Invariant operators and semi-holonomic Verma modules

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Abstract

This brief survey aims to discuss a recently discovered relation between invariant operators in certain geometries, non-holonomic jet prolongations and the corresponding non-holonomic generalizations of the usual Verma modules. Since we wish to provide a sort of bridge between the differential geometric and representational theoretic backgrounds, we also recall a few elementary concepts on both sides. A more detailed information can be found in [EasS], [CSS1], [CSS2], [Slo], and further papers will follow.

1 Cartan geometries

Roughly speaking, the geometries introduced by Cartan under the name 'espace generalisé' are curved deformations of homogeneous spaces G/P where P is a (closed) subgroup in a Lie group G. All such possibilities for G and P give the *flat models* $G \to G/P$ of the geometries in question. The properties of G are encoded in the (left) Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ and the latter form is the subject of the deformations we have in mind. Thus instead of the principal P-bundle $G \to G/P$ we shall deal with a general principal P-bundle $\mathcal{G} \to M$, equipped with a one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, subject to the following properties (ζ_X denotes the fundamental vector field given by X):

Curved geometry $\mathcal{G} \to M$, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$	**Flat model** $G \to G/P$, Maurer-Cartan form ω
$\omega(\zeta_X) = X$ for all $X \in \mathfrak{p}$	$\omega(\zeta_X) = X$ for all $X \in \mathfrak{g}$
$(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega \ \forall b \in P$	$(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega \ \forall b \in G$
$\omega_{\vert T_u\mathcal{G}} \colon T_u\mathcal{G} \to \mathfrak{g}$ iso $\forall u \in \mathcal{G}$	$\omega_{\vert T_u G} \colon T_u G \to \mathfrak{g}$ iso $\forall u \in G$

A form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with the three properties listed above is called *Cartan connection (of the type G/P)*. Let us notice in particular the third property, which yields the *horizontal vector field* $\omega^{-1}(X)$ on $\mathcal{G}$ for each element $X \in \mathfrak{g}$. The first condition then tells that the latter fields are the fundamental fields ζ_X for all $X \in \mathfrak{p}$. The extent of the deformation is measured by the *curvature of the Cartan connection*, the two-form $\kappa \in \Omega^2(\mathcal{G}, \mathfrak{g})$ given by the *structure equation*

$$d\omega + \frac{1}{2}[\omega, \omega] = \kappa.$$

In particular, $(\mathcal{G}, \omega)$ is locally isomorphic to (G, ω) if and only if κ vanishes. It follows immediately from the definition, that κ is a horizontal form. Due to the presence of the horizontal vector fields we can view κ as a function valued in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, i.e. $\kappa \in C^\infty(\mathcal{G}, \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})$.

All these old ideas go back to E. Cartan and his concept of 'generalized spaces', see e.g. [Car1], [Car2]. See also the recent book [Sha] for many illuminating comments.

We are mostly interested in the special class of such geometries where either $\mathfrak{p}$ is a parabolic subalgebra in a complex semisimple Lie algebra $\mathfrak{g}$, or $\mathfrak{p}$ and $\mathfrak{g}$ represent a real form of such a situation. Following Fefferman and Graham, we are using the name *parabolic geometries* in this context, cf. [FefG], [Gra].

More explicitly, we deal with a pair $(\mathfrak{g}, \mathfrak{p})$ where $\mathfrak{g}$ is a (real or complex) semisimple Lie algebra of the Lie group G equipped with a finite grading $\mathfrak{g} = \mathfrak{g}_{-\ell} \oplus \cdots \oplus \mathfrak{g}_\ell$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\ell$. The group P is then the Lie subgroup corresponding to the subalgebra $\mathfrak{p}$. We also write $\mathfrak{g}_-$ for $\mathfrak{g}_{-\ell} \oplus \cdots \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p}_+$ for $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$. Then $\mathfrak{g}_- \simeq (\mathfrak{g}/\mathfrak{p})$ and the Killing form yields $(\mathfrak{g}_-)^* \simeq \mathfrak{p}_+$. In the case of complex Lie groups $P \subset G$ this just means that P is a parabolic subgroup.

The original aim of the Cartan connections was the equivalence problem and they were constructed from some simpler underlying data. Such a construction for our class of geometries was discussed extensively in the literature, see e.g. [Mor], [Tan], [Yam], and the definitive and most complete version is available in [CSch].

