3, so $\mathfrak{g} = \mathfrak{sl}(q+1,\mathbb{R})$. In this case the cohomologies $H^{2,-1}(\mathfrak{g}_{-1},\mathfrak{g})$ and $H^{2,1}(\mathfrak{g}_{-1},\mathfrak{g})$ are trivial, while $H^{2,0}(\mathfrak{g}_{-1},\mathfrak{g})$ is nonzero in general. Thus in this case, there always is a torsion free connection γ on $P_0 \to M$ in the distinguished class, and the only obstruction against the flatness comes from κ_0. Thus the vanishing of the Weyl curvature tensor W^i_{jkl} of γ (cf. the end of 3.6) is equivalent to the local flatness of the projective structures in dimensions greater than two.

4.3. Structures related to the quaternionic manifolds. We deal with another special case of the almost Grasmannian structures where $\mathfrak{g} = \mathfrak{sl}(2,q,\mathbb{R})$, $q \geq 2$.

First assume $q = 2$ (so that we consider a real form of $\mathfrak{so}(6,\mathbb{C})$). Only the cohomology $H^{2,0}(\mathfrak{g}_{-1},\mathfrak{g})$ is nonzero. Thus the only obstruction against the local flatness is the Weyl curvature tensor of any of the underlying linear connections on P_0.

If $q > 2$, then the cohomology $H^{2,1}(\mathfrak{g}_{-1},\mathfrak{g})$ is trivial, while both $H^{2,-1}(\mathfrak{g}_{-1},\mathfrak{g})$ and $H^{2,0}(\mathfrak{g}_{-1},\mathfrak{g})$ are nonzero in general. Thus there are two tensors which obstruct the flatness of the structure: the torsion and the Weyl curvature tensor of any of the underlying linear connections on P_0.

4.4. Higher dimensional Grassmannian structures. In the cases of $\mathfrak{g} = \mathfrak{sl}(p+q,\mathbb{R})$, $3 \leq p \leq q$, the only nonzero second cohomology is $H^{2,-1}(\mathfrak{g}_{-1},\mathfrak{g})$. Thus the only obstruction against the flatness is the torsion of the underlying connections. This means that the structure in question is locally flat if and only if it admits a torsion free linear connection.

Now the remaining structures from the main series:

4.5. Conformal Riemannian structures. The Lie algebra in question is $\mathfrak{g} = \mathfrak{so}(p+1,q+1,\mathbb{R})$, where $p+q = m > 2$ is the dimension of the manifolds. In the case of three-dimensional conformal pseudo–Riemannian structures the only nonzero second cohomology is $H^{2,1}(\mathfrak{g}_{-1},\mathfrak{g})$, so that we are in a situation analogous to that of two–dimensional projective structures. Thus the only obstruction comes from κ_1 and it is given by formula (10). However, now the deformation tensor Γ is the tensor

$$\Gamma_{ij} = \frac{-1}{m-2}\left(R_{ij} - \frac{\delta_{ij}}{2(m-1)}R\right).$$

Thus we get the well known obstruction against local flatness, the Cotton–York tensor $\Gamma_{ij;k} - \Gamma_{ik;j}$.

If the dimension is bigger than three, then the only nonzero second cohomology is $H^{2,0}(\mathfrak{g}_{-1},\mathfrak{g})$ and so we have recovered the well known fact that a conformal pseudo–Riemannian manifold of dimension $m \geq 4$ is locally (conformally) flat if and only if the Weyl curvature of one (and thus any) Riemannian connection from the conformal class vanishes.

4.6. Almost Lagrangian structures. The corresponding Lie algebra is $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{R})$, the manifolds are modelled over $S^2\mathbb{R}^n$. The three-dimensional case (i.e. $n = 2$) is isomorphic to that one of the three dimensional conformal Riemannian structures, so the appropriate obstruction is the Cotton-York tensor.

In all higher dimensions, the only nonzero second cohomology is $H^{2,-1}(\mathfrak{g}_{-1},\mathfrak{g})$, thus the only obstruction is the existence of a torsion free linear connection of the structure.

4.7. Almost spinorial structures. Now $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{R})$, $n \geq 5$ (the lower dimensional cases coincide with some previous ones, e.g. we get the six-dimensional conformal Riemannian structures for $n = 4$). Also in this case the only nonzero second cohomology is $H^{2,-1}(\mathfrak{g}_{-1},\mathfrak{g})$. Thus the almost spinorial structures are locally flat if and only if they admit a torsion free linear connection.

We have not studied in detail the cases of the $|1|$–graded exceptional Lie algebras, but the general theory applies as well.

References

Baston, R. J., *Almost Hermitian symmetric manifolds, I: Local twistor theory*, Duke Math. J. **63** (1991), 81–111.

Čap, A.; Slovák, J.; Souček, V., *Invariant operators on manifolds with almost hermitian symmetric structures, I. invariant differentiation*, Preprint ESI **186** (1994).

Čap, A.; Slovák, J.; Souček, V., *Invariant operators on manifolds with almost hermitian symmetric structures, II. normal Cartan connections*, Preprint ESI **194** (1995).

Kostant, B., *Lie algebra cohomology and the generalized Borel–Weil theorem*, Ann. Math. **74** (1961), 329–387.

Ochiai, T., *Geometry associated with semisimple flat homogeneous spaces*, Trans. Amer. Math. Soc. **152** (1970), 159–193.

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, 1090 Wien, Austria

Department of Algebra and Geometry, Masaryk University in Brno, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic