INVARIANT OPERATORS ON MANIFOLDS WITH ALMOST HERMITIAN SYMMETRIC STRUCTURES, III. STANDARD OPERATORS.

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Abstract. This paper demonstrates the power of the calculus developed in the two previous parts of the series for all real forms of the almost Hermitian symmetric structures on smooth manifolds, including e.g. conformal Riemannian and almost quaternionic geometries. Exploiting some finite dimensional representation theory of simple Lie algebras, we give explicit formulae for distinguished invariant curved analogues of the standard operators in terms of the linear connections belonging to the structures in question, so in particular we prove their existence. Moreover, we prove that these formulae for kth order standard operators, $k = 1, 2, \dots$, are universal for all geometries in question.

1. Introduction

As generally known, several geometries share surprisingly many properties with the conformal Riemannian structures and projective structures. For example the almost quaternionic ones. Following the old ideas by Cartan, and some more recent development by Baston, Eastwood, Gindikin, Goncharov, Ochiai, Tanaka, and others, we have started the project of building a good calculus for all of them. This paper presents the first major application of the technique developed so far for the so called AHS-structures in the first two parts of this series, [CSS1, CSS2].

In [F], Fegan described all conformally invariant operators of the first order on conformal Riemannian manifolds. We use the invariant differentiation with respect to Cartan connections developed in [CSS1], together with some representation theory of simple Lie algebras, in order to extend Fegan's methods to operators of all orders. This new technique works for a wide class of geometries and, using the explicit computations of the canonical Cartan connections in [CSS2], we obtain formulae for all these invariant operators in terms of covariant derivatives with respect to the linear connections belonging to the structures and their curvatures. Moreover, a simple recursive procedure for the computation of the correction terms for standard operators is described.

In such a way, the abstract indication of the existence of the standard invariant linear differential operators on manifolds with almost Hermitian symmetric structures given in [B] is replaced by an explicit and transparent construction, which provides even formulae in closed forms. Surprisingly enough, these universal formulae do not depend on the particular geometry at all.

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In order to make the paper more self-contained, we have included a brief review of some background from [CSS1]. This concerns the short section 2 where we also fix the notation used in the sequel. The sections 3 through 5 provide the necessary development in representation theory. In order to address a wider audience among differential geometers, we try to be quite detailed here. Section 6 gives the main existence result (Theorem 6.5) and the explicit formulae are established in section 7 (Theorems 7.4 and 7.9). Some technical points are postponed to two appendices.

2. A calculus for Cartan connections

The aim of this section is to summarize for convenience of the reader the main development from [CSS1]. Full details and proofs can be found there.

2.1 AHS structures. A basic datum distinguishing a particular AHS structure is a real simple Lie group G with the Lie algebra $\mathfrak{g}$, which is $|1|$-graded, i.e.

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

with $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$; $\mathfrak{g}_j = \{0\}, j \neq -1, 0, 1$. There is a list of all simple real $|1|$-graded Lie algebras (see [KN]). Their complexification is a semisimple $|1|$-graded complex Lie algebra. The classification of complex simple $|1|$-graded Lie algebras corresponds to the well known list of Hermitian symmetric spaces. The latter fact has been the origin of the name A(lmost) H(ermitian) S(ymmetric) we use.

The subalgebras $\mathfrak{g}_{\pm 1}$ are commutative and dual to each other with respect to the Killing form. The algebra $\mathfrak{g}_0$ is reductive with one-dimensional center, which is generated by the grading element E, which is characterized by the fact that each of the subalgebras $\mathfrak{g}_j$, $j = -1, 0, 1$, ist the eigenspaces for the adjoint action of E with eigenvalue j. The semisimple part $[\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{g}_0^s$.

The subgroups P, resp. P_1 of G correspond to the Lie algebra $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, resp. $\mathfrak{g}_1$. The group P_1 is a normal subgroup of P and the group $G_0 = P/P_1$ has the Lie algebra $\mathfrak{g}_0$. Let us mention that we have used the letter B instead of P in [CSS1].

