ARCHIVUM MATHEMATICUM (BRNO)
Tomus 32 (1996), 327 – 344

PRINCIPAL PROLONGATIONS AND GEOMETRIES MODELED ON HOMOGENEOUS SPACES

Jan Slovák

To Ivan Kolář, on the occasion of his 60th birthday.

Abstract. We discuss frame bundles and canonical forms for geometries modeled on homogeneous spaces. Our aim is to introduce a geometric picture based on the non-holonomic jet bundles and principal prolongations as introduced in [Kolář, 71]. The paper has a partly expository character and we focus on very general aspects only. In the final section, various links to known results on the parabolic geometries are given briefly and some directions for further investigations are roughly indicated.

Introduction

The classical G-structures are defined as reductions of the frame bundles P^rM to structure groups G (usually called higher order structures if $r > 1$). As a consequence, certain torsions of such structures vanish. These notions generalize easily to reductions of semi-holonomic frame bundles, and even to reductions of holonomic, semi-holonomic, or non-holonomic principal prolongations as reviewed below. Then we can deal with all torsions quite nicely. These ideas can be traced back up to Cartan and Ehresmann and an explicit treatment of them was given in [Kolář, 71]). Several authors used similar constructions later.

Here we aim to discuss a very general framework for curved geometries modeled on a given homogeneous space G/B, viewed as certain deformations of the Maurer-Cartan form on G. Thus our objects will be principal fiber bundles P with the structure group B equipped with a B-equivariant absolute parallelism $TP \to \mathfrak{g}$ reproducing fundamental fields on P. Such objects are usually called Cartan connections of the type G/B. This paper has been inspired by our recent study of the so called parabolic geometries, i.e. the cases where B is a parabolic subgroup in a semisimple group, we restrict ourselves to very general aspects however. We hope to describe a general setting suitable for a wider range of problems after appropriate refinements. Actually, the papers [Tanaka, 79], [Morimoto, 93], [Čap,

1991 *Mathematics Subject Classification*: 58A20, 53A55.
Key words and phrases: jet prolongation, principal prolongation, Cartan connection.
This paper was worked out during the author's ARC Research Fellowship at the University of Adelaide. Research also supported by GAČR, grant Nr. 201/96/0079 .

Schichl] present essentially complete solutions to our problems in the parabolic case.

Since dealing with a very general setting, we do not present any deep theorems. Rather we focus at geometric constructions of objects which we believe to be useful. Any application to a particular geometrical problem requires further refinements of our objects. We try to indicate certain possibilities for such modifications in the final section.

Let us illustrate our attempts on most simple but rather typical examples, the conformal Riemannian and almost Grassmannian geometries. Both can be defined as reductions of the linear frame bundles to the appropriate subgroups in the general linear group and the above mentioned principal bundles P equipped with the Cartan connections ω are constructed from these data, see e.g. [Tanaka, 70], [Baston, 91], [Čap, Slovák, Souček, 95]. There are two basic options for such constructions, either we try to construct P and ω as abstract objects without any auxiliary bundles, or we try to localize them as reductions of certain 'universal bundles' equipped with canonical forms. All the above mentioned papers took the first option, here we discuss a fairly general background for the other approach. Classically, the higher order (holonomic) frame bundles were considered, which was applicable under vanishing of certain torsions, see e.g. [Kobayashi, 72]. However, the existence of a non-vanishing torsion excludes this approach even for the almost Grasmannian geometries. Moreover, already in the conformal case we cannot obtain the canonical Cartan connections via reductions of higher order holonomic frame bundles, in general. On the other hand, the canonical Cartan connections for both these structures are available via reductions of third order semi-holonomic frame bundles. In the third section of this paper, we show that each principal fiber bundle equipped with a Cartan connection ω can be uniquely obtained in a similar way. Thus the semi-holonomic frame bundles can be considered as the universal bundles.

The first section is preparatory, we discuss the homogeneous spaces as suitable canonical reductions of (holonomic) frame bundles. Next, we study the canonical forms on the semi-holonomic frame bundles. Most ideas in Section 2 appeared at least implicitly in [Kolář, 71a], [Kolář, 71b], [Kolář, 75a], [Kolář, 75b]. The infinite semi-holonomic frame bundles are in fact essentially equivalent to a special case of the universal towers in [Morimoto, 93], but we believe that our categorical treatment will allow a wider range of refinements.

The author would like to thank Andreas Čap and Ivan Kolář for helpful discussions.

1. Homogeneous spaces as reductions of frame bundles

1.1. Our first goal is to describe homogeneous spaces as canonical reductions of the frame bundles. Let us fix a finite dimensional Lie group G and its closed subgroup $B \subset G$, and write $m = \dim G/B$. We also choose a fixed complementary vector space $\mathfrak{n}_- \subset \mathfrak{g}$ to the subalgebra $\mathfrak{b}$ so that $\exp_{|\mathfrak{n}_-}$ is a locally defined diffeomorphism $\mathfrak{n}_- \to G/B$ on a neighborhood of zero. In order to relate the left Maurer-Cartan form on G with the canonical forms on the (holonomic) frame bundles, we need to fix the right identification of B and $\mathfrak{b}$ with subgroups and subalgebras in the jet groups and jet algebras G^k_m and $\mathfrak{g}^k_m$, respectively.

Let us write $P^kM = \operatorname{inv} J^k_0(\mathbb{R}^m, M)$ for the kth order (holonomic) frame bundle on m-dimensional manifolds M. Since $P^kM \subset J^1(\mathbb{R}^m, P^{k-1}M)$ in a canonical way, there is the canonical form $\theta^{(k-1)} \in \Omega^1(P^kM, \mathbb{R}^m \oplus \mathfrak{g}^{k-1}_m)$, see the detailed discussion of more general concepts in Section 2 below.

The left multiplication ℓ_g by elements in G determines the canonical mappings

$$\begin{aligned}
\nu^k &: G \to P^k(G/B)\\
&\quad g \mapsto j^k_0(\ell_g \circ \exp_{|\mathfrak{n}_-}) \in J^k_0(\mathfrak{n}_-, G/B)_{[g]}\\
\iota^k &: B \to G^k_m\\
&\quad b \mapsto j^k_0\big((\exp_{|\mathfrak{n}_-})^{-1} \circ \ell_b \circ \exp_{|\mathfrak{n}_-}\big) \in J^k_0(\mathfrak{n}_-, \mathfrak{n}_-)_0.
\end{aligned}$$

On a neighborhood of the unit, the mapping ι^k extends by the same formula to a mapping $\iota^k : U \subset G \to J^k_0(\mathfrak{n}_-, \mathfrak{n}_-)$. The tangent mapping to ι^k at the unit in G provides the canonical mapping $\mathfrak{g} \to T_{j^k_0 \operatorname{id}}(P^k\mathfrak{n}_-)$

$$\mathfrak{g} \ni X = \tfrac{\partial}{\partial t}_{|0}(\exp tX) \mapsto \tfrac{\partial}{\partial t}_{|0} j^k_0\big((\exp_{|\mathfrak{n}_-})^{-1} \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-}\big)$$

which is always injective on $\mathfrak{n}_-$. We shall also write $\iota^k : \mathfrak{g} \to \mathbb{R}^m \oplus \mathfrak{g}^k_m$ for this mapping, as well as for its restriction $\mathfrak{b} \to \mathfrak{g}^k_m$ to the Lie algebra $\mathfrak{b}$. Since the principal fiber bundle automorphisms $\ell_g : G \to G$ correspond to right invariant vector fields on G on the Lie algebra level, ι^k can be also described as the projection of these vector fields onto the k-jets of the underlying vector fields. The elements in $T_{j^k_0 \operatorname{id}}(P^k\mathfrak{n}_-) \simeq \mathbb{R}^m \oplus \mathfrak{g}^k_m$ can be viewed as right invariant vector fields along the fiber over $0 \in \mathfrak{n}_-$. Let us write ℓ_b for the obvious action of elements $b \in G^{k+1}_m$ on these vector fields.

Obviously, $\nu^k : G \to P^k(G/B)$ are homomorphisms of principal fiber bundles with the corresponding homomorphisms ι^k between the structure groups.

