

Semi-holonomic Verma Modules

Michael Eastwood

*Department of Pure Mathematics, University of Adelaide,
South Australia 5005*

and

Jan Slovák

*Department of Algebra and Geometry, Masaryk University in Brno,
Janáčkovo Nám. 2a, 662 95 Brno, Czech Republic*

Verma modules arise geometrically through the jets of homogeneous vector bundles. We consider in this article, the modules that arise from the semi-holonomic jets of a homogeneous vector bundle. We are particularly concerned with the case of a sphere under Möbius transformations. In this case there are immediate applications in the theory of conformally invariant differential operators.

1 Introduction

The motivation for this article comes from conformal differential geometry. This aspect, however, will be confined to an appendix—the main body of the article will be concerned with purely algebraic results. Suppose G is a Lie group with Lie subgroup P . It is well-known that the space of formal jets of sections of a homogeneous vector bundle on G/P is dual to the corresponding induced module constructed algebraically from the complexified Lie algebras $\mathfrak{p} \subset \mathfrak{g}$ and the inducing representation of P . The G -invariant linear differential operators between homogeneous vector bundles are then in bijective correspondence with the homomorphisms of these modules (see, e.g. [11, 17, 18]). In particular, if G is semisimple and P is parabolic, the induced modules in

This research was begun during a visit of the second author to the University of Adelaide. Some of the writing was undertaken during a visit of the first author to the Erwin Schrödinger Institute. Support from the Australian Research Council, the ESI, and grant number 201/96/0310 of the GAČR is gratefully acknowledged.

question are the (generalized) Verma modules and the structure of their homomorphisms is understood in many cases (see, e.g. [5, 6]). For the moment, suffice it to say that the ‘semi-holonomic’ Verma modules of this article arise using semi-holonomic jets on G/P rather than the usual (holonomic) jets.

Our results and their proofs are partially inspired by Lemma 4.7.1 in [2] but Baston’s proof is rather incomplete and unclear. Our proof closely follows the ‘curved translation principle’ in [9, 11]. Conversations with Andreas Čap and Justin Sawon have been extremely useful. We also thank the referee for some very helpful suggestions.

2 Semi-holonomic modules

Suppose G is a Lie group with Lie subgroup P . We shall denote by \mathfrak{g} and \mathfrak{p} their complexified Lie algebras and by $\mathfrak{U}(\mathfrak{g})$ and $\mathfrak{U}(\mathfrak{p})$, the universal enveloping algebras of \mathfrak{g} and \mathfrak{p} , respectively. Let \mathbb{E} be a finite-dimensional complex representation of P . Regarding $\mathfrak{U}(\mathfrak{g})$ as a left $\mathfrak{U}(\mathfrak{g})$ -module and a right $\mathfrak{U}(\mathfrak{p})$ -module, we may define a $(\mathfrak{U}(\mathfrak{g}), P)$ -module $V(\mathbb{E}) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E}^*$. In this generality, $V(\mathbb{E})$ is known as an induced module (see, e.g. [21]) but when G is semisimple, P is parabolic, and \mathbb{E} is irreducible, $V(\mathbb{E})$ is a (generalised) Verma module (see, e.g. [16]).

Following Baston [2], define an algebra $\bar{\mathfrak{U}}(\mathfrak{g})$ by

$$\bar{\mathfrak{U}}(\mathfrak{g}) = \otimes \mathfrak{g} / \langle X \otimes Y - Y \otimes X - [X, Y] \text{ for } X \in \mathfrak{p} \text{ and } Y \in \mathfrak{g} \rangle.$$

It differs from $\mathfrak{U}(\mathfrak{g})$ in that one is only allowed to commute elements of \mathfrak{p} around using the commutation relations of \mathfrak{g} rather than arbitrary elements. In particular, $\mathfrak{U}(\mathfrak{p})$ is a subalgebra of $\bar{\mathfrak{U}}(\mathfrak{g})$. Thus, $\bar{\mathfrak{U}}(\mathfrak{g})$ is a left $\bar{\mathfrak{U}}(\mathfrak{g})$ -module and a right $\mathfrak{U}(\mathfrak{p})$ -module and we may define the semi-holonomic induced module

$$\bar{V}(\mathbb{E}) = \bar{\mathfrak{U}}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E}^*.$$

It is a $(\bar{\mathfrak{U}}(\mathfrak{g}), P)$ -module in the sense that it is both a $\bar{\mathfrak{U}}(\mathfrak{g})$ -module and a representation of P such that the two induced actions of $\mathfrak{U}(\mathfrak{p})$, obtained by restriction or differentiation, agree. Like $\mathfrak{U}(\mathfrak{g})$, as a vector space, $\bar{\mathfrak{U}}(\mathfrak{g})$ may be filtered by degree. The modules $\bar{V}(\mathbb{E})$ are correspondingly filtered

$$\mathbb{E}^* = \bar{V}_0(\mathbb{E}) \subset \bar{V}_1(\mathbb{E}) \subset \bar{V}_2(\mathbb{E}) \subset \cdots \subset \bar{V}_k(\mathbb{E}) \subset \cdots \subset \bar{V}(\mathbb{E})$$

with exact sequences of P -modules

$$0 \rightarrow \bar{V}_{k-1}(\mathbb{E}) \rightarrow \bar{V}_k(\mathbb{E}) \rightarrow \bigotimes^k(\mathfrak{g}/\mathfrak{p}) \otimes \mathbb{E}^* \rightarrow 0.$$

The analogous statements for induced modules are well-known. They are related by a surjection of $(\bar{\mathfrak{U}}(\mathfrak{g}), P)$ -modules $\bar{V}(\mathbb{E}) \rightarrow V(\mathbb{E})$ and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bar{V}_{k-1}(\mathbb{E}) & \longrightarrow & \bar{V}_k(\mathbb{E}) & \longrightarrow & \bigotimes^k(\mathfrak{g}/\mathfrak{p}) \otimes \mathbb{E}^* \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & V_{k-1}(\mathbb{E}) & \longrightarrow & V_k(\mathbb{E}) & \longrightarrow & \odot^k(\mathfrak{g}/\mathfrak{p}) \otimes \mathbb{E}^* \longrightarrow 0 \end{array} \quad (1)$$

where \odot denotes symmetric tensor product.

For any finite-dimensional representation \mathbb{F} of P , there is a canonical isomorphism

$$\mathrm{Hom}_P(\mathbb{F}^*, V(\mathbb{E})) = \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}), V(\mathbb{E}))$$

known as Frobenius reciprocity (see, e.g. [21]). There is an analogous result for semi-holonomic modules:

Proposition 1 *For any finite-dimensional representation \mathbb{F} of P , there is a canonical isomorphism*

$$\mathrm{Hom}_P(\mathbb{F}^*, \bar{V}(\mathbb{E})) = \mathrm{Hom}_{(\bar{\mathfrak{U}}(\mathfrak{g}), P)}(\bar{V}(\mathbb{F}), \bar{V}(\mathbb{E})).$$

Proof. Given $D \in \mathrm{Hom}_{(\bar{\mathfrak{U}}(\mathfrak{g}), P)}(\bar{V}(\mathbb{F}), \bar{V}(\mathbb{E}))$, define $d : \mathbb{F}^* \rightarrow \bar{V}(\mathbb{E})$ by restriction. Conversely, the formula

$$D(x \otimes f) = x \otimes df \quad \text{for } x \in \bar{\mathfrak{U}}(\mathfrak{g}) \text{ and } f \in \mathbb{F}^*$$

clearly characterises D in terms of d . To complete the proof, notice that

$$D(X \otimes f - 1 \otimes Xf) = X \otimes df - 1 \otimes d(Xf) = X \otimes df - 1 \otimes Xdf$$

for $X \in \mathfrak{p}$ and $f \in \mathbb{F}^*$ so D is well-defined for any $d \in \mathrm{Hom}_P(\mathbb{F}^*, \bar{V}(\mathbb{E}))$. \square

In certain cases (see, e.g. [5, 6]) the spaces $\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}), V(\mathbb{E}))$ are well-understood. However, even in the simplest cases,

$$\mathrm{Hom}_{(\bar{\mathfrak{U}}(\mathfrak{g}), P)}(\bar{V}(\mathbb{F}), \bar{V}(\mathbb{E}))$$

is more mysterious. Instead, we may ask when a given homomorphism $V(\mathbb{F}) \rightarrow V(\mathbb{E})$ can be lifted to $\bar{V}(\mathbb{F}) \rightarrow \bar{V}(\mathbb{E})$. Specific examples show that such a lifting is generally not unique. Also, there can be homomorphisms of the semi-holonomic modules even when lifting a holonomic morphism is impossible (as in Proposition 5). In the light of Frobenius reciprocity and Proposition 1, the lifting problem is equivalent to the question of completing the following diagram of P -modules

$$\begin{array}{ccc} & & \bar{V}(\mathbb{E}) \\ & \nearrow \text{---} & \downarrow \\ \mathbb{F}^* & \xrightarrow{\text{---}} & V(\mathbb{E}) \end{array}$$

In this article we shall answer this question completely for irreducible \mathbb{E} and \mathbb{F} when $G = \text{Spin}_o(n+1, 1)$ acting on the sphere S^n by Möbius transformations and P is the stabilizer subgroup of this action.