The typical and best understood example of AHS structures is a conformal structure on a manifold M. A standard way to define it is a reduction of the frame bundle of M to the conformal group $G_0 = CO(n, \mathbb{R})$. A classical theorem going back to Cartan gives a construction of a P-principal bundle $\mathcal{G}$ (where P is a semidirect product of G_0 and $\mathbb{R}^n$) over M and a uniquely defined Cartan connection ω on $\mathcal{G}$. Such data were considered by Cartan as a curved analogue of the flat model G/P (an example of his 'espaces généralisés'). The characteristic properties of the Cartan connection ω are a simple generalization of properties of the Maurer-Cartan form ω on G/P.

Following previous results by Tanaka, Ochiai, and Baston, a simple and transparent principal bundle approach to a canonical construction of the principal bundle $\mathcal{G}$ with structure group P and of the Cartan connection ω on $\mathcal{G}$ from the standard first order G_0-structure on M was described in [CSS2]. We shall not need the construction here and we shall start with $\mathcal{G}$ and ω as with a given prescribed data, giving to M the structure of an AHS manifold.

2.2 The Cartan connection and the invariant differential. So we suppose that a P-principal bundle $\mathcal{G}$ on M and the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is given on $\mathcal{G}$ (for the definition and properties of the Cartan connections, see [CSS1]).

Any Cartan connection defines an absolute parallelism of $\mathcal{G}$ and for any vector space $\mathbb{V}$, we can define the *invariant differential*

$$\nabla^\omega : \mathcal{C}^\infty(\mathcal{G}, \mathbb{V}) \to \mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^*_{-1} \otimes \mathbb{V})$$

by

$$\nabla^\omega s(u)(X) \equiv \nabla^\omega_X s(u) := [\omega^{-1}(X)s](u)$$

where $\omega^{-1}(X)$ is the constant vector field on $\mathcal{G}$ given by $X \in \mathfrak{g}_{-1}$ and ω. Notice also $TM = \mathcal{G} \times_P \mathfrak{g}_{-1}$, $T^*M = \mathcal{G} \times_P \mathfrak{g}_1$ in a canonical way.

If $\mathbb{V}$ is a (finite dimensional) P-module, than the space $\mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P$ of equivariant maps is a 'frame form' of the space $\Gamma(M, V)$ of smooth sections of the associated vector bundle $V = \mathcal{G} \times_P \mathbb{V}$. We would like to use ∇^ω for a construction of invariant differential operators. Unfortunately, the map $\nabla^\omega s$, $s \in \mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P$, does not usually belong to $\mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^*_{-1} \otimes \mathbb{V})^P$, it is not the frame form of a section of a suitable associated vector bundle over M. So ∇^ω does not define directly a differential operator on M.

A very useful procedure how to improve the situation is to introduce a functorial way how to define a structure of a P-module on the space

$$J^1(\mathbb{V}) := \mathbb{V} \oplus (\mathfrak{g}^*_{-1} \otimes \mathbb{V})$$

in such a way that the map

$$s \in \mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P \mapsto (s, \nabla^\omega s) \in \mathcal{C}^\infty(\mathcal{G}, J^1(\mathbb{V}))^P$$

has again values in the space of equivariant maps. The P-module structure on $J^1(\mathbb{V})$ can be deduced easily from the corresponding homogeneous case (where it is just the representation inducing the homogeneous bundle $J^1(V)$ of 1-jets of sections of V). Moreover, the Cartan connection ω introduces the natural identifications of the first jet prolongations of the associated bundles $V = \mathcal{G} \times_P \mathbb{V}$ with $\mathcal{G} \times_P J^1(\mathbb{V})$.

Consequently, any P-module homomorphism $\Phi : J^1(\mathbb{V}) \to \mathbb{V}'$ induces a well defined differential operator from the space of sections of the bundle V to the space of sections of the bundle V'. Due to the fact that the Cartan connection is uniquely defined by the AHS structure, the corresponding operator is invariant with respect to any of the usual definitions of invariant operators (details on relations between various possible definitions of invariant operators can be found in [Slo]).