1.2. Lemma. *The following diagram commutes for all $b \in B$ and $k \geq 0$:*

$$\begin{array}{ccccc}
\mathfrak{g} & \xrightarrow{\iota^k} & T_{j^k_0 \operatorname{id}}(P^k\mathfrak{n}_-) & = & \mathbb{R}^m \oplus \mathfrak{g}^k_m\\
\Big\downarrow{\scriptstyle \operatorname{Ad}_b} & & & & \Big\downarrow{\scriptstyle \ell_{\iota^{k+1}(b)}}\\
\mathfrak{g} & \xrightarrow{\iota^k} & T_{j^k_0 \operatorname{id}}(P^k\mathfrak{n}_-) & = & \mathbb{R}^m \oplus \mathfrak{g}^k_m
\end{array}$$

Proof. Let us compute $\ell_{\iota^{k+1}(b)} \circ \iota^k(\frac{\partial}{\partial t}_{|0} \exp tX)$.

$$
\begin{aligned}
\tfrac{\partial}{\partial t}_{|0} \exp tX \mapsto\ & \ell_{j_0^{k+1}(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_b \circ \exp_{|\mathfrak{n}_-})}\big(\tfrac{\partial}{\partial t}_{|0} j_0^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-})\big) \\
&= T_{j_0^k \mathrm{id}}(P^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_b \circ \exp_{|\mathfrak{n}_-}))\big(\tfrac{\partial}{\partial t}_{|0} j_0^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-})\big) \\
&= \tfrac{\partial}{\partial t}_{|0} j_0^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_b \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-}) \in T_{\iota^k(b)} P^k(\mathfrak{n}_-) \\
&\simeq Tr^{\iota^k(b^{-1})} . \tfrac{\partial}{\partial t}_{|0} j_0^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_b \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-}) \in T_{j_0^k \mathrm{id}} P^k(\mathfrak{n}_-) \\
&= \tfrac{\partial}{\partial t}_{|0} j_0^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_b \circ \ell_{\exp tX} \circ \ell_{b^{-1}} \circ \exp_{|\mathfrak{n}_-}) \\
&= \iota^k \circ \mathrm{Ad}_b(X) \quad \square
\end{aligned}
$$

1.3. The trivial filtrations. The Lie group B and its Lie algebra $\mathfrak{b}$ carry the compatible filtrations $B = F^0B \supset F^1B \supset \dots$ and $\mathfrak{b} = F^0\mathfrak{b} \supset F^1\mathfrak{b} \supset \dots$ determined by the exact sequences

$$1 \longrightarrow F^k\mathfrak{b} \hookrightarrow \mathfrak{b} \xrightarrow{\iota^k} \mathfrak{g}_m^k \longrightarrow 1$$

$$1 \longrightarrow F^kB \hookrightarrow B \xrightarrow{\iota^k} G_m^k \longrightarrow 1$$

We set $F^{-1}G = G$, $F^{-1}\mathfrak{g} = \mathfrak{g}$ and $F^kG = F^kB$, $F^k\mathfrak{g} = F^k\mathfrak{b}$, $k \geq 0$. So G becomes a filtered Lie group and F^kG are normal subgroups in F^0G with Lie algebras $F^k\mathfrak{g}$ for all $k > 0$.

We say[1] that the *order of the homogeneous space* is k, if k is the smallest integer with $F^k\mathfrak{g} = \{0\}$. The homogeneous space is said to be *infinitesimally effective* if $\cap_{k=0}^{\infty} F^k\mathfrak{g} = \{0\}$. An infinitesimally effective space G/B which does not have any finite order is said to have order ∞.

By definition, if the order of G/B is k then the map ν^k is a reduction of the frame bundle $P^k(G/B)$ in the sense that the structure group might be a covering of a subgroup in G_m^k (like the spin groups in Riemannian geometries).

1.4. Lemma. *Assume the order of G/B is k. Then, under the identification $\iota^k\colon \mathfrak{g} \simeq \iota^k(\mathfrak{g}) \subset \mathbb{R}^m \oplus \mathfrak{g}_m^k$, the pullback $(\nu^{(k+1)})^*(\theta^{(k)}) \in \Omega^1(G, \mathfrak{g})$ is the left Maurer-Cartan form on G.*

Proof. We have to prove that for each $X \in \mathfrak{g}$, $g \in G$,

$$(\nu^{k+1})^*(\theta^{(k)})(\tfrac{\partial}{\partial t}_{|0} g.\exp tX) = X.$$

Let us consider $X \in \mathfrak{g}$ and the vector $\xi = \frac{\partial}{\partial t}_{|0} g.\exp tX \in T_gG$. By definition

$$
\begin{aligned}
(\nu^{k+1})^*\theta^{(k)}(\xi) &= T(P^k \exp_{|\mathfrak{n}_-}^{-1} \circ P^k \ell_{\mathfrak{g}}^{-1}). \tfrac{\partial}{\partial t}_{|0}\big(P^k(\ell_g \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-})(j_0^k \mathrm{id})\big) \\
&= \tfrac{\partial}{\partial t}_{|0} P^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-})(j_0^k \mathrm{id}) \\
&= \tfrac{\partial}{\partial t}_{|0} j_0^k(\exp_{|\mathfrak{n}_-}^{-1} \circ \ell_{\exp tX} \circ \exp_{|\mathfrak{n}_-}) \\
&= \iota^k(X) \in \mathbb{R}^m \oplus \mathfrak{g}_m^k. \quad \square
\end{aligned}
$$

[1] Our definition of the order is very closely related to the *order of isotropy* of the homogeneous space as defined in [Kolář, 71b]. In fact, this is always finite under the condition that G acts effectively on G/B.

Let us notice the role of the chosen extension of the Lie algebra homomorphism $\iota^k \colon \mathfrak{b} \to \mathfrak{g}^k_m$ to the mapping $\mathfrak{g} \to \mathbb{R}^m \oplus \mathfrak{g}^k_m$. Since ν^{k+1} is a principal fiber bundle homomorphism over the injective ι^{k+1}, the image of the left invariant field given by X is $\zeta_{\iota^{k+1}(X)}$ on $P^{k+1}M$. Thus obviously $(\nu^{k+1})^*\theta^{(k)}(X) = \iota^k(X)$ as required in the formulation of the above lemma. However, the statement for $X \in \mathfrak{n}_-$ relies heavily on our choice. In particular, the other obvious identification $\mathfrak{n}_- \simeq \mathbb{R}^m \oplus \{0\}$ does not work, in general.

2. Jet bundles and principal prolongations

2.1. The functors $W^r, \bar{W}^r, \tilde{W}^r$. There are two basic functors in our development: the functor J^1 associating to each fibered manifold the bundle of 1-jets of local sections and acting on morphisms of fibered manifolds over locally invertible morphisms on the bases, and the functor W^1 which maps each principal bundle P with a fixed structure group into the principal fiber bundle of jets of the local trivializations of P. The action on morphisms is given by jet composition. The fiber over $0 \in \mathbb{R}^m$ in $W^1(\mathbb{R}^m \times G)$ is the Lie group $W^1_m G := (G^1_m \times G) \rtimes (\mathbb{R}^m \otimes \mathfrak{g})$ and jet composition defines the structure of a principal fiber bundle with structure group $W^1_m G$ on $W^1 P$. In particular, the group G embeds into the structure group of $W^1 P$, the projection $p^1_0 \colon W^1(\mathbb{R}^m \times G) \to \mathbb{R}^m \times G$ restricts to a group homomorphism, and p^1_0 is a principal fiber bundle homomorphism. Both functors can be iterated to create the so called non-holonomic rth order jet prolongations $\tilde{J}^r$ and principal prolongations $\tilde{W}^r$ of fibered manifolds and principal bundles, respectively. While the jet prolongations are heavily used in modern geometry, the idea of the principal prolongation introduced in [Kolář, 1971] appears only from time to time under various names.

As pointed out already by Ehresmann, the non-holonomic prolongations offer a general tool to deal with higher order torsions of geometric structures. Since the general non-holonomic prolongations are too big and redundant for most practical problems, the so called semi-holonomic prolongations $\bar{J}^r$, $\bar{W}^r$ have to be introduced. We first define $\bar{J}^1 = J^1$, $\bar{W}^1 = W^1$ and notice that there are canonical natural transformations $(\bar{\pi}^1_0)_Y \colon J^1 Y \to Y$, $(\bar{p}^1_0)_P \colon W^1 P \to P$ to the identity functors. The action of the functor J^1 on the mappings $(\bar{\pi}^1_0)$ defines the natural transformation $(J^1\bar{\pi}^1_0)_Y \colon J^1 J^1 \to J^1$ and $\bar{J}^2$ is defined as the equalizer of two natural transformations $\pi^1_0, J^1\bar{\pi}^1_0 \colon J^1 J^1 \to J^1$. We have defined the functor W^1 only on the category of principal fiber bundles with a fixed structure group G, but we can obviously extend its action to a wider class of morphisms. Indeed, if $\varphi \colon P \to P'$ is a homomorphism over a group homomorphism $\varphi_0 \colon G \to G'$ where the structure group G' is at the same time a subgroup in G, then we define $W^1\varphi(j^1_{(0,e)}\psi) = j^1_{(0,e)}(\varphi \circ \psi)_{|(\mathbb{R}^m \times G')}$. This action is also functorial for all appropriate morphisms. We have seen that $p^1_0 \colon W^1 P \to P$ satisfies these conditions and so $W^1 p^1_0$ is a well defined natural transformation $W^1 W^1 \to W^1$. Now, $\bar{W}^2$ is defined as the equalizer of two natural transformations $p^1_0, W^1 p^1_0 \colon W^1 W^1 \to W^1$.