3 Statement of results

For the rest of this article, G will denote $\text{Spin}_o(n+1, 1)$. Acting on $\mathbb{R}\mathbb{P}_{n+1}$, it has three orbits according to whether the corresponding vector is time-like, null, or space-like. Let P be the stabilizer subgroup for some choice of basepoint on the null orbit. Then G/P may be identified with the sphere S^n and G with the double cover of its group of conformal motions (see, e.g. [10] for further discussion). This aspect will be taken up in the appendix where applications to conformal geometry will be considered. In this context, G is often referred to as the group of Möbius transformations since when $n = 2$ we have $\text{Spin}_o(3, 1) \cong \text{SL}(2, \mathbb{C})$ and $S^2 \cong \mathbb{C}\mathbb{P}_1$, the Riemann sphere.

In order to state our results, we firstly need to state what is known concerning the homomorphisms of Verma modules in this case. They are completely classified [6, (3.1)] and the answer is as follows. Choose a positive root system for \mathfrak{g} compatible with \mathfrak{p} (the Cartan subalgebra and all the positive root spaces are contained in \mathfrak{p}).

3.1 The case n even

Write $n = 2m$. Let λ be a dominant integral weight for \mathfrak{g} . Then λ is also dominant for \mathfrak{p} and we shall denote by \mathbb{E}^0 the irreducible representation of P with lowest weight $-\lambda$. Under the affine action of the Weyl

group of \mathfrak{g} (namely $\lambda \mapsto w(\lambda + \rho) - \rho$ where ρ is half the sum of the positive roots) we obtain $2m + 2$ weights which are dominant for \mathfrak{p} . We shall denote the corresponding finite-dimensional representations of P by $\mathbb{E}^0, \mathbb{E}^1, \dots, \mathbb{E}^{m-1}, \mathbb{E}_-^m, \mathbb{E}_+^m, \mathbb{E}^{m+1}, \dots, \mathbb{E}^{2m-1}, \mathbb{E}^{2m}$, where the superscript is the length of the corresponding Weyl group element. Then, there are non-trivial Verma module homomorphisms which may be laid out in accordance with the appropriate Hasse diagram (see, e.g. [4]):

$$\begin{array}{ccccccc}
 & & & V(\mathbb{E}_+^m) & & & \\
 & & \nearrow & & \searrow & & \\
 V(\mathbb{E}^{2m}) & \rightarrow & \dots & \rightarrow & V(\mathbb{E}^{m+1}) & & \\
 & & \searrow & & \nearrow & & \\
 & & & V(\mathbb{E}_-^m) & & & \\
 & & & & & & V(\mathbb{E}^{m-1}) \rightarrow \dots \rightarrow V(\mathbb{E}^1) \rightarrow V(\mathbb{E}^0).
 \end{array} \tag{2}$$

In fact, writing $\mathbb{E}^m = \mathbb{E}_+^m \oplus \mathbb{E}_-^m$, this is the (generalised) Bernstein-Gelfand-Gelfand resolution $V(\mathbb{E}^\bullet)$ of the representation of G with highest weight λ (see, e.g. [4, 16]). We shall refer to these homomorphisms as standard. In addition, the composition $V(\mathbb{E}^{m+1}) \rightarrow V(\mathbb{E}^{m-1})$ through $V(\mathbb{E}_\pm^m)$ is non-trivial and also standard. There are also non-standard homomorphisms

$$V(\mathbb{E}^{m+k}) \longrightarrow V(\mathbb{E}^{m-k}) \quad \text{for } k = 2, 3, \dots, m. \tag{3}$$

The homomorphisms listed so far are known as non-singular.

The singular homomorphisms are obtained by taking λ to be an integral weight so that λ is not dominant for \mathfrak{g} but $\lambda + \rho$ is. We may still define \mathbb{E}^k as before when the appropriate weight is dominant for \mathfrak{p} . However, it is easily verified (e.g. using the algorithms of [4]) that if λ lies on the non-dominant side of two or more walls, then $w(\lambda + \rho) - \rho$ is never dominant for \mathfrak{p} with one exception, namely when the two walls correspond to the circled nodes



of the Dynkin diagram. This case is completely degenerate, however, with equalities $V(\mathbb{E}^{m+1}) = V(\mathbb{E}_-^m) = V(\mathbb{E}_+^m) = V(\mathbb{E}^{m-1})$. The remaining $m + 1$ cases correspond to the $m + 1$ walls of the dominant chamber. There are two standard singular homomorphisms $V(\mathbb{E}^{m+1}) \rightarrow V(\mathbb{E}_\mp^m)$ (or, equivalently, $V(\mathbb{E}_\pm^m) \rightarrow V(\mathbb{E}^{m-1})$), again corresponding to the circled nodes of the Dynkin diagram, and $m - 1$ non-standard singular homomorphisms

$$V(\mathbb{E}^{m+k+1}) = V(\mathbb{E}^{m+k}) \longrightarrow V(\mathbb{E}^{m-k}) = V(\mathbb{E}^{m-k-1}) \quad \text{for } k = 1, 2, \dots, m - 1.$$

Up to scale, this is a complete list of the non-trivial Verma module homomorphisms. (Recall that a Verma module is induced from an *irreducible* representation of P .) We may now state the main theorem for n even.

Theorem 1 *For $n \geq 4$, no non-standard non-singular homomorphism*

$$V(\mathbb{E}^n) \longrightarrow V(\mathbb{E}^0)$$

lifts to a homomorphism $\bar{V}(\mathbb{E}^n) \rightarrow \bar{V}(\mathbb{E}^0)$. All other Verma module homomorphisms lift to the corresponding semi-holonomic modules.

3.2 The case n odd

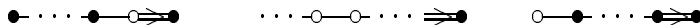
Write $n = 2m + 1$. There are slight but essential differences to the even dimensional case. The orbit of a dominant integral weight λ for \mathfrak{g} under the affine action of the Weyl group involves exactly $2m + 2$ weights dominant for \mathfrak{p} . Let us denote the corresponding finite-dimensional representations of P by $\mathbb{E}^0, \mathbb{E}^1, \dots, \mathbb{E}^{2m}, \mathbb{E}^{2m+1}$. The generalised Bernstein-Gelfand-Gelfand resolution $V(\mathbb{E}^\bullet)$ consists of the non-trivial Verma module homomorphisms

$$V(\mathbb{E}^{2m+1}) \rightarrow V(\mathbb{E}^m) \rightarrow \dots \rightarrow V(\mathbb{E}^1) \rightarrow V(\mathbb{E}^0). \quad (4)$$

These are the standard homomorphisms and there are no other non-trivial homomorphisms between the modules in the pattern.

If we start with a weight λ with $\lambda + \rho$ sitting on a wall of the dominant Weyl chamber, then there are still weights dominant for \mathfrak{p} in its orbit under the affine action of the Weyl group; however, there are no non-trivial homomorphisms between them. In other words, there are no singular homomorphisms in the odd dimensional case.

There are, however, some non-standard operators obtained from a half-integral weight λ with $\lambda + \rho$ in the dominant Weyl chamber. There are $m + 1$ possibilities where the half-integral coefficients appear only with respect to one or two walls, as indicated by the circled nodes in the following Dynkin diagrams:



We obtain corresponding homomorphisms

$$V(\mathbb{E}^{m+k}) \rightarrow V(\mathbb{E}^{m+1-k}), \quad \text{for } k = 1, 2, \dots, m + 1.$$

For $n \geq 5$, each pattern of modules with the same central character and some half-integral coefficients involves only one such homomorphism. For $n = 3$ all four prospective modules exist with two operators between them but these should be regarded as giving two separate families.

Up to scale, this is a complete list of the non-trivial Verma module homomorphisms and we may now state the main theorem for n odd.

Theorem 2 *When n is odd, all Verma module homomorphisms lift to the corresponding semi-holonomic modules.*

4 Proof of results

Firstly, some general remarks. Suppose $D : V(\mathbb{F}) \rightarrow V(\mathbb{E})$ is a homomorphism of induced modules. Since \mathbb{F}^* is finite-dimensional, its image under D is contained in $V_k(\mathbb{E})$ for some k . The least k for which this is the case is called the order of D .