The situation most commonly considered is the case when $\mathbb{V}$ and $\mathbb{V}'$ are irreducible P-modules. It means that $\mathbb{V}$ (resp. $\mathbb{V}'$) are irreducible G_0-modules with the trivial action of the nilpotent part of P. In such a case, natural candidates for P-homomorphisms Φ are projections from the space $\mathfrak{g}^*_{-1} \otimes \mathbb{V}$ (considered as an $\mathfrak{g}^s_0$-module) onto its irreducible factors, extended by zero on the $\mathbb{V}$ part of the module $J^1(\mathbb{V})$. We shall see below that for any such projection, there is just one specific value for the action of the grading element E for which the corresponding projection is a P-homomorphism and that any invariant first order differential operator on a manifold with a given AHS structure is obtained by this construction. For conformal structures, this was exactly the content of the classification theorem obtained by Fegan in [F] (see 7.2 below).

2.3 Iterated differentiation, semiholonomic jets. Iteratively, we can define the functor $\bar{J}^k(-)$ (the k-th semi-holonomic prolongation) mapping any P-module $\mathbb{V}$ to a submodule $\bar{J}^k(\mathbb{V})$ of the P-module $J^1(\bar{J}^{k-1}(\mathbb{V}))$. Considered as a G_0-module, it looks like

$$\bar{J}^k(\mathbb{V}) = \mathbb{V} \oplus (\mathfrak{g}^*_{-1} \otimes \mathbb{V}) \oplus ... \oplus (\otimes^k(\mathfrak{g}^*_{-1}) \otimes \mathbb{V}).$$

As in the first order case, the iterated invariant differential $(\nabla^\omega)^k$ defines the map

$$j^k_\omega : s \in \mathcal{C}^\infty(\mathcal{G}, \mathbb{V})^P \mapsto (s, \nabla^\omega s, \dots, (\nabla^\omega)^k s) \in \mathcal{C}^\infty(\mathcal{G}, \bar{J}^k(\mathbb{V}))^P.$$

Moreover, if $V = \mathcal{G} \times_P \mathbb{V}$ is the bundle associated to $\mathbb{V}$, then its kth semi-holonomic jet prolongation $\bar{J}^k(V)$ is the bundle associated to the representation $\bar{J}^k(\mathbb{V})$. Thus construction of a large class of higher order invariant differential operators is now possible as it was in the first order case: It is sufficient to take any P-homomorphism from $\bar{J}^k(\mathbb{V})$ to a P-module $\mathbb{V}'$ and to compose it with the map j^k_ω.

The question to be answered is how to construct such P-module homomorphisms. If $\mathbb{V}$ is an irreducible P-module, then it is easy to find all G_0–module homomorphisms between the corresponding modules using representation theory. An explicit criterion showing when such a G_0–homomorphisms is actually a P–module homomorphism, was proved in [CSS1] and will be used below to prove existence results for invariant operators (see 5.2 for more details).

2.4 Distinguished connections, the deformation tensor. Invariant operators are given as a composition of a suitable P-homomorphism and the Cartan connection. To express the result in standard terms (covariant derivatives, curvature terms) and to find explicit formulas for it, we need more information.

Let us recall first the relation between the original first order structure $\mathcal{G}_0$ on M (e.g. a conformal one in the best known example) and the P-principal bundle $\mathcal{G}$ constructed from it. If P_1 is the Lie group corresponding to the Lie algebra $\mathfrak{g}_1$, then $\mathcal{G}_0 \simeq \mathcal{G}/P_1$. The value of the Cartan connection ω can be split with respect to the grading of $\mathfrak{g}$ as $\omega = \omega_{-1} + \omega_0 + \omega_1$. For any G_0-equivariant section $\sigma \colon \mathcal{G}_0 \to \mathcal{G}$ (which always exists), the pullback $\sigma^*\omega_0$ is a principal connection on $\mathcal{G}_0$. The space of all such connections is an affine space modeled on the space of 1-forms on M. We have got in such a way a distinguished class of connections on M which are completely characterized by the requirements that they have to belong to $\mathcal{G}_0$, and their torsion has to coincide with the $\mathfrak{g}_{-1}$-component of the curvature of ω. In the conformal case, for example, this class consists of all Weyl geometries (thus contains all Levi-Civita connections corresponding to any Riemannian metric chosen inside the given conformal class, in particular). The associated covariant derivatives are standard tools used for description of differential operators.