The higher order semi-holonomic jet prolongations are usually defined recursively. Assume $\bar{J}^k$ comes equipped with the canonical transformation $\bar{J}^k \to \bar{J}^{k-1}$, so that there are two canonical transformations $J^1\bar J^k \to J^1\bar J^{k-1}$, and define $\bar J^{k+1}$ as the equalizer of those two transformations. A simple check shows that this is equivalent to the definition $\bar J^{k+1}Y = \bar J^2(\bar J^{k-1}Y)\cap J^1(\bar J^kY)$. The latter definition can be modified for the principal prolongations as well, i.e. we define

$$\bar W^{k+1}P = \bar W^2(\bar W^{k-1}P)\cap W^1(\bar W^kP).$$

In particular, we obtain a sequence of natural transformations

$$\dots \xrightarrow{\bar p^{r+1}_r} \bar W^r \xrightarrow{\bar p^r_{r-1}} \bar W^{r-1} \xrightarrow{\bar p^{r-1}_{r-2}} \dots \xrightarrow{\bar p^2_1} \bar W^1 \xrightarrow{\bar p^1_0} \mathrm{Id}$$

which are given by the restrictions of the target jet projections. We shall write $\bar p^k_l$ for the composition $\bar p^k_{k-1}\circ\dots\circ\bar p^{l+1}_l$ for all $k > l \geq 0$.

The holonomic rth order principal prolongation W^r is defined exactly as W^1, on replacing 1-jets by r-jets. Clearly W^rP is identified canonically as a subspace $W^rP \subset \bar W^rP \subset \tilde W^rP$.

2.2. Let us also recall the functors $T^r_m = J^r_0(\mathbb R^m,\ )$, $\bar T^r_m$, $\tilde T^r_m = (T^1_m)^r$ of the holonomic, semi-holonomic, and non-holonomic rth order m-velocities, respectively. The principal prolongations W^rP may be viewed as subbundles in the bundles T^r_mP where m is the dimension of the base manifolds. Just observe that each principal bundle morphism (i.e. a local trivialization) $\varphi\colon \mathbb R^m\times G\to P$ is determined by the restriction $\varphi_{|\mathbb R^m\times\{e\}}\colon \mathbb R^m\to P$. As discussed above, $\bar T^r_mM = \bar T^2_m(\bar T^{r-2}_mM)\cap T^1_m(\bar T^{r-1}_mM)$ for all manifolds M and $r\geq 2$. Since the action of all the functors in question is given by the jet compositions, it is easy to see that an element in $\tilde W^rP$ is semi-holonomic or holonomic if and only if it sits in $\bar T^r_mP$ or T^r_mP, respectively.

If we start with the trivial principal fiber bundle $\mathrm{id}_M\colon M\to M$ with the structure group $\{e\}$, we obtain the holonomic rth order frame bundles $P^rM\subset T^r_mM$ on the manifold M, and the semi-holonomic frame bundles[2] $\bar W^rM\subset\bar T^r_mM$.

As already mentioned, the holonomic principal prolongation of a trivial principal bundle $\mathbb R^m\times G$ has the form (up to the natural identifications)

$$W^r(\mathbb R^m\times G) = \mathbb R^m\times (p^r_0)^{-1}_{\mathbb R^m\times G}(0) =: \mathbb R^m\times W^r_mG$$

where W^r_mG turns out to be the structure group of W^rP, for all principal fiber bundles P over m-dimensional manifolds with structure group G. Iterating this observation, we obtain the structure groups $(W^1_m)^r(G)$ of non-holonomic principal prolongations.

Since the semi-holonomic principal prolongations $\bar W^rP$ are defined by means of equalizers of natural transformations, they obey principal fiber bundle structures

[2]These frame bundles can be also constructed directly, without any reference to the more general concept of principal prolongation. The notation $\bar W^rM$ underlines our point of view, while Ehresmann used $\bar H^rM$, and $\bar P^rM$ would be more compatible with the notation in [Kolář, Michor, Slovák, 93].

and their structure groups $\bar{W}^r_m G$ are again the fibers over zero in $\bar{W}^r(\mathbb{R}^m \times G)$. By definition, the Lie groups $\bar{W}^r_m G$ are equipped with the projections $\bar{p}^r_{r-1}\colon \bar{W}^r_m G \to \bar{W}^{r-1}_m G$ given by the two coinciding projections $W^1\bar{p}^1_0$ and $\bar{p}^1_0$.

In particular, starting with the trivial group $G = \{e\}$, we arrive at the structure groups of the semi-holonomic frame bundles, the groups $\bar{G}^r_m$ of all invertible jets in $\bar{J}^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$. This structure groups come equipped with a filtration obtained from the exact sequences

$$(1) \qquad 1 \to \bar{N}^{r,k}_m \to \bar{G}^r_m \to \bar{G}^{k-1}_m \to 1, \qquad r \geq k > 1.$$

The kernel $\bar{N}^{r,r}_m = \otimes^r \mathbb{R}^{m*} \otimes \mathbb{R}^m$ is an abelian normal subgroup in $\bar{G}^r_m$. We shall write $\bar{\mathfrak{g}}^r_m$, $\bar{\mathfrak{n}}^{r,k}_m$ for the corresponding Lie algebras and we might omit the indices r and m, if clear from the context.

Since the element $j^1_{(0,e)}\varphi \in W^1P$, $\varphi\colon \mathbb{R}^m \times G \to P$ is determined equivalently by any jet $j^1_{(0,b)}\varphi$, $b \in G$ we shall often use the 'fiber jet' notation $\mathbf{j}^1_0\varphi$ for the elements in W^1P.

2.3. The canonical forms. Let us review first the canonical forms on the general principal prolongations. Let $P \to M$ be a principal fiber bundle with structure group G, $\dim M = m$, $p^1_0\colon W^1P \to P$ be the target jet projection. The vector space $\mathbb{R}^m \oplus \mathfrak{g}$ can be identified with the space of right invariant vector fields on $\mathbb{R}^m \times G$ along the fiber over zero, let us write ℓ for the canonical action of $W^1_m G$ on these vector fields. The form $\theta = \theta_{\mathbb{R}^m} \oplus \theta_{\mathfrak{g}} \in \Omega^1(W^1P, \mathbb{R}^m \oplus \mathfrak{g})$ is defined for each $\xi \in T_{j^1_{(0,e)}\varphi}(W^1P)$ by $\theta(\xi) = (T_{(0,e)}\varphi)^{-1}(Tp^1_0(\xi)) \in T_{(0,e)}(\mathbb{R}^m \times G)$. So we can view θ as a one-form with values in the right-invariant vector fields mentioned above.

A straightforward computation shows nice properties of these canonical forms see e.g. [Kolář, Michor, Slovák, 93, p. 155] for details.

(1) $\theta_{\mathbb{R}^m}(\xi) = 0$ if and only if ξ is a vertical vector
(2) for each element $X + Y + Z \in \mathfrak{w}^1_m(\mathfrak{g}) \simeq \mathfrak{g}^1_m + \mathfrak{g} + (\mathbb{R}^{m*} \otimes \mathfrak{g})$ we have $\theta_{\mathfrak{g}}(\zeta_{X+Y+Z}) = Y$
(3) θ is equivariant with respect to ℓ i.e. $(r^a)^*(\theta) = \ell_{a^{-1}} \circ \theta$ for all $a \in G$.