Proposition 2 *A homomorphism $V(\mathbb{F}) \rightarrow V(\mathbb{E})$ of order 2 or less always lifts to a homomorphism of the corresponding semi-holonomic modules.*

Proof. By Frobenius reciprocity and Proposition 1, it suffices to show that the surjection of P -modules $\bar{V}_2(\mathbb{E}) \rightarrow V_2(\mathbb{E})$ admits a P -equivariant splitting. Such a splitting may be defined as the identity on $V_1(\mathbb{E}) = \bar{V}_1(\mathbb{E})$ and further characterised by

$$V_2(\mathbb{E}) \ni XYe \mapsto \frac{1}{2}(XY + YX + [X, Y])e \in \bar{V}_2(\mathbb{E})$$

for $X, Y \in \mathfrak{g}$ and $e \in \mathbb{E}^*$. It is elementary to check that this is well-defined and P -equivariant. \square

Generally, the symbol of a k^{th} order homomorphism $D : V(\mathbb{F}) \rightarrow V(\mathbb{E})$ is the P -module homomorphism

$$\sigma(D) : \mathbb{F}^* \rightarrow V_k(\mathbb{E})/V_{k-1}(\mathbb{E}) = \odot^k(\mathfrak{g}/\mathfrak{p}) \otimes \mathbb{E}^*.$$

In our case (for G and P as fixed at the beginning of Section 3), this is especially simple as follows. Choose a Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}_+$ and write $\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{l} \oplus \mathfrak{u}_+$, as usual. (See, e.g. [10] for an explicit description.) The algebras \mathfrak{u}_\pm are Abelian and irreducible as representations of \mathfrak{l} . Using

the Poincaré-Birkhoff-Witt procedure to put elements of $\mathfrak{U}(\mathfrak{g})$ into standard order, we may identify

$$V(\mathbb{E}) = \mathfrak{U}(\mathfrak{u}_-) \otimes \mathbb{E}^* = \odot \mathfrak{u}_- \otimes \mathbb{E}^*$$

as an \mathfrak{l} -module.

Proposition 3 *When \mathbb{F} is irreducible, a homomorphism of induced modules $V(\mathbb{F}) \rightarrow V(\mathbb{E})$ is determined by its symbol.*

Proof. The centre of \mathfrak{l} is one-dimensional and acts non-trivially on \mathfrak{u}_- . (We shall be more explicit about this in Proposition 7.) Therefore, the image of \mathbb{F}^* lies in $\odot^k \mathfrak{u}_- \otimes \mathbb{E}^*$. \square

Similar considerations apply to the non-holonomic case. There is a symbol

$$\mathbb{F}^* \longrightarrow \bar{V}_k(\mathbb{E})/\bar{V}_{k-1}(\mathbb{E}) = \otimes^k (\mathfrak{g}/\mathfrak{p}) \otimes \mathbb{E}^* = \otimes^k \mathfrak{u}_- \otimes \mathbb{E}^*$$

which determines a given homomorphism of semi-holonomic induced modules when \mathbb{F} is irreducible.

Proposition 4 *A k^{th} order homomorphism of Verma modules $V(\mathbb{F}) \rightarrow V(\mathbb{E})$ lifts to the corresponding semi-holonomic modules if and only if the diagram of P -modules*

$$\begin{array}{ccc} & & \bar{V}_k(\mathbb{E}) \\ & \nearrow & \downarrow \\ \mathbb{F}^* & \xrightarrow{\quad} & V_k(\mathbb{E}) \end{array}$$

may be completed as shown.

Proof. For general induced modules, it is conceivable that one would be able to lift a P -module homomorphism with image in $V_k(\mathbb{E})$ to $\bar{V}_l(\mathbb{E})$ for some $l > k$ without being able to lift to $\bar{V}_k(\mathbb{E})$. Proposition 3 and the corresponding result for semi-holonomic Verma modules, prevent this in our case. \square

As a further equivalent formulation of the lifting problem, observe that, by Frobenius reciprocity, a homomorphism of induced modules $V(\mathbb{F}) \rightarrow V(\mathbb{E})$, when \mathbb{F} is irreducible, is equivalent to a maximal weight vector in $V(\mathbb{E})$, namely the image of a highest weight vector of \mathbb{F}^* . By Proposition 1, the same remark applies to the semi-holonomic case. Our lifting problem, therefore, in the case when \mathbb{F} is irreducible, is the problem of trying to lift a given maximal weight vector in $V(\mathbb{E})$ to a maximal vector in $\bar{V}(\mathbb{E})$. This is the point of view adopted by Baston [2]. As already observed, the particular weight corresponding to the centre of \mathfrak{l} forces the maximal weight into $\odot^k \mathfrak{u}_- \otimes \mathbb{E}^*$ and a prospective lift into $\otimes^k \mathfrak{u}_- \otimes \mathbb{E}^*$. This is only rarely achieved by the tautological embedding $\odot^k \mathfrak{u}_- \hookrightarrow \otimes^k \mathfrak{u}_-$.

Now we can prove the following special case of Theorem 1. (It is the case $\lambda = 0$ in the discussion of §3.1.) Let Λ^1 denote \mathfrak{u}_- as a representation of P and let Λ^k denote its k^{th} exterior power.

Proposition 5 *For $n \geq 4$, the homomorphism $V(\Lambda^n) \rightarrow V(\Lambda^0)$ does not lift to the corresponding semi-holonomic modules.*

Proof. We need a formula for the action of \mathfrak{u}_+ on $\otimes^k \mathfrak{u}_- \otimes \mathbb{E}^*$ regarded as an \mathfrak{l} -submodule of $\bar{V}(\mathbb{E})$. Elements of $\otimes^k \mathfrak{u}_- \otimes \mathbb{E}^*$ may be written as linear combinations of simple ones:

$$y_1 \otimes y_2 \otimes \cdots \otimes y_k \otimes e \quad \text{for } y_1, y_2, \dots, y_k \in \mathfrak{u}_- \text{ and } e \in \mathbb{E}^*.$$

By using the Poincaré-Birkhoff-Witt procedure, the action of $x \in \mathfrak{u}_+$ on such an element is given by

$$\begin{aligned} & \sum_{1 \leq p < q \leq k} y_1 \otimes \cdots \otimes \hat{y}_p \otimes \cdots \otimes [[x, y_p], y_q] \hat{y}_q \otimes \cdots \otimes y_k \otimes e \\ & + \sum_{1 \leq p \leq k} y_1 \otimes \cdots \otimes \hat{y}_p \otimes \cdots \otimes y_k \otimes [x, y_p] e \end{aligned} \in \otimes^{k-1} \mathfrak{u}_- \otimes \mathbb{E}^*.$$
(5)

Following [10], elements of $\mathfrak{g} = \mathfrak{so}(n+1, 1)$ may be written

$$\begin{pmatrix} \lambda & x_a & 0 \\ y^b & m_a^b & -x^b \\ 0 & -y_a & -\lambda \end{pmatrix} \quad (6)$$

where $m_a^b \in \mathfrak{so}(n)$. With these conventions (5) gives rise to a tensor equation

$$\begin{aligned} (x(y \otimes e))_{a_1 a_2 \dots a_{k-1}} = & \\ & \sum_{1 \leq p < q \leq k} \left[\begin{array}{l} x_{a_{q-1}} y_{a_1 \dots a_{p-1}}{}^b{}_{a_p \dots a_{q-2} b a_q \dots a_{k-1}} \\ - x^b y_{a_1 \dots a_{p-1} b a_p \dots a_{q-2} a_{q-1} a_q \dots a_{k-1}} \\ - x^b y_{a_1 \dots a_{p-1} a_{q-1} a_p \dots a_{q-2} b a_q \dots a_{k-1}} \end{array} \right] \otimes e \\ & + \sum_{1 \leq p \leq k} \left[\begin{array}{l} x_c y_{a_1 \dots a_{p-1} d a_p \dots a_{k-1}} - x_d y_{a_1 \dots a_{p-1} c a_p \dots a_{k-1}} \\ + x^b y_{a_1 \dots a_{p-1} b a_p \dots a_{k-1}} \end{array} \right] e \end{aligned}$$

for $y_{a_1 a_2 \dots a_k} \otimes e \in \bigotimes^k \mathfrak{u}_- \otimes \mathbb{E}^*$, where $[m_{cd} + \lambda]e$ is action of $\mathfrak{l} = \mathfrak{so}(n) \oplus \mathbb{C}$ on \mathbb{E}^* , indices are raised and lowered with the standard metric g_{ab} on \mathbb{R}^n , and repeated indices are summed, following Einstein's convention for tensors. When \mathbb{E} is trivial, as it is for $\bar{V}(\Lambda^0)$, the second sum drops out. Of course, the action of \mathfrak{u}_+ on $\omega \in \bigodot^k \mathfrak{u}_- \subset V(\Lambda^0)$ is obtained by symmetrizing:

$$(x\omega)_{a_1 a_2 \dots a_{k-1}} = \frac{k(k-1)}{2} [x_{(a_1} \omega_{a_2 \dots a_{k-1})}{}^b{}_b - 2x^b \omega_{b a_1 a_2 \dots a_{k-1}}] \quad (7)$$

where parentheses on indices take the symmetric part. Up to scale, this is the formula (4.11) of [10] with $w = 0$, where it is also observed that

$$\omega_{a_1 a_2 a_3 a_4 \dots a_{n-1} a_n} = g_{(a_1 a_2} g_{a_3 a_4} \dots g_{a_{n-1} a_n)}$$

is the highest weight corresponding to the homomorphism $V(\Lambda^n) \rightarrow V(\Lambda^0)$. Indeed, with this choice of ω ,

$$\omega_{a_2 \dots a_{n-1}}{}^b{}_b = 2g_{(a_2 a_3} \dots g_{a_{n-2} a_{n-1})}$$

whence substitution in (7) gives zero. By Weyl's classical invariant theory, the general lift of this to an \mathfrak{l} -invariant vector in $\bigotimes^n \mathfrak{u}_-$ has the form

$$\omega_{a_1 a_2 \dots a_n} = \sum_{\sigma \in \mathcal{S}_n} c_\sigma g_{a_{\sigma(1)} a_{\sigma(2)}} g_{a_{\sigma(3)} a_{\sigma(4)}} \dots g_{a_{\sigma(n-1)} a_{\sigma(n)}} \quad \text{where } \sum_{\sigma} c_\sigma = 1.$$

Thus, to complete the proof, we need to show that for no such ω can we have

$$\bigotimes^{n-1} \mathfrak{u}_- \ni \sum_{1 \leq p < q \leq n} \left[\begin{array}{l} x_{a_{q-1}} \omega_{a_1 \dots a_{p-1}}{}^b{}_{a_p \dots a_{q-2} b a_q \dots a_{n-1}} \\ - x^b \omega_{a_1 \dots a_{p-1} b a_p \dots a_{q-2} a_{q-1} a_q \dots a_{n-1}} \\ - x^b \omega_{a_1 \dots a_{p-1} a_{q-1} a_p \dots a_{q-2} b a_q \dots a_{n-1}} \end{array} \right] = 0$$

for all $x \in \mathfrak{u}_+$. If we symmetrize this expression over its first $n - 2$ indices $a_1 a_2 \cdots a_{n-2}$, then we obtain terms of the following two types

$$g_{(a_1 a_2 \cdots a_{n-3} a_{n-2})} x_{a_{n-1}} \quad x_{(a_1 g_{a_2 a_3} \cdots g_{a_{n-2}}) a_{n-1}}$$

and it is straightforward to check[‡] that, for every term in ω , there are $n - 2$ of the former type, independent of σ . Bearing in mind that $\sum c_\sigma = 1$, it follows that

$$\begin{aligned} (x\omega)_{(a_1 a_2 \cdots a_{n-2}) a_{n-1}} &= \\ &= (n - 2) g_{(a_1 a_2 \cdots a_{n-3} a_{n-2})} x_{a_{n-1}} + (2 - n) x_{(a_1 g_{a_2 a_3} \cdots g_{a_{n-2}}) a_{n-1}} \end{aligned}$$

which is non-zero for $n \geq 4$. \square

The key technique in our proofs is the translation principle of Zuckerman [22] and others. The idea is as follows. Suppose \mathbf{W} is a finite-dimensional representation of G . Use the same symbol for the restriction of this representation to P .

Proposition 6 *There is a canonical isomorphism of $(\mathfrak{U}(\mathfrak{g}), P)$ -modules*

$$V(\mathbb{E} \otimes \mathbf{W}) = V(\mathbb{E}) \otimes \mathbf{W}^*.$$

Proof. We may view $\mathfrak{U}(\mathfrak{g}) \otimes \mathbb{E}^* \otimes \mathbf{W}^*$ as a \mathfrak{g} -module in two different ways:

1. $X(x \otimes e \otimes w) = Xx \otimes e \otimes w$
2. $X(x \otimes e \otimes w) = Xx \otimes e \otimes w + x \otimes e \otimes Xw$.

There is a \mathfrak{g} -homomorphism between these two modules characterised as the identity on elements of the form $1 \otimes e \otimes w$ for $e \in \mathbb{E}^*$ and $w \in \mathbf{W}^*$. This descends to the required isomorphism of induced modules. \square

A non-trivial finite-dimensional irreducible representation of G is never irreducible as a representation of P but enjoys a composition series

$$\mathbf{W} = \mathbb{W}_{-\ell} + \mathbb{W}_{1-\ell} + \cdots + \mathbb{W}_{\ell-1} + \mathbb{W}_\ell \quad (8)$$

[‡]These computations are easily done using Penrose's bug notation [19, Appendix].

and each $\mathbb{E} \otimes \mathbb{W}_i$ splits as a direct sum of irreducibles, say $\mathbb{E}_{i,j}$. Thus, we obtain

$$V(\mathbb{E} \otimes \mathbb{W}) = \begin{pmatrix} V(\mathbb{E}_{\ell,1}) \\ \oplus \\ V(\mathbb{E}_{\ell,2}) \\ \oplus \\ \vdots \end{pmatrix} + \begin{pmatrix} V(\mathbb{E}_{\ell-1,1}) \\ \oplus \\ V(\mathbb{E}_{\ell-1,2}) \\ \oplus \\ \vdots \end{pmatrix} + \cdots + \begin{pmatrix} V(\mathbb{E}_{1-\ell,1}) \\ \oplus \\ V(\mathbb{E}_{1-\ell,2}) \\ \oplus \\ \vdots \end{pmatrix} + \begin{pmatrix} V(\mathbb{E}_{-\ell,1}) \\ \oplus \\ V(\mathbb{E}_{-\ell,2}) \\ \oplus \\ \vdots \end{pmatrix} \quad (10)$$

as $(\mathfrak{U}(\mathfrak{g}), P)$ -modules. Fix attention on one particular factor $V(\mathbb{E}')$ occurring on the right hand side. Under suitable circumstances, we may split off this Verma module as a direct summand. Since a Verma module is a highest weight module, elements in the centre $Z(\mathfrak{U}(\mathfrak{g}))$ of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ act by scalar multiplication. The resulting algebra homomorphism $Z(\mathfrak{U}(\mathfrak{g})) \rightarrow \mathbb{C}$ is called the central character of this Verma module.

Proposition 8 *Suppose that $V(\mathbb{E}')$ has distinct central character from all the other $V(\mathbb{E}_{ij})$ occurring on the right hand side of (10). Then $V(\mathbb{E}')$ canonically splits off from $V(\mathbb{E} \otimes \mathbb{W})$ as a direct summand.*

Proof. The inclusion $V(\mathbb{E}') \hookrightarrow V(\mathbb{E} \otimes \mathbb{W})$ is defined by mapping to the joint eigenspace of the central character of $V(\mathbb{E}')$. The complementary subspace is the direct sum of the generalised eigenspaces for the remaining central characters. \square

This proposition may be used in conjunction with:

Theorem 3 (Harish-Chandra) *Two Verma modules $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character if and only if their highest weights are related under the affine action of the Weyl group of \mathfrak{g} .*

Proof. See, for example, [15]. \square

Specifically, we may take the highest weight λ of \mathbb{E}^* and use the Weyl group to bring $\lambda + \rho$ into the dominant chamber. If we do the same to μ , the highest weight of \mathbb{F}^* , then the resulting dominant weights coincide if and only if $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character.