The existence of the horizontal fields $\omega^{-1}(X)$ for each $X \in \mathfrak{g}$ yields the so called *invariant differential* ∇^ω acting on all vector-valued functions $C^\infty(\mathcal{G}, \mathbb{E})$, in particular on the sections of the corresponding associated bundles. Simply, the differential is defined for all $u \in \mathcal{G}$, $X \in \mathfrak{g}_-$, by the Lie derivative of functions

$$\nabla^\omega \colon C^\infty(\mathcal{G}, \mathbb{E}) \to C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{E}), \quad \nabla^\omega s(u)(X) = \mathcal{L}_{\omega^{-1}(X)} s(u).$$

For our purposes, we shall assume the Cartan connection on $\mathcal{G} \to M$ is given. Then a rich geometrical structure is induced and the main source for various underlying concepts and their relations lies in the grading of the Lie algebra $\mathfrak{g}$.

2 Jet-modules

Let us fix some notation. For each representation λ of P, we shall write $\mathbb{E}_\lambda$ for the corresponding P-module. Given the principal P-bundle $\mathcal{G}$ and such a module $\mathbb{E}_\lambda$, we write $\mathcal{E}(\mathbb{E}_\lambda)$, or just $\mathcal{E}(\lambda)$ for the induced associated vector bundle. The kth jet prolongation of such bundles $\mathcal{E}$, i.e. the vector spaces of all jets of their local sections will be denoted by $J^k\mathcal{E}$.

In the special case of the homogeneous vector bundle $\mathcal{E}(\lambda) = G \times_P \mathbb{E}_\lambda$ over the homogeneous space $G \to G/P$, the jet prolongations $J^k\mathcal{E}(\lambda)$ inherit the action of G. If we view sections in $C^\infty\mathcal{E}(\lambda)$ as P-equivariant functions $s \in C^\infty(\mathcal{G}, \mathbb{E}_\lambda)^P$, then the 1-jets of sections at the distinguished point $o \in G/P$ are identified with 1-jets of these equivariant functions at the unit $e \in G$ and the action is given by $g.(j^1_e s) = j^1_e(s \circ \ell_{g^{-1}})$ for all $g \in G$. Indeed, for each section s and any local section u of $\mathcal{G}$ we have

$$\begin{aligned} g.j^1_o(x \mapsto \{u(x), s(u(x))\}) &= j^1_o(x \mapsto \{\ell_g \circ \ell_{g^{-1}} \circ u(x), s(\ell_{g^{-1}} \circ u(x))\}) \\ &= j^1_0(x \mapsto \{u(x), s \circ \ell_{g^{-1}}(u(x))\}). \end{aligned}$$

Thus the induced action of $Z \in \mathfrak{p}$ on the section s is given by differentiation in the direction of the right invariant vector field ζ^R_Z on G, $Z.j^1_e s = -j^1_e \zeta^R_Z.s$.

We shall write the jets as $j^1_e s = (v, \varphi) \in \mathbb{E}_\lambda \oplus (\mathfrak{g}^*_- \otimes \mathbb{E}_\lambda)$, where we identify $T_e G \simeq \mathfrak{g}_- \oplus \mathfrak{p}$ via ω, i.e. $v = s(e)$ and $\varphi(X) = \omega^{-1}(X).s(e)$. Now we can express the action of $Z \in \mathfrak{p}$ in these terms (we use the commuting of the left invariant fields and the right invariant fields and the fact that their values at e coincide)

$$\begin{aligned} (v,\varphi) \mapsto -j^1_e\zeta^R_Z.s &= \big(-\zeta^R_Z.s(e), (X \mapsto -\omega^{-1}(X).(\zeta^R_Z.s)(e))\big) \\ &= \big(-\omega^{-1}(Z).s(e), (X \mapsto -\omega^{-1}(Z).(\omega^{-1}(X).s)(e))\big) \\ &= \big(\lambda(Z)(v), (X \mapsto -\omega^{-1}(X).(\omega^{-1}(Z).s)(e) - [\omega^{-1}(Z), \omega^{-1}(X)].s(e))\big) \\ &= \big(\lambda(Z)(v), (X \mapsto \lambda(Z) \circ \varphi(X) - \varphi \circ \mathrm{ad}_-(Z)(X) + \lambda(\mathrm{ad}_\mathfrak{p}(Z)(X)(v)))\big) \\ &= \big(\lambda(Z)(v), \lambda(Z) \circ \varphi - \varphi \circ \mathrm{ad}_-(Z) + \lambda(\mathrm{ad}_\mathfrak{p}(Z)(\_))(v)\big). \end{aligned} \tag{1}$$

So we define the $\mathfrak{p}$-module $J^1\mathbb{E}_\lambda$ as $\mathbb{E}_\lambda \oplus (\mathfrak{g}^*_- \otimes \mathbb{E}_\lambda)$ with the $\mathfrak{p}$-action given by (1). We call this module the *first jet prolongation* of $\mathbb{E}_\lambda$.