The canonical forms on $\bar{W}^{r+1}M$ are defined as restrictions of the forms θ on $W^1(\bar{W}^r M)$ to the tangent spaces $T\bar{W}^{r+1}M$. As a corollary we get quite detailed information on the canonical forms on the semi-holonomic frame bundles:

2.4. Proposition. *The semi-holonomic frame bundles $\bar{W}^{r+1}M$, $r \geq 0$ come equipped with the canonical forms $\theta^{(r)} = \theta_- \oplus \theta_{\bar{\mathfrak{g}}^r_m} \in \Omega^1(\bar{W}^{r+1}M, \mathbb{R}^m \oplus \bar{\mathfrak{g}}^r_m)$ with the following properties*

(1) *$\theta_-(\xi) = 0$ if and only if ξ is a vertical vector*
(2) *for each element $Y \in \bar{\mathfrak{g}}^{r+1}_m$, $\theta_{\bar{\mathfrak{g}}^r_m}(\zeta_Y) = \bar{p}^{r+1}_r(Y) \in \bar{\mathfrak{g}}^r_m$*
(3) *$\theta^{(r)}$ is equivariant with respect to the action ℓ of $\bar{G}^{r+1}_m$ on $\mathbb{R}^m \oplus \bar{\mathfrak{g}}^r_m$, i.e.*

$$(r^b)^*\theta^{(r)} = \ell_{b^{-1}} \circ \theta^{(r)} \quad \textit{for all } b \in \bar{G}^{r+1}_m$$

(4) *for each $k \geq r$ we have the commutative diagram*

$$\begin{array}{ccc} T\bar{W}^{k+1}M & \xrightarrow{\theta^{(k)}} & \mathbb{R}^m \oplus \bar{\mathfrak{g}}^k_m \\ {\scriptstyle T\bar{p}^{k+1}_{r+1}}\downarrow & & \downarrow{\scriptstyle \bar{p}^k_r} \\ T\bar{W}^{r+1}M & \xrightarrow{\theta^{(r)}} & \mathbb{R}^m \oplus \bar{\mathfrak{g}}^r_m \end{array}$$

Proof. Remember $\theta^{(r)}$ is the restriction of the canonical form θ on $W^1(\bar{W}^r M)$ to the tangent space of $\bar{W}^{r+1}M$. Thus (1) and (3) are obvious and (2) follows from 2.3.(2) and 2.2.(1).

In order to prove (4), it suffices to deal with the case $r = k - 1$. Moreover, $\theta^{(r)}$ is the restriction of the canonical form θ on $\bar{W}^2(\bar{W}^{r-1}M)$, so we can discuss the case $r = 2$ with a general bundle P instead of the frame bundles. Let G be its structure group. Any vector $\xi \in T\bar{W}^2P$ is of the form $\frac{\partial}{\partial t}_{|0}\mathbf{j}^1_0\varphi^t$ with $\varphi^t : u \mapsto \mathbf{j}^1_0\psi^t_u$ such that

$$\begin{array}{ccc} W^1(\mathbb{R}^m \times G) & \xrightarrow{\varphi^t} & W^1P \\ \downarrow{\scriptstyle \bar{p}^1_0} & & \downarrow{\scriptstyle \bar{p}^1_0} \\ \mathbb{R}^m \times G & \xrightarrow{\psi^t_{(0,e)}} & P \end{array}$$

commutes (the dashed arrow is the canonical embedding). The definition of the canonical form says we have to take the projection of ξ to W^1P, i.e. $\frac{\partial}{\partial t}_{|0}\varphi^t(\mathbf{j}^1_0\,\mathrm{id}) = \frac{\partial}{\partial t}_{|0}\mathbf{j}^1_0(\bar{p}^1_0 \circ \varphi^t) \in TW^1P$ and then to interpret this as an element in $T_{\mathbf{j}^1_0\,\mathrm{id}}W^1(\mathbb{R}^m \times G)$ via the tangent mapping to φ^0. Similarly for the projection $T\bar{p}^1_0(\xi)$. The situation is described in the following diagram

$$\begin{array}{ccccc} T_{\mathbf{j}^1_0\,\mathrm{id}}W^1(\mathbb{R}^m \times G) & \xrightarrow{T\varphi^0} & TW^1P & \ni & \frac{\partial}{\partial t}_{|0}\bar{p}^1_0\mathbf{j}^1_0\varphi^t \\ \downarrow{\scriptstyle T\bar{p}^1_0} & & \downarrow{\scriptstyle T\bar{p}^1_0} & & \\ T_{(0,e)}(\mathbb{R}^m \times G) & \xrightarrow{T\psi^0_{(0,e)}} & TP & \ni & T\bar{p}^1_0(\frac{\partial}{\partial t}_{|0}\bar{p}^1_0\mathbf{j}^1_0\varphi^t) \end{array}$$

The choice of φ^t and ψ^t guarantees the required commutativity. □

2.5. The infinite semi-holonomic prolongation. The semi-holonomic frame bundles with the canonical projections build a sequence of principal fiber bundles

(1) $$\dots \xrightarrow{\bar{p}^{r+1}_r} \bar{W}^r M \xrightarrow{\bar{p}^r_{r-1}} \bar{W}^{r-1}M \xrightarrow{\bar{p}^{r-1}_{r-2}} \dots \xrightarrow{\bar{p}^2_1} \bar{W}^1M \xrightarrow{\bar{p}^1_0} M.$$

and by restriction we obtain the sequence

(2) $$\dots \xrightarrow{\bar{p}^{r+1}_r} \bar{G}^r_m \xrightarrow{\bar{p}^r_{r-1}} \bar{G}^{r-1}_m \xrightarrow{\bar{p}^{r-1}_{r-2}} \dots \xrightarrow{\bar{p}^2_1} \bar{G}^1_m \xrightarrow{\bar{p}^1_0} 1.$$

By definition, if $\mathbf{j}^1_0\psi \in \bar W^k M$, $\mathbf{j}^1_0\varphi \in \bar G^k_m$ then

$$\begin{aligned}\bar p^k_{k-1}(\mathbf{j}^1_0\psi\circ\mathbf{j}^1_0\varphi) &= \mathbf{j}^1_0(p^1_0\circ\psi\circ\varphi)_{|(\mathbb{R}^m\times\bar G^{k-2}_m)}\\ &= \mathbf{j}^1_0\big((p^1_0\circ\psi)_{|(\mathbb{R}^m\times\bar G^{k-2}_m)}\circ(p^1_0\circ\varphi)_{|(\mathbb{R}^m\times\bar G^{k-2}_m)}\big)\\ &= (\bar p^k_{k-1}\mathbf{j}^1_0\psi)\circ(\bar p^k_{k-1}\mathbf{j}^1_0\varphi).\end{aligned}$$

Thus, also the principal actions are compatible and so the inverse limit $\bar W^\infty_m$ of sequence (1) is a principal fiber bundle with structure group $\bar G^\infty_m$, the inverse limit of sequence (2) of Lie groups. Let us denote the canonical projections $\bar W^\infty M \to \bar W^k M$ by $\bar p^\infty_k$, and the same for $\bar G^\infty_m \to \bar G^k_m$. So in particular, for all $g \in \bar G^\infty_m$, $u \in \bar W^\infty M$ we have $\bar p^\infty_k(u.g) = \bar p^\infty_k(u).\bar p^\infty_k(g)$.

Similarly, the tangent mappings to the projections $T\bar p^{k+1}_k$ define the inverse limit structure on the tangent bundle $T\bar W^\infty M$ and we also get such structures on the trivial principal fiber bundle $\mathbb{R}^m\times\bar G^\infty_m$ and the space $\mathbb{R}^m\oplus\bar{\mathfrak g}^\infty_m \simeq T_{\mathrm{id}}\bar W^\infty\mathbb{R}^m$ of constant right invariant vector fields on the latter bundle (i.e. right invariant fields along the fiber over zero). Let us check that the actions ℓ of $\bar G^{k+1}_m$ on $\mathbb{R}^m\oplus\bar{\mathfrak g}^k_m$ are compatible with the projections $\bar p^k_{k-1}$ and so they define the action ℓ of $\bar G^\infty_m$ on the space $\mathbb{R}^m\oplus\bar{\mathfrak g}^\infty_m$. Indeed, choose $\psi\colon W^1(\mathbb{R}^m\times\bar G^k_m)\to W^1(\mathbb{R}^m\times\bar G^k_m)$ with $\mathbf{j}^1_0\psi\in\bar G^{k+1}_m$ and $\frac{\partial}{\partial t}_{|0}\mathbf{j}^1_0\varphi_t \in T_{\mathbf{j}^1_0\mathrm{id}}\bar W^k\mathbb{R}^m$. Then $p^1_0(\ell_{\mathbf{j}^1_0\psi}\frac{\partial}{\partial t}_{|0}\mathbf{j}^1_0\varphi_t) = \frac{\partial}{\partial t}_{|0}p^1_0\circ\psi(\mathbf{j}^1_0\varphi_t)$ while $\ell_{p^1_0\mathbf{j}^1_0\psi}(Tp^1_0(\frac{\partial}{\partial t}_{|0}\mathbf{j}^1_0\varphi_t)) = \frac{\partial}{\partial t}_{|0}\big(p^1_0\circ\psi_{|\bar W^k\mathbb{R}^m}(\mathbf{j}^1_0(p^1_0\circ\varphi_t)_{|\mathbb{R}^m\times\bar G^{k-1}_m})\big)$. Since the principal fiber bundle morphisms are determined by values on a section, we really obtain the required equality $\ell_{\bar p^{k+1}_k g}(T\bar p^k_{k-1}(X)) = T\bar p^k_{k-1}(\ell_g(X))$ for all $g\in\bar G^{k+1}_m$, $X\in\mathbb{R}^m\oplus\bar{\mathfrak g}^k_m$.