For a non-trivial homomorphism of Verma modules $V(\mathbb{F}) \rightarrow V(\mathbb{E})$, it is evident that $V(\mathbb{E})$ and $V(\mathbb{F})$ must have the same central character. By construction, the modules (2) or (4) all have the same central character and there are no others. The translation principle aims to relate such a pattern of Verma modules and their homomorphisms to the corresponding pattern with different central character. So, suppose $V(\mathbb{F}) \rightarrow V(\mathbb{E})$ is a non-trivial homomorphism. Let \mathbf{W} be a finite-dimensional irreducible representation of G . By Proposition 6, we obtain a homomorphism

$$V(\mathbb{F} \otimes \mathbf{W}) = V(\mathbb{F}) \otimes \mathbf{W}^* \rightarrow V(\mathbb{E}) \otimes \mathbf{W}^* = V(\mathbb{E} \otimes \mathbf{W}).$$

Now suppose that $V(\mathbb{F}')$ occurs as a composition factor of $V(\mathbb{F} \otimes \mathbf{W})$ with distinct central character from all other factors. Suppose, moreover, that $V(\mathbb{E}')$ has the same central character as $V(\mathbb{F}')$ and occurs as a composition factor of $V(\mathbb{E} \otimes \mathbf{W})$ but that no other factor has this particular central character. Then, by Proposition 8, we obtain $V(\mathbb{F}') \rightarrow V(\mathbb{E}')$ as the composition

$$V(\mathbb{F}') \rightarrow V(\mathbb{F} \otimes \mathbf{W}) \rightarrow V(\mathbb{E} \otimes \mathbf{W}) \rightarrow V(\mathbb{E}').$$

The result of this process, namely

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}), V(\mathbb{E})) \longrightarrow \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}'), V(\mathbb{E}'))$$

is called translation. In the best cases, it is an isomorphism:

Proposition 9 *Suppose that $V(\mathbb{E})$ and $V(\mathbb{F})$ have the same central character. Suppose that $V(\mathbb{E}')$ and $V(\mathbb{F}')$ have the same central character. Let \mathbf{W} be a finite-dimensional irreducible representation of G and suppose that*

- $V(\mathbb{F}')$ occurs in the composition series for $V(\mathbb{F} \otimes \mathbf{W})$ and has distinct central character from all other factors;
- $V(\mathbb{E}')$ occurs in the composition series for $V(\mathbb{E} \otimes \mathbf{W})$ and has distinct central character from all other factors.

It follows that $V(\mathbb{F})$ occurs in the composition series for $V(\mathbb{F}' \otimes \mathbf{W}^)$ and that $V(\mathbb{E})$ occurs in the composition series for $V(\mathbb{E}' \otimes \mathbf{W}^*)$. We suppose further that*

- *all other composition factors of $V(\mathbb{F}' \otimes \mathbf{W}^*)$ have central character distinct from $V(\mathbb{F})$;*

- all other composition factors of $V(\mathbb{E}' \otimes \mathbf{W}^*)$ have central character distinct from $V(\mathbb{E})$.

Then translation gives an isomorphism

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}), V(\mathbb{E})) \xrightarrow{\cong} \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}'), V(\mathbb{E}'))$$

(whose inverse is given by translation using \mathbf{W}^* .)

Proof. Straightforward, using the tautological isomorphisms

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}) \otimes \mathbf{W}^*, V(\mathbb{E}')) = \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}), V(\mathbb{E}') \otimes \mathbf{W}) \quad (11)$$

and

$$\mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}') \otimes \mathbf{W}, V(\mathbb{E})) = \mathrm{Hom}_{(\mathfrak{U}(\mathfrak{g}), P)}(V(\mathbb{F}'), V(\mathbb{E}) \otimes \mathbf{W}^*).$$

See [21] for details. Essentially the same reasoning but in a more subtle situation is employed towards the end of §4.1. \square

The classification of Verma module homomorphisms described in Section 3 may be achieved using this proposition. We shall come back to this shortly.

We have described translation in detail in order that we may follow the same procedure, as far as is possible, in the semi-holonomic case. Since $\bar{\mathfrak{U}}(\mathfrak{g})$ has only trivial centre, we may expect much less power from a semi-holonomic translation principle. In particular, there is no hope for a classification of homomorphisms. Nevertheless, some elements remain. As usual, suppose \mathbb{E} is a finite-dimensional representation of P and \mathbf{W} is a finite-dimensional representation of G .

Proposition 10 *There is a canonical isomorphism of $(\bar{\mathfrak{U}}(\mathfrak{g}), P)$ -modules*

$$\bar{V}(\mathbb{E} \otimes \mathbf{W}) = \bar{V}(\mathbb{E}) \otimes \mathbf{W}^*.$$

Proof. The proof of Proposition 6 easily extends. \square

For any finite-dimensional irreducible representation \mathbf{W} of G , we shall refer to the integer 2ℓ which occurs in (8) as the length of \mathbf{W} .

Proposition 11 *Suppose that \mathbb{E} is an irreducible representation of P and that $V(\mathbb{E}')$ splits from $V(\mathbb{E} \otimes \mathbf{W})$ as in Proposition 8. Then, the orders of the splitting homomorphisms $V(\mathbb{E}') \rightleftharpoons V(\mathbb{E} \otimes \mathbf{W})$ are bounded by the length of \mathbf{W} .*

Proof. Since \mathbb{E}' is a composition factor of $\mathbb{E} \otimes \mathbf{W}$,

$$\ell(\mathbb{E}) - \ell \leq \ell(\mathbb{E}') \leq \ell(\mathbb{E}) + \ell.$$

A non-zero symbol $\mathbb{E}'^* \rightarrow \odot^k \mathbf{u}_- \otimes \mathbb{E}^* \otimes \mathbf{W}^*$ forces $-\ell(\mathbb{E}') \leq -k - \ell(\mathbb{E}) + \ell$. Combining these inequalities,

$$k + \ell(\mathbb{E}) - \ell \leq \ell(\mathbb{E}') \leq \ell(\mathbb{E}) + \ell$$

and $k \leq 2\ell$, as required. Similarly, a non-zero symbol $\mathbb{E}^* \otimes \mathbf{W}^* \rightarrow \odot^k \mathbf{u}_- \otimes \mathbb{E}'^*$ implies that $-\ell(\mathbb{E}) - \ell \leq -k - \ell(\mathbb{E}')$ and, again, $k \leq 2\ell$. \square

Corollary 1 *If \mathbf{W} has length less than or equal to 2, then these splittings lift to the semi-holonomic modules: $\bar{V}(\mathbb{E}') \rightleftharpoons \bar{V}(\mathbb{E} \otimes \mathbf{W})$.*

Proof. Immediate from Proposition 2. \square

We shall refer to as fundamental those finite-dimensional irreducible representations of G corresponding to the fundamental dominant weights of \mathfrak{g} . In the notation of [4], this entails having a 1 over some node of the Dynkin diagram and 0's over the others.

Proposition 12 *The fundamental representations of G have length less than or equal to 2.*

Proof. Elementary computation. In fact, the fundamental spin representations have length 1 and the others have length 2. The examples of (9) are typical. \square

Our main result concerning translation is as follows:

Theorem 4 *Suppose \mathbf{W} is a finite-dimensional representation of G of length less than or equal to 2. Suppose that \mathbb{E} , \mathbb{F} , \mathbb{E}' , and \mathbb{F}' are finite-dimensional irreducible representations of P subject to the assumptions of Proposition 9. Then a homomorphism of Verma modules $D : V(\mathbb{F}) \rightarrow V(\mathbb{E})$ lifts to a homomorphism $\bar{D} : \bar{V}(\mathbb{F}) \rightarrow \bar{V}(\mathbb{E})$ of the corresponding semi-holonomic modules if and only if the same is true of the translated homomorphism $D' : V(\mathbb{F}') \rightarrow V(\mathbb{E}')$.*

Proof. If \bar{D} exists, then Propositions 6 and 10, and Corollary 1, give a commutative diagram

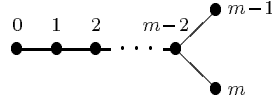
$$\begin{array}{ccccccc}
\bar{V}(\mathbb{F}') \rightarrow \bar{V}(\mathbb{F} \otimes \mathbf{W}) = \bar{V}(\mathbb{F}) \otimes \mathbf{W}^* & \xrightarrow{D \otimes 1} & \bar{V}(\mathbb{E}) \otimes \mathbf{W}^* = \bar{V}(\mathbb{E} \otimes \mathbf{W}) & \rightarrow & \bar{V}(\mathbb{E}') \\
\downarrow & & \downarrow & & \downarrow \\
V(\mathbb{F}') \rightarrow V(\mathbb{F} \otimes \mathbf{W}) = V(\mathbb{F}) \otimes \mathbf{W}^* & \xrightarrow[D']{D \otimes 1} & V(\mathbb{E}) \otimes \mathbf{W}^* = V(\mathbb{E} \otimes \mathbf{W}) & \rightarrow & V(\mathbb{E}') \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow
\end{array}$$

and composition along the top row lifts D' . \square

We now use this semi-holonomic translation to prove the non-existence part of Theorem 1. Let w_\circ denote the longest element of the Weyl group of \mathfrak{g} such that $w_\circ \rho$ is dominant for \mathfrak{p} .