By the construction, there always is the P-module structure on $J^1\mathbb{E}$ compatible with (1). In our case, with $P \subset G$ parabolic, the whole P is a semidirect product of its reductive part G_0 and the nilpotent exponential image of $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$.

Now, $J^1\mathbb{E}_\lambda = \mathbb{E}_\lambda \oplus (\mathfrak{g}_-^* \otimes \mathbb{E}_\lambda)$ as G_0-module and we can read the P-module structure off (1) as well.

Obviously, for each P-module homomorphism $\alpha\colon \mathbb{E}_\lambda \to \mathbb{E}_\mu$ the mapping

$$J^1\alpha\colon (v, \varphi) \mapsto (\alpha(v), \alpha \circ \varphi)$$

is a well defined P-module homomorphism $J^1\mathbb{E}_\lambda \to J^1\mathbb{E}_\mu$. Thus J^1 is a functor on P-modules. We shall also write $J^1\lambda$ for the corresponding representation.

As well known from the classical differential geometry, the iteration of the jet functors leads to the non-holonomic jet prolongations and the semi-holonomic and holonomic ones are defined as certain equalizers. Since we have posed no restrictions on the representation λ above, we can also iterate the functors J^1 on the P-modules.

Let us look more carefully at $J^1J^1\mathbb{E}_\lambda$ and the non-holonomic second order prolongation $J^1J^1\mathcal{E}(\lambda)$. There are two obvious $\mathfrak{p}$-module homomorphisms $J^1J^1\mathbb{E}_\lambda \to J^1\mathbb{E}_\lambda$, the first one given by the projection $p_\lambda\colon (v, \varphi) \mapsto v$ defined on each first jet prolongation and the other obtained by the action of J^1 on p_λ. Thus there is the submodule $\bar{J}^2\mathbb{E}_\lambda$ in $J^1J^1\mathbb{E}_\lambda$ on which these two projections coincide. As a vector space, this is

$$\bar{J}^2\mathbb{E}_\lambda = \mathbb{E}_\lambda \oplus (\mathfrak{g}_-^* \otimes \mathbb{E}_\lambda) \oplus (\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathbb{E}_\lambda).$$

The two P-module homomorphisms J^1p_λ, $p_{J^1\lambda}$ give rise to fiber bundle morphisms $J^1J^1\mathcal{E}(\lambda) \to J^1\mathcal{E}(\lambda)$ which are just the two standard projections on second non-holonomic jet prolongations.

Iterating this procedure, we obtain the *kth semi-holonomic jet modules* $\bar{J}^k\mathbb{E}_\lambda$ and their prolongations $J^1(\bar{J}^k\mathbb{E}_\lambda)$ equipped with two natural projections onto $\bar{J}^k\mathbb{E}_\lambda$, which correspond to the usual projections on the first jet prolongation of semi-holonomic jets. Their equalizer is then the P-submodule $\bar{J}^{k+1}\mathbb{E}_\lambda$. As a vector space (and G_0-module),

$$\bar{J}^k\mathbb{E}_\lambda = \bigoplus_{i=0}^{k} (\otimes^i \mathfrak{g}_-^* \otimes \mathbb{E}_\lambda).$$

Now, the existence of the Cartan connection ω on our principal bundle $\mathcal{G}$ provides important geometrical identifications: In particular, we have the mapping

$$\iota\colon C^\infty(\mathcal{G}, \mathbb{E}_\lambda)^P \to C^\infty(\mathcal{G}, J^1\mathbb{E}_\lambda)^P, \quad \iota(s)(u) = (s(u), (X \mapsto \nabla^\omega s(u)(X)))$$

which yields a diffeomorphism $J^1\mathcal{E}(\lambda) \simeq \mathcal{G} \times_P J^1\mathbb{E}_\lambda$.