Now, the compatible inverse limit structures on $T\bar W^\infty M$ and $\mathbb{R}^m\times\bar{\mathfrak g}^\infty_m$, and 2.4.(4) yield the canonical form $\theta^{(\infty)} = \theta_-\oplus\theta_{\bar{\mathfrak g}^\infty_m}\in\Omega^1(\bar W^\infty M,\mathbb{R}^m\oplus\bar{\mathfrak g}^\infty_m)$

$$\bar p^\infty_k(\theta^{(\infty)}(X)) = \theta^{(k)}(T\bar p^\infty_{k+1}(X)).$$

We shall not go into details on the smooth manifold structure of these projective limits of finite dimensional manifolds, all important aspects can be found in the forthcoming book [Kriegl, Michor, 97]. Let us mention just that smooth curves are exactly those mappings which project to smooth curves by all $\bar p^\infty_k$ and smooth mappings are those which prolong smoothly the smooth curves. In particular the canonical form $\theta^{(\infty)}$ is smooth.

2.6. Proposition. *The canonical form $\theta^{(\infty)}$ satisfies*

(1) *θ_- is the pullback of the canonical soldering form on $P^1M = \bar W^1M$, in particular $\theta_-(\xi)=0$ if and only if ξ is a vertical vector*
(2) *$\theta^{(\infty)}$ reproduces fundamental vector fields, i.e. for each element $Y\in\bar{\mathfrak g}^\infty_m$, $\theta_{\bar{\mathfrak g}^\infty_m}(\zeta_Y)=Y$*
(3) *$\theta^{(\infty)}$ is equivariant with respect to the induced action ℓ of $\bar G^\infty_m$ on $\mathbb{R}^m\oplus\bar{\mathfrak g}^\infty_m$, the space of right invariant vector fields on the fiber of $\mathbb{R}^m\times\bar G^\infty_m$ over 0.*
(4) *$\theta^{(\infty)}$ defines an absolute parallelism on $\bar W^\infty M$.*

Proof. By definition, $\theta_-(X)=0$ if and only if $\theta_-^{(k)}=0$ for all k. Now (1) follows from 2.4.(4) with $r=0$. In order to see (2) notice

$$\bar p_k^\infty(\theta^{(\infty)}(\zeta_Y)) = \theta^{(k)}(\bar p_{k+1}^\infty\zeta_Y) = \theta^{(k)}(\zeta_{\bar p_{k+1}^\infty Y}) = \bar p_k^\infty Y.$$

Further we have

$$\begin{aligned}\bar p_k^\infty(\theta^{(\infty)}(u.g)(Tr^g.Y)) &= \theta^{(k)}(\bar p_{k+1}^\infty(u.g))(Tr^{\bar p_{k+1}^\infty g}.T\bar p_{k+1}^\infty Y)\\ &= \ell_{\bar p_{k+1}^\infty g^{-1}}\circ\theta^{(k)}(\bar p_{k+1}^\infty(u))(T\bar p_{k+1}^\infty Y)\end{aligned}$$

so that $\theta^{(\infty)}(u.g)(Tr^g.Y)=\ell_{g^{-1}}\circ\theta^{(\infty)}(u)(Y)$ as required in (3).

The last item is obvious. □

One-forms on a manifold P equipped with a Lie group action, which are reproducing the fundamental vector fields of the action and define an absolute parallelism on TP are usually called *Cartan connections*, see e.g. [Alekseevsky, Michor, 95] or [Kobayashi, 72]. Thus we call $\theta^{(\infty)}\in\bar W^\infty M$ the *canonical Cartan connection on the semi-holonomic infinite order frame bundle of* M. Let us notice we should view $\mathbb{R}^m\oplus\bar{\mathfrak g}_m^\infty$ as a $\bar G_m^\infty$-module, rather than the Lie algebra of constant vector fields since the bracket in the latter algebra is not completely compatible with the action ℓ. This is clearly reflected in the structure equations below.

2.7. The structure equation. Each absolute parallelism $\theta\in\Omega^1(P,V)$ on a manifold defines the structure equation $d\theta=\alpha(\theta,\theta)$ with a unique function $\alpha\in C^\infty(P,\Lambda^2V^*\otimes V)$. In our case, $\theta^{(\infty)}$ is right invariant and it reproduces fundamental vector fields, which implies that the corresponding function $\alpha^{(\infty)}$ is also right invariant and for $Y\in\bar g_m^\infty$ we obtain $i_Y\circ\alpha^{(\infty)}=-\ell'_Y\circ\theta^{(\infty)}$, i.e. it restricts to the action of Y on $\mathbb{R}^m\oplus\bar g_m^\infty$ via the tangent mapping to the action ℓ. Now we can split the values $\alpha^{(\infty)}(X,Y)$ on vectors $X=X_-+X_{\bar{\mathfrak g}_m^\infty}$, $Y=Y_-+Y_{\bar{\mathfrak g}_m^\infty}$ into

$$\begin{aligned}\alpha^{(\infty)}(X,Y)= &-\ell'(X_{\bar{\mathfrak g}_m^\infty})(Y_-)+\ell'(Y_{\bar{\mathfrak g}_m^\infty})(X_-)\\ &-[X_{\bar{\mathfrak g}_m^\infty},Y_{\bar{\mathfrak g}_m^\infty}]_{\mathbb{R}^m\oplus\bar{\mathfrak g}_m^\infty}+\alpha^{(\infty)}(X_-,Y_-)\end{aligned}$$

in order to obtain the non-trivial horizontal part of the exterior differential of $\theta^{(\infty)}$. Let us denote the summands on the first line by $-\lambda^{(\infty)}(X,Y)$. Our observations lead to the structure equation of the canonical Cartan connection

$$\begin{aligned}d\theta^{(\infty)} &= \alpha(\theta^{(\infty)},\theta^{(\infty)})\\ &= -\frac{1}{2}[\theta^{(\infty)},\theta^{(\infty)}]_{\mathbb{R}^m\oplus\bar{\mathfrak g}_m^\infty}-\lambda^{(\infty)}(\theta^{(\infty)},\theta^{(\infty)})+\kappa^{(\infty)}(\theta_-^{(\infty)},\theta_-^{(\infty)})\end{aligned}\tag{1}$$

where the bracket is given by the bracket of the corresponding vector fields (i.e. the bracket of the vertical components), λ expresses the interaction of the vertical and horizontal parts of the arguments via the ℓ'-action, and the curvature κ is a $(\Lambda^2\mathbb{R}^{m*}\otimes(\mathbb{R}^m\oplus\bar{\mathfrak g}_m^\infty))$-valued function on $\bar W^{(\infty)}M$ which is $\bar G_m^\infty$ equivariant with respect to ℓ.

2.8. Let us now inspect how much of the curvature $\kappa^{(\infty)}$ is visible already on $\bar W^k M$. Its equivariance and horizontality yield

$$\begin{aligned}\kappa^{(\infty)}(u.g)(X,Y) &= \ell_{g^{-1}}\kappa^{(\infty)}(u)(\bar p_0^\infty(\ell_g X), \bar p_0^\infty(\ell_g Y))\\ &= \ell_{g^{-1}}\kappa^{(\infty)}(u)(\ell_{\bar p_1^\infty g}X, \ell_{\bar p_1^\infty g}Y)\end{aligned}$$

where $\ell_{\bar p_1^\infty g}X$ means the standard action of G^1_m on $\mathbb{R}^m$ and the result is interpreted as a horizontal vector in $\mathbb{R}^m \oplus \bar{\mathfrak g}^\infty_m$.

Since $g \in \bar N^{\infty,k+1}$ implies $\bar p_k^\infty \circ \ell_g = \bar p_k^\infty$, the projections $\bar p_k^\infty \kappa^{(\infty)}(u.g)(X,Y)$ do not depend on the choice of $g \in \bar N_m^{\infty,k+1}$ for any $k \geq 1$. In particular, for all $k \geq 1$ we can define the function

$$\begin{aligned}&\kappa^{(k)} \in C^\infty(\bar W^{k+1}M, \Lambda^2\mathbb{R}^{m*} \otimes (\mathbb{R}^m \oplus \bar{\mathfrak g}^k_m))\\ &\kappa^{(k)}(\bar p^\infty_{k+1}u)(X,Y) = \bar p_k^\infty \kappa^{(\infty)}(u)(X,Y).\end{aligned}$$

Further notice that $\lambda^{(k)}(\bar p^\infty_{k+1}Y, \bar p^\infty_{k+1}X) := \bar p_k^\infty \lambda^{(\infty)}(X,Y)$ is well defined.