Proposition 13 *Let \mathbb{E}^0 be the irreducible representation of P with lowest weight $-\lambda$ for λ a dominant integral weight for \mathfrak{g} . Let \mathbb{E}^n denote the irreducible representation of P with lowest weight $-w_\circ(\lambda + \rho) + \rho$. Then the homomorphism $V(\mathbb{E}^n) \rightarrow V(\mathbb{E}^0)$ does not lift to the corresponding semi-holonomic modules.*

Proof. Write $\lambda_0, \lambda_1, \dots, \lambda_m$ for the fundamental weights of \mathfrak{g} in accordance with labelling the Dynkin diagrams thus:



A weight of \mathfrak{g} dominant and integral with respect to \mathfrak{p} is of the form $\lambda = a_0 \lambda_0 + a_1 \lambda_1 + \dots + a_m \lambda_m$ for non-negative integers a_1, \dots, a_m . We shall write \mathbb{E}_λ for the finite-dimensional irreducible representation of P with $-\lambda$ as lowest weight. If a_0 is also a non-negative integer, then λ is dominant integral for \mathfrak{g} and we shall write \mathbf{W}_λ for the finite-dimensional irreducible representation of G with lowest weight $-\lambda$. Suppose λ is dominant integral for \mathfrak{g} . Then

$$\mathbb{E}_\lambda \otimes \mathbf{W}_{\lambda_j} = \left(\begin{array}{c} \mathbb{E}_{\lambda + \lambda_j} \\ \oplus \\ \vdots \end{array} \right) + \dots \quad (12)$$

and we maintain that $\mathbb{E}_{\lambda + \lambda_j}$ has distinct central character from all other P -irreducibles occurring in this composition series. Also,

$$\mathbb{E}_{\lambda + \lambda_j} \otimes \mathbf{W}_{\lambda_j}^* = \dots + \left(\begin{array}{c} \vdots \\ \oplus \\ \mathbb{E}_\lambda \end{array} \right) \quad (13)$$

and we maintain that \mathbb{E}_λ has distinct central character. To see the first of these, consider the distance of $\lambda + \lambda_j$ from the origin in \mathfrak{g}^* . Since both λ and λ_j are \mathfrak{g} -dominant, there are no weights of $\mathbb{E}_\lambda \otimes \mathbf{W}_{\lambda_j}$ of greater distance from the origin and after translating by ρ the inequality is strict, i.e. $\lambda + \lambda_j + \rho$ is farther from the origin than any other ρ -translated weight of $\mathbb{E}_\lambda \otimes \mathbf{W}_{\lambda_j}$. Since the Weyl group acts by isometries, Theorem 3 completes the argument. To see that \mathbb{E}_λ has distinct central character in (13) notice that $\ell(\mathbb{E}_\lambda)$ is greater than or equal to the value of ℓ on the other irreducibles. However, it is easy to check that when λ is \mathfrak{g} -dominant, ℓ is strictly minimized on the affine Weyl group orbit by $\ell(\mathbb{E}_\lambda)$.

Now consider the representation $\mathbb{E}_{w_\circ(\lambda+\rho)-\rho}$ for λ a dominant integral weight of \mathfrak{g} . It has the same central character as \mathbb{E}_λ and, indeed, there is a non-trivial Verma module homomorphism $V(\mathbb{E}_{w_\circ(\lambda+\rho)-\rho}) \rightarrow V(\mathbb{E}_\lambda)$. The composition series

$$\mathbb{E}_{w_\circ(\lambda+\rho)-\rho} \otimes \mathbf{W}_{\lambda_j} = \cdots + \begin{pmatrix} \mathbb{E}_{w_\circ(\lambda+\lambda_j+\rho)-\rho} \\ \oplus \\ \vdots \end{pmatrix}$$

has irreducible factors related to (12) under the affine action of w_\circ and hence with the same central characters. Similarly for

$$\mathbb{E}_{w_\circ(\lambda+\lambda_j+\rho)-\rho} \otimes \mathbf{W}_{\lambda_j}^* = \begin{pmatrix} \vdots \\ \oplus \\ \mathbb{E}_{w_\circ(\lambda+\rho)-\rho} \end{pmatrix} + \cdots$$

as compared with (13). Thus, we are in the situation covered by Theorem 4 and we may conclude that for λ a dominant integral weight of \mathfrak{g} and λ_j a fundamental weight, the homomorphism

$$V(\mathbb{E}_{w_\circ(\lambda+\rho)-\rho}) \rightarrow V(\mathbb{E}_\lambda)$$

lifts to the corresponding semi-holonomic modules if and only if the same is true of

$$V(\mathbb{E}_{w_\circ(\lambda+\lambda_j+\rho)-\rho}) \rightarrow V(\mathbb{E}_{\lambda+\lambda_j}).$$

Repeated application of this conclusion reduces to the case $\lambda = 0$. This is precisely Proposition 5. \square

4.1 Completing the proof of Theorem 1

The idea is exactly as in the proof of Proposition 13 and we shall omit many details. The first thing to note is that when \mathbb{E}^0 is trivial all the Verma module homomorphisms of (2) are first order. By Proposition 2, these lift to the corresponding semi-holonomic modules and now it is straightforward to check that, using the semi-holonomic translation of Theorem 4 by suitably chosen fundamental representations, we may lift the general diagram (2).

Next consider the following special singular homomorphisms—they are the most degenerate of the $m + 1$ different types listed in §3.1:

$$V(\mathbb{E}_{\lambda_{m-1}-(m+1)\lambda_0}) \rightarrow V(\mathbb{E}_{\lambda_m-m\lambda_0}), \quad V(\mathbb{E}_{\lambda_m-(m+1)\lambda_0}) \rightarrow V(\mathbb{E}_{\lambda_{m-1}-m\lambda_0}), \quad (14)$$

and,

$$V(\mathbb{E}_{-(m+k)\lambda_0}) \rightarrow V(\mathbb{E}_{-(m-k)\lambda_0}) \quad \text{for } k = 1, 2, \dots, m-1. \quad (15)$$

The two homomorphisms (14) are first order and therefore lift. The homomorphism (15) when $k = 1$ is second order and therefore lifts. (Alternatively, it may be obtained by translating either of (14) by an appropriate fundamental spin representation.) Now we may use \mathbf{W}_{λ_0} to move along the series (15) as follows.

$$\mathbb{E}_{-(m-k)\lambda_0} \otimes \mathbf{W}_{\lambda_0} = \mathbb{E}_{-(m-k-1)\lambda_0} + \mathbb{E}_{\lambda_1-(m-k+1)\lambda_0} + \mathbb{E}_{-(m-k+1)\lambda_0} \quad (16)$$

and under the Weyl group

$$\begin{aligned} -(m-k-1)\lambda_0 + \rho &\mapsto \begin{cases} -\lambda_{m-k-2} + \rho & \text{for } k = 1, \dots, m-2 \\ \rho & \text{for } k = m-1 \end{cases} \\ \lambda_1 - (m-k+1)\lambda_0 + \rho &\mapsto \begin{cases} \lambda_0 - \lambda_{m-k-1} + \rho & \text{for } k = 1, \dots, m-2 \\ \rho & \text{for } k = m-1 \end{cases} \\ -(m-k+1)\lambda_0 + \rho &\mapsto \begin{cases} -\lambda_{m-1} - \lambda_m + \rho & \text{for } k = 1 \\ -\lambda_{m-k} + \rho & \text{for } k = 2, \dots, m-1 \end{cases} \end{aligned}$$

with dominant results. Therefore, the three factors on the right hand side of (16) have mutually distinct central characters for $k = 1, \dots, m-2$ whilst for $k = m-1$ the first two factors have equal central character differing from the third. A similar analysis applies to $\mathbb{E}_{-(m+k)\lambda_0}$ and, observing that $\mathbf{W}_{\lambda_0}^* \cong \mathbf{W}_{\lambda_0}$, we are now in a position to apply Theorem 4. It follows that

all the homomorphisms (15) lift to the corresponding semi-holonomic modules. (It is interesting to note that, in accordance with Proposition 5, the process breaks down just when it would lift $V(\Lambda^n) \rightarrow V(\Lambda^0)$.) It is now straightforward to check that semi-holonomic translation with suitably chosen fundamental representations lifts all the singular homomorphisms from these $m + 1$ basic examples. Not only that, but the homomorphisms (15) may also be translated into the non-singular regime as follows.