Indeed, the mapping $\iota\colon s \mapsto (s, \nabla^\omega s)$ is well defined and depends on first jets only, but we have to check its equivariance. This means exactly the commuting with the derivatives in the directions of fundamental vector fields $\omega^{-1}(Z)$, $Z \in \mathfrak{p}$. So we aim at $-\zeta_Z \iota(s)(u) = J^1\lambda(Z) \circ \iota(s)(u)$. To see this, we just have to copy the computation in (1) and to remember that the curvature of any Cartan connection is horizontal:

$$\begin{aligned} -\zeta_Z.\iota(s)(u) &= \big(-\omega^{-1}(Z).s(u), (X \mapsto -\omega^{-1}(Z).(\omega^{-1}(X).s)(u))\big) \\ &= \big(\lambda(Z)(s(u)), (X \mapsto -\omega^{-1}(X).(\omega^{-1}(Z).s)(u) - \omega^{-1}([Z,X]).s(u))\big) \\ &= \big(\lambda(Z)(s(u)), (X \mapsto \lambda(Z)\circ\nabla^\omega s(u)(X) - \nabla^\omega s(u)\circ \mathrm{ad}_-(Z)(X) \\ &\qquad + \lambda(\mathrm{ad}_{\mathfrak{p}}(Z)(X))(s(u)))\big) \end{aligned}$$

Clearly, we have constructed a diffeomorphism $J^1\mathcal{E}(\lambda) \to \mathcal{G}\times_P J^1\mathbb{E}_\lambda$.

Now, consider a homomorphism $\alpha\colon \mathbb{E}_\lambda \to \mathbb{E}_\mu$. The corresponding homomorphism $f\colon \mathcal{E}(\lambda) \to \mathcal{E}(\mu)$ is defined by $\{u,v\} \mapsto \{u,\alpha(v)\}$, and so the induced action on sections is $(x \mapsto \{u(x), s(u(x))\}) \mapsto (x \mapsto \{u(x), \alpha\circ s(u(x))\})$. Taking 1-jet of this expression we obtain just the homomorphism $J^1\alpha$. Thus, for each fiber bundle morphism $f\colon \mathcal{E}(\lambda)\to\mathcal{E}(\mu)$ given by a P-module homomorphism $\alpha\colon \mathbb{E}_\lambda\to\mathbb{E}_\mu$, the first jet prolongation $J^1 f$ corresponds to the P-module homomorphism $J^1\alpha$.

Since the higher order semi-holonomic jets are built by means of iterations, we obtain the following consequences for higher orders: For each integer $k>0$, the kth semi-holonomic jet prolongation $\bar{J}^k\mathcal{E}(\lambda)$ carries the natural structure of the associated fiber bundle $\mathcal{G}\times_P \bar{J}^k\mathbb{E}_\lambda$. Moreover, the invariant differential defines the natural embedding

$$J^k\mathcal{E}(\lambda) \ni j^k_u s \mapsto \{u, (s(u), \nabla^\omega s(u), \dots, (\nabla^\omega)^k s(u))\} \in \bar{J}^k E_\lambda \simeq \mathcal{G}\times_P \bar{J}^k\mathbb{E}_\lambda.$$

It is just the existence of the natural associated bundle structure on $\bar{J}^k\mathcal{E}(\lambda)$ (i.e. depending on ω only) which gives rise to the differential operator $D_\Phi\colon C^\infty\mathcal{E}(\lambda)\to C^\infty\mathcal{E}(\mu)$ for each P-module homomorphism $\Phi\colon \bar{J}^k\mathbb{E}_\lambda \to \mathbb{E}_\mu$. In view of the existence of the canonical Cartan connections on the parabolic geometries, this means that each such P-module homomorphism defines an invariantly defined operator on manifolds with the appropriate geometric structures. On the other hand, not all invariant operators arise in this way, see e.g. [CSS1, EasS, Slo].

The existence of the canonical embedding provided by the iterated differential suggests a straightforward method for explicit constructions of such operators. Given a P-module homomorphism $\Phi\colon \bar{J}^k\mathbb{E}_\lambda\to\mathbb{E}_\mu$ we compose this with the iterated differentials to obtain quite explicit analytic expressions for the operators. On the other hand, we can also start with an arbitrary G_0-module homomorphism Φ, compose it with the differentials and discuss the equivariance of the resulting expression. Its expansion in terms of the underlying generalized Weyl geometries yields an algorithmic method for finding operators, see e.g. [CSS1, Slo].

While the semi-holonomic prolongations $\bar{J}^k\mathcal{E}(\lambda)$ are constructed by a purely algebraic construction, the embedding of $J^k\mathcal{E}(\lambda)$ depends of course heavily on the curvature of the Cartan connection. This makes the discussion on the algebraic conditions for the existence of invariant operators which are not coming from P-module homomorphisms much more difficult.