If ω and $\tilde{\omega}$ are two Cartan connections which differ only in the $\mathfrak{g}_1$-component, there exists an equivariant map $\Gamma \in \mathcal{C}^\infty(\mathcal{G}, \mathfrak{g}^*_{-1} \otimes \mathfrak{g}_1)$ such that $\tilde{\omega} = \omega - \Gamma \circ \omega_{-1}$. The map Γ is the P-equivariant representation on $\mathcal{G}$ of a tensor on M, which is called the *deformation tensor*. In particular, once we fix the Cartan connection ω and the G_0-equivariant section $\sigma \colon \mathcal{G}_0 \to \mathcal{G}$, there is the unique Cartan connection $\tilde{\omega}$ which is σ-related to the pullback $\sigma^*(\omega_{-1} + \omega_0)$. This is the Cartan connection whose invariant derivative $\nabla^{\tilde{\omega}}$ is as close to the covariant derivative $\nabla^{\sigma^*\omega_0}$ as possible. The corresponding deformation tensor Γ then gives the full remaining comparison. For conformal structures, this is just the well known 'rho–tensor' having the following expression in terms of the Ricci curvature:

$$\Gamma_{ij} = \frac{-1}{m-2}\big(R_{ij} - \frac{\delta_{ij}}{2(m-1)}R\big),$$

where R_{ij} and R are the P-equivariant pull-backs of the Ricci tensor and the scalar curvature to $\mathcal{G}$ and m is the dimension of the manifold M. Thus Γ is a generalization of the 'rho–tensor' to all AHS structures. Similar explicit formulae for these rho-tensors for most AHS structures have been computed in [CSS2].

Now, the value $\nabla^\omega s$ of the invariant differential on a section s can be described in more familiar terms, using ∇^γ and the deformation tensor Γ as follows. The choice of σ defines the trivialization of the bundle $p\colon \mathcal{G} \to \mathcal{G}_0$ expressed by the second coordinate $\tau\colon \mathcal{G} \to \mathfrak{g}_1$, which can be characterized by the formula $u = \sigma(p(u))\cdot\exp(\tau(u))$. Let $\mathbb{V}$ be an irreducible P-module, $V = \mathcal{G}\times_P \mathbb{V} \simeq \mathcal{G}_0 \times_{G_0} \mathbb{V}$ the corresponding associated vector bundle. Sections $s\in\Gamma(V)$ will be represented by means of equivariant maps $s\in\mathcal{C}^\infty(\mathcal{G}_0,\mathbb{V})^{G_0}$ or equivalently as $p^*s\in\mathcal{C}^\infty(\mathcal{G},\mathbb{V})^P$. Then we have for all $u\in P$, $X\in\mathfrak{g}_{-1}$

$$\left(\nabla^\omega(p^*s)(u)\right)(X) = (p^*(\nabla^\gamma s))(u)(X) + [X,\tau(u)]\cdot((p^*s)(u))$$

where the bracket $[X,\tau(u)]\in\mathfrak{g}_0$ acts on the element of the $\mathfrak{g}_0$-module $\mathbb{V}$.

All terms in the formula are G_0-equivariant, but only the first one is also P_1-equivariant (i.e. constant along fibers of p). It is the map τ in the second term, which is not P_1-equivariant (it varies when $u\in\mathcal{G}$ changes its position in the fiber). This shows again that the invariant differential $\nabla^\omega s$ is not P-equivariant even if s itself is. In many cases we can find a homomorphism Φ in such a way that the term containing τ is killed by Φ and the resulting composition is an invariant operator.