Now, applying the projection $\bar p_k^\infty$ to structure equation 2.7 (1), we obtain

Proposition. *The structure equation at $u \in \bar W^{k+1}M$ is*

$$d\theta^{(k)}(u) \simeq \begin{pmatrix} -\dfrac{1}{2}[\theta^{(k)}(u), \theta^{(k)}(u)]_{\mathbb{R}^m\oplus\bar{\mathfrak g}^k_m} - \lambda^{(k-1)}(\theta^{(k)}(u), \theta^{(k)}(u)) \\ + \kappa^{(k)}(u)(\theta_-, \theta_-) \end{pmatrix} (\operatorname{mod} \bar n^{k,k}_m)$$

2.9. Remark. Each semi-holonomic jet $j^1_0 f \in \bar T^2_m M$ determines the so called *difference tensor* $\Delta j^1_0 f \in \Lambda^2\mathbb{R}^{m*} \otimes T_{f(0)}M$ which is the obstruction to the holonomicity of $j^1_0 f$, see [Kolář, 71]. Now, each element $u \in \bar W^{k+1}M$, $u = \mathbf{j}^1_0\varphi$ is also viewed as $j^1_0(\varphi_{|(\mathbb{R}^m\times\{e\})}) \in \bar T^2_m(\bar W^{k-1}M)$ and, moreover, it determines the identification $T_{(0,e)}\varphi\colon \mathbb{R}^m \oplus \bar{\mathfrak g}^k_m \simeq T_{\bar p^{k+1}_k(u)}\bar W^k M$. In particular, there is the 'horizontal subspace' $u(\mathbb{R}^m) \subset T_{\bar p^{k+1}_k(u)}\bar W^k M$ identified with $u(\mathbb{R}^m) := T_{(0,e)}\varphi(\mathbb{R}^m \oplus \{0\})$. A direct computation shows that the restriction of $d\theta^{(k-1)}$ to $u(\mathbb{R}^m)$ is given by the difference tensor $\Delta(u) \in \Lambda^2\mathbb{R}^{m*} \otimes (\mathbb{R}^m \oplus \bar{\mathfrak g}^{k-1}_m)$, where u is viewed as an element in $\bar W^2(\bar W^{k-1}M)$, see [Kolář, 75a] for details.

3. Geometries modeled on homogeneous spaces

Let G/B be of order k, $k \leq \infty$, together with the fixed complementary subspace $\mathfrak n_- \subset \mathfrak g$ to $\mathfrak b$. The principal fiber bundle homomorphisms $\nu^r\colon G \to P^r(G/B) \subset \bar W^r(G/B)$ over the group homomorphisms $\iota^r\colon B \to G^r_m \subset \bar G^r_m$ are compatible with the projections $\bar p^{r+1}_r$, so we always obtain the reduction ν^∞ of $\bar W^\infty(G/B)$ to the structure group B and in fact all ν^r, $r \geq k$ are reductions.

We intend to discuss geometries described by suitable reductions $\varphi\colon P \to \bar W^\infty M$ of the semi-holonomic frame bundles which should mimic basic features of homogeneous spaces G/B. So not only they should be reductions to the subgroup B over the fixed embeddings $\iota^k\colon B \to \bar{G}^k_m$ but additionally the pullbacks of the canonical forms on the semi-holonomic frame bundles should equip P with a Cartan connection of the type G/B. We have seen that this is the case on the homogeneous space itself, where the left-invariant Maurer-Cartan form is restored in this way. Obviously, if the images of the pullbacks of the canonical forms happen to be in $\iota^k(\mathfrak{g})$, then the latter requirement will be achieved.

We shall start with the somewhat inverse question: *Given a principal fiber bundle $p\colon P \to M$ with structure group B and a Cartan connection ω on P, is there a 'canonical' reduction $P \to \bar{W}^\infty M$ with the above required properties?*

As shown in the proof below, the answer is given by a simple construction which is essentially complete after getting the reduction $\varphi_{k+1}\colon P \to \bar{W}^{k+1}M$, where k is the order of G/B.

3.1. Proposition. *Let G/B be of order $k \le \infty$, $P \to M$ be a principal fiber bundle with structure group B, and let $\omega \in \Omega^1(P,\mathfrak{g})$ be a Cartan connection of type G/B. Then there is a unique reduction $\varphi\colon P \to \bar{W}^\infty M$ such that $\varphi^*\theta^{(\infty)} = \iota^{(\infty)} \circ \omega$. Moreover for all $r \ge k$, $\varphi_{r+1} = \bar{p}^\infty_{r+1} \circ \varphi\colon P \to \bar{W}^{r+1}M$ are reductions to B and $\varphi^*_{r+1}\theta^{(r)} = \iota^{(r)} \circ \omega$.*

Proof. Let us first consider the quotient projection $\pi \circ \omega : TP \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{b}$. Since ω reproduces fundamental vector fields, we obtain the induced linear isomorphism $\omega_0(u)\colon T_{p(u)}M \simeq T_uP/V_uP \to \mathfrak{g}/\mathfrak{b}$ at each point $u \in P$.

$$
\begin{tikzcd}
 & T_uP \arrow[r, "\omega"] \arrow[d, "p"'] & \mathfrak{g} \arrow[d] & \\
T_{p(u)}M \arrow[r, "\simeq"] & T_uP/V_uP \arrow[r, dashed, "\omega_0"] & \mathfrak{g}/\mathfrak{b} \arrow[r, "\simeq"] & \mathfrak{n}_-
\end{tikzcd}
$$

Thus we have defined the mapping

$$\varphi_1\colon P \to \bar{W}^1M = P^1M, \quad u \mapsto \omega_0(u)^{-1} \in \mathfrak{n}^*_- \otimes T_{p(u)}M = P^1_{p(u)}M.$$

A change of the point u to $u.b$, $b \in B$ results in $\omega_0(u.b) = \mathrm{Ad}^0(b^{-1}) \circ \omega_0(u)$ where Ad^0 means the induced adjoint action on the quotient. According to Lemma 1.2, $\mathrm{Ad}^0(b^{-1})$ corresponds to the action $\ell_{\iota^1(b^{-1})}$ under the identification $\mathfrak{n}_- \subset \mathbb{R}^m \oplus \mathfrak{g}^0_m \simeq \mathbb{R}^m$ given by ι^0. Thus φ_1 is a principal fiber bundle homomorphism over the group homomorphism ι^1 and we have got a reduction of the standard first order frame bundle to the structure group B/F^1B. Notice that $\bar{p}^1_0 \circ \varphi_1 = p$ and so for each $\xi \in T_uP$, $\theta^{(0)}(T\varphi_1.\xi) = \omega_0(Tp.\xi)$. Thus the pullback of the canonical form $\theta^{(0)}$ coincides with ω_0 under the chosen identification.

In particular $\xi \in T_uP$ is vertical if and only if $T\varphi_1.\xi$ is vertical and consequently the horizontal subspaces $H(u) := \omega^{-1}(u)(\mathfrak{n}_-) \subset T_uP$ have horizontal images $H_1(u) = T\varphi_1 \subset \bar{W}^1M$.

Now, for every principal fiber bundle $Q \to M$ with any structure group G, any horizontal subspace $H \subset T_u W^1 Q$ determines an element $U_H \in \bar{W}^2 Q$ as follows:

$$\begin{tikzcd}
H \arrow[r, hook] & T_u W^1 Q \arrow[d, "Tp_0^1"] & T_{\mathbf{j}_0^1\,\mathrm{id}} W^1(\mathbb{R}^m \times G) \arrow[l, dashed, "\tilde{U}_H"'] & \\
Tp_0^1(H) \arrow[r, hook] & T_{p_0^1(u)} Q & \mathbb{R}^m \oplus \mathfrak{g} \arrow[l, "\tilde{u}"'] \arrow[u] & \tilde{u}^{-1}(Tp_0^1(H)) \arrow[l, hook'] \arrow[ul, dashed, hook]
\end{tikzcd}$$

$Tp_0^1(H)$ is horizontal in $T_{p_0^1(u)}Q$ and u provides the mapping $\tilde{u}^{-1}\colon Tp_0^1 Q \to \mathbb{R}^m \oplus \mathfrak{g}$ which identifies H with a horizontal subspace $\tilde{U}^{-1}(H)$ in $\mathbb{R}^m \oplus \mathfrak{g} \subset T_{\mathbf{j}_0^1\,\mathrm{id}} W^1(\mathbb{R}^m \times G)$. This determines $\tilde{U}_H$ uniquely since it has to respect the fundamental fields. By the construction, this mapping defines a point in $\bar{W}^2 Q$. Let us also notice how an element $b \in W_m^1 G$ acts on U_H. By definition, $r^b(U_H)$ is given by the composition $Tr^{p_0^1 b} \circ \tilde{U}_H \circ \ell_b$.