3 Semi-holonomic Verma modules and the translation principle

Instead of seeking for P-module homomorphisms $\Phi\colon \bar{J}^k\mathbb{E}_\lambda \to \mathbb{E}_\mu$, we can pass to their dual morphisms $\Phi^*\colon \mathbb{E}^*_\mu \to (\bar{J}^k\mathbb{E}_\lambda)^*$. This simple observation has amazing consequences.

Let us first recall the notion of the (generalized) Verma modules. It is well known for holonomic jets that the dual modules $(J^k\mathbb{E}_\lambda)^*$ enjoy the nice algebraic structure of the finite dimensional parts of the filtered induced module

$$V_{\mathfrak{p}}(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E}^*_\lambda.$$

In particular, the (topological) dual to the inverse limit $J^\infty\mathbb{E}_\lambda$ is the whole *generalized Verma module* $V_{\mathfrak{p}}(\lambda)$. Thus, instead of looking for homomorphisms defined on the highly reducible P-modules $J^k\mathbb{E}_\lambda$ we have only to discuss morphisms defined on the highest weight modules $V_{\mathfrak{p}}(\lambda)$ and the whole task is reduced to the discussion on the so called singular vectors in the Verma modules. These are quite well known in representation theory and the description of the invariant operators on the homogeneous bundles is reduced to a purely algebraic question in this way. See e.g. [Dob], [BasE] for many further links. Fortunately, our 'less symmetric' P-modules $\bar{J}^k\mathbb{E}_\lambda$ have quite similar duals, first studied in [EasS].

We start with a modification of the definition of $\mathfrak{U}(\mathfrak{g})$. Our algebra $\bar{\mathfrak{U}}(\mathfrak{g})$ is defined as the quotient of the free tensor algebra $T(\mathfrak{g})$ by the two-sided ideal I which is generated by $\{X \otimes Y - Y \otimes X - [X, Y];\text{ for all } X \in \mathfrak{p}, Y \in \mathfrak{g}\}$. Thus, we force the compatibility of the commutator with the bracket only for those brackets with at least one element in $\mathfrak{p}$.

The *semi-holonomic Verma module* induced from the P-module $\mathbb{E}_\lambda$ is the $(\mathfrak{g}, P)$-module

$$\bar{V}_{\mathfrak{p}}(\lambda) = \bar{\mathfrak{U}}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E}^*_\lambda.$$

The obvious filtration of $\bar{U}(\mathfrak{g})$ gives rise to the filtration of the semi-holonomic Verma module

$$\mathbb{E}^*_\lambda = \mathcal{F}_0\bar{V}_{\mathfrak{p}}(\lambda) \subset \mathcal{F}_1\bar{V}_{\mathfrak{p}}(\lambda) \subset \dots$$

by $\mathfrak{U}(\mathfrak{p})$-modules, with

$$\mathcal{F}_{k+1}\bar{V}_{\mathfrak{p}}(\lambda)/\mathcal{F}_k\bar{V}_{\mathfrak{p}}(\lambda) \simeq \otimes^{k+1}\mathfrak{g}_- \otimes \mathbb{E}^*_\lambda$$

as G_0–modules.

A straightforward computation shows that the $\mathfrak{U}(\mathfrak{p})$-modules $(\bar{J}^k\mathbb{E}_\lambda)^*$ dual to the semi-holonomic jet modules are naturally identified with the $\mathfrak{U}(\mathfrak{p})$-submodules $\mathcal{F}_k\bar{V}_\mathfrak{p}(\lambda)$ in the semi-holonomic Verma module $\bar{V}_\mathfrak{p}(\lambda)$.

The next ingredient we need is the translation principle for homomorphisms of Verma modules which is based on the properties of the central (or infinitesimal) character. We do not have a straightforward analogy of the central character for semi-holonomic Verma modules, but we can still extend the translation functors due to Zuckerman and Jantzen. In fact we can show that once the essential homomorphisms building these functors would exist in the semi-holonomic setting, then also the whole translation principle worked as well. Special situations were studied in great detail in [EasS].