Consider the homomorphism $V(\mathbb{E}_{-(n-1)\lambda_0}) \rightarrow V(\mathbb{E}_{-\lambda_0})$. This is (15) when $k = m - 1$. According to (16),

$$\mathbb{E}_{-\lambda_0} \otimes \mathbf{W}_{\lambda_0} = \mathbb{E}_0 + \mathbb{E}_{\lambda_1 - 2\lambda_0} + \mathbb{E}_{-2\lambda_0}.$$

Central character does not split $V(\mathbb{E}_{\lambda_1 - 2\lambda_0})$ off from $V(\mathbb{E}_{-\lambda_0} \otimes \mathbf{W}_{\lambda_0})$ but does provide a surjection

$$V(\mathbb{E}_{-\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \longrightarrow V(\mathbb{E}_{\lambda_1 - 2\lambda_0}).$$

Similarly, there is a homomorphism

$$\begin{aligned} V(\mathbb{E}_{\lambda_1 - n\lambda_0}) &\longrightarrow V(\mathbb{E}_{-(n-1)\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \\ &= V(\mathbb{E}_{\lambda_1 - n\lambda_0} + \mathbb{E}_{-n\lambda_0}) \oplus V(\mathbb{E}_{-(n-2)\lambda_0}) \end{aligned} \quad (17)$$

injecting into the first summand and, hence, a well-defined composition

$$V(\mathbb{E}_{\lambda_1 - n\lambda_0}) \rightarrow V(\mathbb{E}_{-(n-1)\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \rightarrow V(\mathbb{E}_{-\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \rightarrow V(\mathbb{E}_{\lambda_1 - 2\lambda_0}) \quad (18)$$

generalising the usual translation principle. We must show that this homomorphism is non-zero. To do this, we attempt to invert the translation as in Proposition 9:

$$\mathbb{E}_{\lambda_1 - 2\lambda_0} \otimes \mathbf{W}_{\lambda_0}^* = \mathbb{E}_{\lambda_1 - \lambda_0} + \begin{pmatrix} \mathbb{E}_{2\lambda_1 - 3\lambda_0} \\ \oplus \\ \mathbb{E}_{\lambda_2 - 2\lambda_0} \\ \oplus \\ \mathbb{E}_{-\lambda_0} \end{pmatrix} + \mathbb{E}_{\lambda_1 - 3\lambda_0}$$

and these composition factors have mutually distinct central character. In particular, $V(\mathbb{E}_{-\lambda_0})$ splits off from $V(\mathbb{E}_{\lambda_1 - 2\lambda_0} \otimes \mathbf{W}_{\lambda_0}^*)$. Now, consider the composition

$$V(\mathbb{E}_{-(n-1)\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \rightarrow V(\mathbb{E}_{-\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \rightarrow V(\mathbb{E}_{\lambda_1 - 2\lambda_0}).$$

We claim that it is non-zero. By (11), it is equivalent to see that

$$V(\mathbb{E}_{-(n-1)\lambda_0}) \rightarrow V(\mathbb{E}_{-\lambda_0}) \rightarrow V(\mathbb{E}_{\lambda_1-2\lambda_0} \otimes \mathbf{W}_{\lambda_0}^*)$$

is non-zero. This follows because, as we have just observed, the second homomorphism is a splitting. Now consider the full composition (18). If it were zero, then from (17) we would have a non-zero homomorphism

$$V(\mathbb{E}_{-n\lambda_0}) \longrightarrow V(\mathbb{E}_{\lambda_1-2\lambda_0}).$$

According to the classification of §3.1, there is no such operator and so (18) is non-zero, as claimed. (It is interesting to note that, in fact, inverse translation fails—the composition

$$V(\mathbb{E}_{-(n-1)\lambda_0}) \rightarrow V(\mathbb{E}_{\lambda_1-n\lambda_0} \otimes \mathbf{W}_{\lambda_0}^*) \rightarrow V(\mathbb{E}_{\lambda_1-2\lambda_0} \otimes \mathbf{W}_{\lambda_0}^*) \rightarrow V(\mathbb{E}_{-\lambda_0})$$

turns out to be zero.)

A similar analysis may be carried out to obtain

$$V(\Lambda^{m+k}) = V(\mathbb{E}_{\lambda_{m-k}-(m+k+1)\lambda_0}) \longrightarrow V(\mathbb{E}_{\lambda_{m-k}-(m-k+1)\lambda_0}) = V(\Lambda^{m-k})$$

for $k = 2, \dots, m-1$ and even $V(\Lambda^{m+1}) \rightarrow V(\Lambda^{m-1})$ by translating (15) with $\mathbf{W}_{\lambda_{m-k-1}}$. These are the nonstandard homomorphisms (3) with $\lambda = 0$ for $k = 2, \dots, m-1$ and also the standard homomorphism $V(\mathbb{E}^{m+1}) \rightarrow V(\mathbb{E}^{m-1})$.

General dominant integral λ can now be obtained by careful translation with fundamental representations. By repeated application of Propositions 2, 6, and 10, (as in the proof of Theorem 4), bearing in mind Proposition 12, it follows that all these Verma modules homomorphisms (3) for $k = 2, \dots, m-1$ admit semi-holonomic lifts, as claimed in Theorem 1.

4.2 The proof of Theorem 2

In fact, in order to conclude the proof we have just to specify some steps which have been already discussed. The discussion turns out to be much simpler than the even dimensional case, since all non-trivial homomorphisms are non-singular now.

In particular, the basic pattern with $\mathbb{E}^0 = \mathbb{E}_{\lambda_0}$ contains only homomorphisms of order one. By Proposition 2, they all lift to the corresponding semi-holonomic Verma modules. All other standard homomorphisms are

now achieved from the basic pattern by translation with the fundamental representations. By Theorem 4, all of them admit a semi-holonomic lift.

Next consider the non-standard homomorphisms. Once more, suitable translation with fundamental representations restricts the discussion of lifting to the cases where \mathbb{E}^0 is as close to the fundamental Weyl chamber and the origin in \mathfrak{g}^* as possible. Labelling the fundamental weights as

$$\begin{array}{ccccccc} 0 & 1 & 2 & & m-1 & m & \\ \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \end{array},$$

we have only to consider the homomorphisms

$$V(\mathbb{E}_{-(m+3/2)\lambda_0+\lambda_m}) \rightarrow V(\mathbb{E}_{-(m+1/2)\lambda_0+\lambda_m})$$

$$V(\mathbb{E}_{-(m+k-1/2)\lambda_0}) \rightarrow V(\mathbb{E}_{-(m-k+3/2)\lambda_0}) \quad \text{for } k = 2, 3, \dots, m, m+1.$$

The homomorphism on the first line is of order one, while the second line with $k = 2$ yields a homomorphism of order two. By Proposition 2, they both lift to the semi-holonomic modules.

Now, the translation

$$\begin{aligned} V(\mathbb{E}_{-(m+k+1/2)\lambda_0}) &\rightarrow V(\mathbb{E}_{-(m+k-1/2)\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \rightarrow \\ &\rightarrow V(\mathbb{E}_{-(m-k+3/2)\lambda_0} \otimes \mathbf{W}_{\lambda_0}) \rightarrow V(\mathbb{E}_{-(m-k+1/2)\lambda_0}) \end{aligned}$$

exists for all $k = 2, 3, \dots, m$ as discussed in detail in proving Proposition 13. Hence, semi-holonomic translation produces all the remaining lifts and the proof of Theorem 2 is complete.

A Applications to conformal geometry

When G is the Möbius group and G/P is the n -sphere, homomorphisms of Verma modules correspond to Möbius-invariant linear differential operators between conformally weighted spinor-tensor bundles on this sphere. The statements for Verma modules in Section 3 may, therefore, be interpreted as a classification of Möbius-invariant differential operators on the sphere. For example, when \mathbb{E}^0 is trivial, (2) corresponds to the de Rham sequence and the splitting $\Lambda^m = \Lambda_+^m \oplus \Lambda_-^m$ is the (conformally invariant) decomposition of m -forms into self-dual and anti-self-dual types.