Now, our construction will proceed by induction. Assume we already have a principal fiber bundle homomorphism $\varphi_k\colon P \to \bar{W}^k M$ over the Lie group homomorphism $\iota^k\colon B \to \bar{G}_m^k$ and let us write $H_k(u) := \varphi_k(\omega(u)^{-1}(\mathfrak{n}_-))$. Then these horizontal subspaces define the mapping $\varphi_{k+1}\colon P \to \bar{W}^{k+1} M$. Let us further assume that $\varphi_k(u)$ is given by the embedding $T\varphi_{k-1}(u) \circ \iota^{k-1}\colon \mathfrak{n}_- \to T_{\varphi_{k-1}(u)} \bar{W}^{k-1} M$. Notice this is the case for $k = 1$. The properties of φ_{k+1} can be read quite easily from the following diagram

$$\begin{tikzcd}
\mathfrak{n}_- \arrow[r, hook] & \mathfrak{g} \arrow[r, hook, "\iota^k"] \arrow[d, "\mathrm{Ad}_b"] & T_{\mathbf{j}_0^1\,\mathrm{id}} W^1(\bar{W}^{k-1}\mathbb{R}^m) \arrow[d, "\ell_{\iota^{k+1}(b)}"] & \\
\mathfrak{n}_- \arrow[r, hook] & \mathfrak{g} \arrow[r, hook, "\iota^k"] \arrow[d, "\omega(u)^{-1}"] & T_{\mathbf{j}_0^1\,\mathrm{id}} W^1(\bar{W}^{k-1}\mathbb{R}^m) \arrow[r] \arrow[d, dashed, "\varphi_{k+1}(u)"] & T_{\mathbf{j}_0^1\,\mathrm{id}} \bar{W}^{k-1}\mathbb{R}^m \arrow[d, "\varphi_k(u)"] \\
\omega(u)^{-1}(\mathfrak{n}_-) \arrow[r, hook] & T_u P \arrow[r, "T\varphi_k"] \arrow[d, "Tr^b"] & T_{\varphi_k(u)} W^1(\bar{W}^{k-1} M) \arrow[r] \arrow[d, "Tr^{\iota^k(b)}"] & T_{\varphi_{k-1}(u)} \bar{W}^{k-1} M \\
\omega(u.b)^{-1}(\mathfrak{n}_-) \arrow[r, hook] & T_{u.b} P \arrow[r, "T\varphi_k"] & T_{\varphi_k(u.b)} W^1(\bar{W}^{k-1} M) &
\end{tikzcd}$$

First, the composition in the second column is $\omega(u.b)^{-1}$ while the composition in the third one is the action of $\iota^{k+1}(b)$ on $\varphi_{k+1}(u)$. Thus φ_{k+1} is a principal fiber bundle homomorphism over ι^{k+1}. Further, $\theta^{(k)}(\varphi_{k+1}(u))(T\varphi_{k+1}.\xi) = \varphi_{k+1}(u)^{-1} \circ T(p_0^1 \circ \varphi_{k+1}).\xi = \varphi_{k+1}(u)^{-1}(T\varphi_k.\xi) = \iota^k \circ \omega(\xi)$. Altogether, we have constructed a sequence of principal fiber bundle homomorphisms $\varphi_k\colon P \to \bar{W}^k M$.

$$\begin{tikzcd}
\dots \arrow[r] & \bar{W}^{k+1} M \arrow[r] & \dots \arrow[r] & \bar{W}^{r+1} M \arrow[r] & \bar{W}^r M \arrow[r] & \dots \arrow[r] & M \\
 & & & P \arrow[ull, "\varphi_{k+1}"] \arrow[u, "\varphi_{r+1}"] \arrow[ur, "\varphi_r"'] \arrow[urrr, "p"'] & & &
\end{tikzcd}$$

If k is the order of G/B, then for all $r > k$ the homomorphisms φ_r are reductions to structure group B and $\varphi_{r+1}^*\theta^{(r)} = \iota^r \circ \omega$.

Let us notice that φ_1 was completely determined by the quotient mappings ω_0 which had to coincide with the pullback of $\theta^{(0)}$. Moreover, the rest of the construction was uniquely determined by the horizontal subspaces given by ω. Thus, the whole construction was determined by our requirements uniquely. □

3.2. Definition. Let G/B be of order k. A *geometric structure of type G/B* is a reduction of $\varphi\colon P \to \bar{W}^{k+1}M$ to the structure group B such that the values of $\theta^{(k)}_{|T\varphi(TP)}$ are in $\iota^k(\mathfrak{g})$. An *infinitesimal structure of type G/B* is a reduction of P^1M to the structure group B/F^1B, i.e. to the effective structure group of the tangent bundle $T(G/B)$.

3.3 Remark. The situation is most simple if the order of the homogeneous space is $k = 1$ and the chosen $\mathfrak{n}_-$ is an ideal, for example in the Riemannian geometries. Then the structures of type G/B coincide with the infinitesimal structures of type G/B and the Cartan connection ω happens to be the canonical linear connection on M.

Let us illustrate the difference between the two definitions on our simplest examples of higher order homogeneous spaces. For both conformal and almost Grassmannian geometries, B/F^1B is exactly the subgroup of the general linear group which is used for the definition of the corresponding reductions. Thus, the infinitesimal structure is just what we are used to. The standard geometrical constructions (well known already to Cartan) then provide a structure of the type G/B for each infinitesimal structure in our sence. However, there are more general structures available, e.g. in the conformal case we might consider 'weak conformal structures' where the distinguished connections share a fixed non-vanishing torsion. Of course, the general calculus developed for such geometries in [Čap, Slovák, Souček, 94] still applies.

If we pass to more general parabolic geometries with reducible tangent bundles, then the data necessary for the reconstruction of the bundles P and the Cartan connection ω are weaker than our infinitesimal structure. We shall provide some more comments on this problem in the next section.

The properties of the canonical forms on $\bar{W}^kM$ and our Definition 3.2 imply immediately the following

3.4. Corollary. *Let $P \to M$ be a structure of type G/B given by the reduction $\varphi\colon P \to \bar{W}^{k+1}M$. The pullback $\omega_P = \varphi^*\theta^{(k)}$ of the canonical form on $\bar{W}^{k+1}M$ is a Cartan connection on P.*

4. Examples, Remarks, and Outlook

Up to now, we discussed a very general setting covering all possible homogeneous spaces. Let us conclude with a few remarks towards more subtle (and interesting) questions. Of course, to deal with them we always have to restrict ourselves to a suitable class of homogeneous spaces. We refer to the parabolic geometries as our basic example, but we do not touch any details, the interested reader should probably consult the original papers.

4.1. Directions of further investigations.

(1) algorithmic procedures constructing the Cartan connections from the infinitesimal data in a 'canonical' way
(2) sets of invariants ensuring local equivalence of given infinitesimal structures
(3) weaker variants of infinitesimal structures
(4) 'calculus' for the Cartan connections similar to the Ricci calculus in Riemannian geometries, suitable for dealing with the invariant operators for the geometries in question

4.2. The basic point is to incorporate some further geometric structures on the homogeneous spaces. Our constructions can be easily modified in order to obtain analogous structures on the bundles $P \to M$ as well. We suggest a simple general model:

Let $\mathcal{C}$ be a category of manifolds which are locally isomorphic to the object M_0 with a fixed point $O \in M_0$, i.e. for each object $M \in \mathcal{C}$ and each point $x \in M$, there is a neighborhood $U \subset M$ of x isomorphic to a neighborhood of $O \in M_0$. Consider the category $\mathcal{PC}$ of principal fiber bundles over objects in $\mathcal{C}$, with morphisms over $\mathcal{C}$-morphisms. The modified functor W^1 then associates to each such principal fiber bundle in $\mathcal{PC}$ with structure group B the fiber bundle of all fiber jets at $O \in M_0$ of local trivializations $M_0 \times B \to P$ in $\mathcal{PC}$. All previous constructions come through with the modified concepts of semi-holonomic jet groups and algebras. Of course, some further refinements could be still necessary.