Roughly speaking, the key observation for the original translations is that for each G-module $\mathbf{W}$ the induced module $V_\mathfrak{p}(\mathbb{E}_\lambda \otimes \mathbf{W})$ is naturally isomorphic to $V_\mathfrak{p}(\mathbb{E}_\lambda)\otimes\mathbf{W}$ and so each homomorphism $\varphi\colon V_\mathfrak{p}(\lambda) \to V_\mathfrak{p}(\mu)$ gives rise to the 'twisted' homomorphism $\varphi \otimes \mathrm{id}_\mathbf{W}$ and the compositions of the latter morphism with the projections and embeddings given by the decompositions by the central character yield the 'translated' operators.

Now, the semi-holonomic Verma modules also satisfy $\bar{V}_\mathfrak{p}(\mathbb{E}_\lambda\otimes\mathbf{W}) \simeq \bar{V}_\mathfrak{p}(\mathbb{E}_\lambda)\otimes\mathbf{W}$ for G-modules $\mathbf{W}$, see [EasS], but new features appear. In general, there are just two main points: (1) We have to find 'initial data' to start the translations with. (2) We have to collect enough of P-module homomorphisms providing the necessary splittings.

There are many homomorphisms of semi-holonomic Verma modules in general. Therefore we restrict our attention only to those which posses a given symbol, we talk about 'liftings' of the homomorphisms between classical Verma modules to the semi-holonomic ones.

In [EasS], this was done in detail for the conformal Riemannian geometries. Both in even and odd dimensions m, the initial data consist of the exterior differentials on forms, the powers of the Laplace operator and the Dirac operator (though even these may be translated, essentially all from just a sigle identity operator — but such translations are highly non-trivial and work only in one direction, as a rule). All the necessary splitting operators are then obtained from the choice of the standard representation $\mathbf{W} = \mathbb{R}^{m+2}$ of $G = Spin(m+1, 1, \mathbb{R})$ and they all exist in the semi-holonomic case since their orders are at most two. In odd dimensions, this works for all powers of the Laplacians and the rest is achieved by translations, while in even dimensions $m = 2n$ the critical nth power of the Laplacian does not exist in the form of a P-module homomorphism of the semi-holonomic Verma modules, see [EasS]. On the other hand, it is known that these critical powers do exist as invariant operators on conformal Riemannian manifolds. Consequently, there are homomorphisms of (holonomic) Verma modules which do not have semi-holonomic lifts.

Thus, we are able to get very general structural results on the existence of the homomorphisms of semi-holonomic Verma modules. On the other hand, even if we find the singular vectors in $\bar{V}_{\mathfrak{p}}(\lambda)$ defining those homomorphisms, it is not evident how to find analytic formulae for the operators in a direct algorithmic way.

Of course, an explicit expression of the singular vector in terms of the rising and lowering operators will provide the analytic formula in terms of the invariant differential. In the flat case, such formulae are discussed in several papers, see e.g. [Dob] and the references therein. From the point of view of differential geometry, expressions of these operators in terms of the underlying linear connections on the manifold in question seem to be more relevant. An algorithmic procedure expanding the invariant differential in these terms is worked out in [CSS1], [Slo].

The combination of the direct discussion on the jet level with the dual picture seems to be most promising. More explicitly, the twisted homomorphisms exist also on the semi-holonomic jet modules and they can be computed in a very explicit way. Also the splittings admit quite concrete expressions. Thus the existence and structural results on the homomorphisms of semi-holonomic Verma modules find their explicit counterparts on the level of the jets. See [Cap] for further information.

4 Final remarks

We skip completely the discussion on the holonomic cases, these are more or less understood in the representation theory.

The treatment of the semi-holonomic modules is in its beginning and the paper [EasS] shows the first steps only. At present, any concrete type of parabolic geometries, i.e. any choice of G and P as discussed above, allows a similar procedure along the lines of the latter paper, but it seems that a satisfactory understanding of the general case of all parabolic geometries requires further new ideas. One of them will definitly be the algebraic understanding of the geometric formal adjointness, as discussed in [Eas].

There is also the possibility of a direct approach on the (finite) jet level. In fact, the 'twisted' operators between semi-holonomic Verma modules do have their counterparts and all the splitting operators are built by a quite nice geometric construction using the canonical Cartan connections. First steps in this direction can be traced in several papers, see e.g. [BEG] discussing the Thomas's D-operators. The general and very explicite constructions are presented in [Cap]. This approach has also proved as very fruitful when applied to the special case of the so called curved analogs of the Bernstein-Gelfand-Gelfand resolutions. In [CSS4], the latter analogs are constructed for all parabollic geometries and all irreducible G-modules, without referring to the flat case.

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