The reason for this correspondence is the duality $V_k(\mathbb{E}) = (J^k\mathbb{E})^*$ for any P -module \mathbb{E} . Here, $J^k\mathbb{E}$ is the representation of P inducing the k^{th} jet bundle of the vector bundle on G/P induced by \mathbb{E} . The association of the k^{th} jet bundle $J^k E$ to a vector bundle E is something that is naturally defined on any smooth manifold. Sometimes, these jet bundles are called holonomic to distinguish them from the semi-holonomic jet bundles $\bar{J}^k E$, generally defined by induction as follows (cf. [12]). Start with $J^0 E = E$ and $\bar{J}^1 E = J^1 E$. We shall define $\bar{J}^k E$ as a sub-bundle of $J^1 \bar{J}^{k-1} E$. It therefore comes equipped with a natural projection $\bar{J}^k E \rightarrow \bar{J}^{k-1} E$. Suppose $\bar{J}^k E$ is already defined. Then there are two natural mappings $J^1 \bar{J}^k E \rightarrow J^1 \bar{J}^{k-1} E$. The first is obtained by applying J^1 to the projection $\bar{J}^k E \rightarrow \bar{J}^{k-1} E$. The second is obtained as the composition $J^1 \bar{J}^k E \rightarrow \bar{J}^k E \hookrightarrow J^1 \bar{J}^{k-1} E$. We define $\bar{J}^{k+1} E$ as the sub-bundle on which these two mappings agree:

$$0 \rightarrow \bar{J}^{k+1} E \rightarrow J^1 \bar{J}^k E \rightrightarrows J^1 \bar{J}^{k-1} E.$$

It is easy to check that there is a tautologically defined homomorphism $J^k E \rightarrow \bar{J}^k E$, equivalently a differential operator of order k . These bundles fit into the commutative diagram (cf. (1))

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \odot^k \Lambda^1 \otimes E & \longrightarrow & J^k E & \longrightarrow & J^{k-1} E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \otimes^k \Lambda^1 \otimes E & \longrightarrow & \bar{J}^k E & \longrightarrow & \bar{J}^{k-1} E & \longrightarrow & 0 \end{array}$$

with exact rows. Notice that the functor $E \mapsto \bar{J}^k E$ is completely determined by the 1-jet functor $F \mapsto J^1 F$. On a homogeneous space G/P , this geometrically defines P -modules $\bar{J}^k \mathbb{E}$ for any P -module \mathbb{E} . It is easy to check that $\bar{V}_k(\mathbb{E}) = (\bar{J}^k \mathbb{E})^*$.

It is well-known that a Möbius invariant differential operator on the sphere may sometimes admit a curved analogue, namely a differential operator invariantly defined for any conformal geometry and reducing to the given operator on the sphere. The best known is perhaps the Yamabe operator

$$\Delta + \frac{n-2}{4(n-1)} R$$

acting on conformal densities of weight $1 - \frac{n}{2}$. Here, Δ is the Laplacian and R the scalar curvature, both computed with respect to a metric in the conformal class (see, e.g. [9]).

Theorem 5 *If the homomorphism of induced modules $V(\mathbb{F}) \rightarrow V(\mathbb{E})$ lifts to the associated semi-holonomic modules, then the corresponding Möbius-invariant differential operator on the sphere admits a curved analogue.*

Proof. E. Cartan's frame bundle (see, e.g. [7]) attaches to each conformal manifold, a principal P -bundle. A representation \mathbb{E} of P therefore gives rise to an associated vector bundle E . If \mathbb{E} is irreducible, then E is a conformally weighted spinor-tensor bundle (see, e.g. [1, 9]). More generally, E enjoys a composition series with conformally weighted spinor-tensor bundles as factors. It is shown in [7] that $J^1 E$ may be canonically identified with the bundle associated to $J^1 \mathbb{E}$. This amounts to the construction of an invariant first order differential operator from E to the bundle associated to $J^1 \mathbb{E}$. It is accomplished using the Cartan connection. An equivalent construction may be given using the methods of T.Y. Thomas described in [1]. Since $\bar{J}^k E$ is constructed purely in terms of 1-jets, it follows immediately that it may be identified with the vector bundle associated to the representation $\bar{J}^k \mathbb{E}$. Therefore, a homomorphism of P -modules $\bar{J}^k \mathbb{E} \rightarrow \mathbb{F}$ gives rise to an invariant homomorphism of vector bundles $\bar{J}^k E \rightarrow F$. The composition $J^k E \rightarrow \bar{J}^k E \rightarrow F$ is the required curved analogue. \square

It is interesting to compare this theorem with results from conformal geometry. The conformal analogues constructed by this theorem are already known to exist (see [11] in four dimensions and [2] generally) but the approach via semi-holonomic homomorphisms seems to be cleaner. The homomorphism $V(\Lambda^n) \rightarrow V(\Lambda^0)$ of Proposition 5 actually has a curved analogue (i.e. a conformally invariant operator with symbol $\Delta^{n/2}$) but its existence is quite subtle [14]. It is conjectured that, when n is even, all other homomorphisms $V(\mathbb{E}^n) \rightarrow V(\mathbb{E}^0)$ do not have curved analogues but the only case where this has been verified is for Δ^3 in four dimensions [13]. It is possible that the semi-holonomic approach will shed light on this conjecture. Some differential geometric aspects are clearly represented in the algebra—the holonomic symbol gives rise to the symbol of the invariant operator and the extra terms involved in a semi-holonomic lift give rise to curvature terms.

The results of this article should generalise to the almost Hermitian symmetric geometries of Baston [3]. The relevant invariant derivatives are certainly present [7, 8]. It remains, therefore, to identify those homomorphisms of Verma modules which lift to their semi-holonomic counterparts. For the exceptional geometry based on E_6 partial results have been obtained by Sawon [20].

References

- [1] T. N. Bailey, M. G. Eastwood, and A. R. Gover, Thomas's structure bundle for conformal, projective, and related structures, *Rocky Mtn. Jour. Math.* **24** (1994), 1191–1217.
- [2] R. J. Baston, Verma modules and differential conformal invariants, *Jour. Diff. Geom.* **32** (1990), 851–898.
- [3] R. J. Baston, Almost Hermitian symmetric manifolds I: Local twistor theory, *Duke Math. Jour.* **63** (1991), 81–137.
- [4] R. J. Baston and M. G. Eastwood, “The Penrose Transform: Its Interaction with Representation Theory,” Oxford University Press, 1989.
- [5] B. D. Boe and D. H. Collingwood, A comparison theory for the structure of induced representations I, *Jour. Alg.* **94** (1985), 511–545.
- [6] B. D. Boe and D. H. Collingwood, A comparison theory for the structure of induced representations II, *Math. Zeit.* **190** (1985), 1–11.
- [7] A. Čap, J. Slovák, and V. Souček, Invariant operators on manifolds with almost Hermitian symmetric structures I: Invariant differentiation, Preprint 1994 (ESI 186 from ftp.esi.ac.at).
- [8] A. Čap, J. Slovák, and V. Souček, Invariant operators on manifolds with almost Hermitian symmetric structures II: Normal Cartan connections, Preprint 1995 (ESI 194 from ftp.esi.ac.at).
- [9] M. G. Eastwood, Notes on conformal differential geometry, in “Proceedings of the Fifteenth Winter School on Geometry and Physics, Srní 1995,” *Suppl. Rend. Circ. Mat. Palermo* (to appear).
- [10] M. G. Eastwood and C. R. Graham, Invariants of conformal densities, *Duke Math. Jour.* **63** (1991), 633–671.
- [11] M. G. Eastwood and J. W. Rice, Conformally invariant differential operators on Minkowski space and their curved analogues, *Commun. Math. Phys.* **109** (1987), 207—228. Erratum, *Commun. Math. Phys.* **144** (1992), 213.

- [12] C. Ehresmann, Extension du calcul des jets aux jets non holonomes, *C. R. Acad. Sci. Paris* **239** (1954), 1762–1764.
- [13] C. R. Graham, Conformally invariant powers of the Laplacian II: Nonexistence, *Jour. Lond. Math. Soc.* **46** (1992), 566–576.
- [14] C. R. Graham, R. Jenne, L. J. Mason, and G. A. J. Sparling, Conformally invariant powers of the Laplacian I: Existence, *Jour. Lond. Math. Soc.* **46** (1992), 557–565.
- [15] J. E. Humphreys, “Introduction to Lie Algebras and Representation Theory,” Grad. Texts Math. vol. 9, Springer, 1972.
- [16] J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, *Jour. Alg.* **49** (1977), 496–511.
- [17] A. A. Kirillov, Invariant operators over geometric quantities, in “Current Problems in Mathematics,” vol 16 (Russian), VINITI, 1980, pp. 3–29.
- [18] B. Kostant, Verma modules and the existence of quasi-invariant differential operators, in “Non-Commutative Harmonic Analysis,” Lect. Notes Math. vol. 466, Springer, 1975, pp. 101–128.
- [19] R. Penrose and W. Rindler, “Spinors and Space-time, vol. 1,” Cambridge University Press, 1984.
- [20] J. Sawon, “Homomorphisms of Semi-holonomic Verma Modules: an Exceptional Case,” M.Sc. thesis, University of Adelaide, 1996.
- [21] D. A. Vogan, “Representations of Real Reductive Lie Groups,” Prog. Math. vol. 15, Birkhäuser, 1981.
- [22] G. Zuckerman, Tensor products of finite and infinite dimensional representations of semisimple Lie groups, *Ann. Math.* **106** (1977), 295–308.