4.3. Parabolic geometries. Let us illustrate briefly the arising problems. Assume G is semisimple and $\mathfrak{b}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ a parabolic subalgebra in the complexfications. Let us fix the root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ so that $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{n}_+$ with $\mathfrak{g}_0$ the reductive part of $\mathfrak{b} = \mathfrak{g}_0 \oplus \mathfrak{n}_+$. Then the powers of $\mathfrak{n}_+$ define the finer B-invariant filtration

$$G = F^{-\mu}G \supset F^{-\mu+1}G \supset \cdots \supset F^0 G = B \supset F^1 G \supset \cdots \supset F^{\mu} G$$

and the compatible grading

$$\mathfrak{g} = \mathfrak{g}_{-\mu} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_{\mu}.$$

In particular, there is the induced filtration on the tangent space of G/B. Thus we may restrict ourselves to manifolds endowed with such filtrations and deal with the local trivializations of our principal fiber bundles respecting these filtrations. A powerful general theory for such objects was worked out in [Morimoto, 93]. From our point of view, he additionally considers the induced finer filtrations F^k on the jet groups as well, and he always factors out the action of F^{k+1} when constructing the kth bundle. This theory is very well suited for equivalence problems on filtered manifolds.

The algebraic structure of $\mathfrak{g}$ shows that the order of G/B is always two. Indeed, we need just to observe that the kernel of the adjoint action of $\mathfrak{b}$ on $\mathfrak{g}/\mathfrak{n}_-$ is just $\mathfrak{g}_\mu$ and the action of $\mathfrak{g}_\mu$ on $\mathfrak{g}_{-\mu}$ is effective, see [Tanaka, 79] for algebraic details.

Since $\mathfrak{n}_-$ is a subalgebra now, the embeddings $\iota^k\colon \mathfrak{n}_- \to \mathfrak{n}_- \oplus \mathfrak{g}^k_m$ can be computed explicitly by means of the Baker-Campbell-Hausdorff formula:

$$\iota^k(X) = j^k_0(Y \mapsto f(e^{\mathrm{adY}})X), \text{ where } f(z) = \tfrac{\log z}{z-1}$$

which is a quite nice polynomial expression in view of the nilpotency of $\mathfrak{n}_-$.

4.4. Let us come back to the indicated directions (1)–(4) and discuss briefly what has been already done for the parabolic geometries.

The algorithmic construction of the Cartan connections from the reduction of the structure group of the tangent bundle is very well known in all cases where the tangent bundle is irreducible, i.e. exactly if $\mathfrak{n}_-$ is abelian, see [Tanaka, 79], [Čap, Slovák, Souček, 95]. Tanaka has also given an essentially complete answer for all parabolic geometries, however only from the point of view of the associated equivalence problem for the infinitesimal structures. A very explicit construction is given in the forthcoming paper [Čap, Schichl], starting from a G_0-structure on the associated graded vector bundle to the tangent space. These constructions also provide a nice answer to question (3): up to some very rare cohomological obstructions, the suitable 'weak infinitesimal structure' should be a reduction of the associated graded vector space to the tangent space to structure group G_0. An application of quite general concepts offering a similar construction can be also found in [Morimoto, 93].

In all these approaches, the Lie algebra cohomology on $\mathfrak{n}_-$ with values in $\mathfrak{g}$ is essential for the normalizations. In terms of the general infinitesimal structures on the tangent space from our point of view this imposes some additional conditions on the torsions, while the general problem has not been solved completely yet from our point of view. On the other hand, there is the general question: *What is the best general geometrical definition of 'parabolic geometries'?*.

A good answer to problem (1) yields essentially solutions to (2), namely the Cartan connections describe explicitly all necessary invariants. Much less is known about (4). As far as we know, only the paper [Čap, Slovák, Souček, 94] offers a version of such a calculus for all parabolic geometries with irreducible tangent bundles.

4.5. Finally, let us comment on the most unpleasant point of our general constructions, the extremely bad encoding of the bracket in $\mathfrak{n}_-$. In fact, it was of no importance in our development and so we can expect really good and simple behavior of our general objects only in the case when $\mathfrak{n}_-$ is abelian. Indeed, in this case, the trivial filtration coincides with the finer one.

In general, there always is the Levi part $G_0 \subset B$ and $B = G_0 \rtimes N_+$ is a semidirect product of G_0 and the nilpotent radical. Obviously, if G/B is infinitesimally effective, then $\iota^1_{|\mathfrak{g}_0}$ is injective since $F^1\mathfrak{b}$ is nilpotent. So we might start with a choice of a reduction $P^{(0)}$ of P^1M to G_0, choose a connection γ on this reduction and consider the Ehresmann prolongation γ^k of this connection. The latter will provide a mapping $P_0 \to \bar{W}^{k+1}M$ which will be G_0-equivariant. Thus the orbit of its image under the action of $\iota^{k+1}(B)$ will be a principal fiber bundle with the appropriate structure group. In the case with $\mathfrak{n}_-$ abelian, we really get a structure of type G/B without any further work and we even can use the special algebraic properties to normalize our choices. At the same time we obtain a class of connections yielding the same bundle on the last but one level, an analogy to the class of linear connections compatible with a conformal Riemannian structure. Since these constructions are given in a much more explicit way for the parabolic geometries with irreducible tangent bundles in [Čap, Slovák, Souček, 95], we shall not go into any details here. We believe that a better understanding of the embeddings $\iota^k : \mathfrak{g} \to \mathbb{R}^m \oplus \mathfrak{g}^k_m$ will enable us to use a similar construction in many other cases as well.

References

Alekseevsky, D.V.; Michor, P.W., *Differential geometry of Cartan connections*, ESI Preprint 39, Publ. Math. Debrecen **47** (1995), 349–375.

Baston, R. J., *Almost Hermitian symmetric manifolds, I: Local twistor theory;*, Duke Math. J. **63** (1991), 81–111.

Čap, A.; Schichl, H., paper in preparation.

Čap, A.; Slovák, J., *On local flatness of AHS–manifolds*, to appear in Rendiconti Circ. Mat. Palermo, Proceedings of the Winter School Geometry and Physics, Srní 1995.

Čap, A.; Slovák, J.; Souček, V., *Invariant Operators on Manifolds with Almost Hermitian Symmetric Structures, I. Invariant Differentiation*, electronically available at `ftp.esi.ac.at`, Preprint ESI 186 (1994), 31 pp.

Čap, A.; Slovák, J.; Souček, V., *Invariant Operators on Manifolds with Almost Hermitian Symmetric Structures, II. Normal Cartan connections*, electronically available at `ftp.esi.ac.at`, Preprint ESI 194 (1995), 16 pp.

Kobayashi, S., *Transformation groups in differential geometry*, Springer-Verlag, Berlin, Heidelberg, New York, 1972.

Kolář, I., *Canonical forms on the prolongations of principal fiber bundles*, Rev. Roumaine Math. Pures Appl. **16** (1971), 1091–1106.

Kolář, I., *Higher order torsions of spaces with Cartan Connection*, Cahiers Topologie Géom. Différentielle **12** (1971), 137–146.

Kolář, I., *Generalized G-structures and G-structures of higher order*, Bollettino U. M. Ital. **12, Suppl. 3** (1975), 245–256.

Kolář, I., *A generalization of the torsion form*, Časopis pro pěstování matematiky **100** (1975), 284–290.

Kolář, I.; Michor, P. W.; Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin Heidelberg New York, 1993.

Kriegl, A.; Michor, P. W., *The Convenient Setting for Global Analysis*, to appear, Surveys and Monographs, AMS, Providence, 1997.

Libermann, P., *Sur les prolongements des fibrés principaux et des groupoides différentiables banachiques*, Analyse global, Sém. Mat. Supérieures, No. 42 (Été, 1969) (1971), 7–108.

Morimoto, T., *Geometric structures on filtered manifolds*, Hokkaido Math. J. **22** (1993), 263–347.

Slovák, J., *The principal prolongation of first order G-structures*, Proceedings of the Winter School Geometry and Physics, Srní 1994, Supplemento ai Rendiconti Circ. Mat. Palermo **39** (1996), 123–131.

Tanaka, N., *On differential systems, graded Lie algebras and pseudo-groups*, J. Math. Kyoto Univ. **10** (1970), 1–82.

Tanaka, N., *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japanese J. Math. **2** (1976), 131–190.

Tanaka, N., *On the equivalence problems associated with simple graded Lie algebras*, Hokkaido Math. J. **8** (1979), 23–84.

Department of Algebra and Geometry
Masaryk University in Brno
Janáčkovo nám. 2a
662 95 Brno, CZECH REPUBLIC

E-mail: slovak@math.muni.cz