2.5 Correction terms and obstruction terms. To construct higher order invariant operators, we have to use higher order iterations of the invariant differential. To understand what is happening in higher orders, the second order case is a representative example. It is possible again to express $(\nabla^\omega)^2 s$ using ∇^γ and Γ. For any section $s\in\mathcal{C}^\infty(\mathcal{G}_0,\mathbb{V})^{G_0}$, we have

$$\left((\nabla^\omega)^2(p^*s)\right) = p^*((\nabla^\gamma)^2 s) + D_0(\gamma,\Gamma) + D_1(\gamma,\Gamma,\tau) + D_2(\gamma,\Gamma,\tau)$$

where

$$\begin{aligned} D_0(\gamma,\Gamma)(u)(X,Y) &= [X,\Gamma(u).Y]\cdot(p^*s(u));\\ D_1(\gamma,\Gamma,\tau)(u)(X,Y) &= [X,\tau(u)]\cdot(p^*(\nabla^\gamma_Y s))(u) + \left([Y,\tau(u)]\cdot(p^*\nabla^\gamma s)(u)\right)(X);\\ D_2(\gamma,\Gamma,\tau)(u)(X,Y) &= \left([Y,\tau(u)]\cdot([\_,\tau(u)]\cdot(p^*s)(u))\right)(X)\\ &\quad -\tfrac{1}{2}[X,[\tau(u),[\tau(u),Y]]]\cdot(p^*s)(u), \end{aligned}$$

and $\cdot$ denotes the appropriate action of an element from $\mathfrak{g}_0$ on the space in question (either $\mathbb{V}$ or $\mathfrak{g}^*_{-1}\otimes\mathbb{V}$). The term D_0 is called the *correction term* and the terms D_i, $i = 1,2$, which are homogeneous of degree i in τ, are called *obstruction terms.*

As for the first order case, the map $(\nabla^\omega)^2(p^*s)$ is only G_0-equivariant and, in general, not P-equivariant. To define an invariant second order operator, it is necessary to kill all obstruction terms by a suitable G_0-homomorphism. If it is possible, then the leading term together with the correction term gives an explicit formula for the corresponding invariant operator (expressed already in standard language).

2.6 The algorithm for higher orders. In fact, it can be shown (see [CSS1]) that vanishing of $D_1(\gamma, \Gamma, \tau)$ implies vanishing of all higher order obstruction terms, so that existence proofs can be simplified. The algebraic condition discussed above is equivalent to vanishing of the sum of certain terms linear in τ, so that it is even more simple condition, but it is only sufficient condition, not necessary one.

To have an explicit algorithm for computation of the form of the correction terms, we need to take into account during the inductive procedure all obstruction terms, not only the linear ones. For that, we can use the algorithm for recurrent computation of the correction and obstruction terms, which was proved in [CSS1] (for more details see 7.4). Using MAPLE, it was easy to implement this algorithm and to compute explicitly the correction and obstruction terms for low orders. The number of terms is growing enormously. For the 6th order, the full formula has 7184 terms and the correction part itself has 328 terms. We shall see later on that for standard operators studied below, further essential simplification is possible and the final formula will have only 10 summands. To write down on paper an explicit form of invariant operators of higher orders is too awkward. Nevertheless, we shall see that for a broad class of operators, the algorithm for the explicit form of the operator can be simplified substantially and that the form of correction terms for standard operators is remarkably stable and universal, independently of the type of AHS structure and the representation $\mathbb{V}$ considered (see section 7).

In the next sections, we shall use representation theory to show how the theory explained above can be used for better understanding of properties of standard invariant operators.

3. G_0-homomorphisms

To construct invariant operators, we have to learn how to construct P-homomorphisms from $\bar{J}^k(\mathbb{V})$ to a P-module $\mathbb{V}'$. The first thing to do is to understand what are the possibilities for G_0-homomorphisms. We shall concentrate on the situation when $\mathbb{V}$ is an irreducible P-module. This implies that $\mathbb{V}$ is an irreducible G_0-module and the nilpotent part acts trivially. Representation theory offers enough tools to classify all G_0-homomorphisms in this case. Any such homomorphism is equivalent to a projection of $\bar{J}^k(\mathbb{V})$ onto one of its irreducible components and a decomposition of the tensor product $\bar{J}^k(\mathbb{V}) = (\otimes^i \mathfrak{g}_{-1}^*) \otimes \mathbb{V}$ to irreducible components is a standard problem studied in representation theory of semi-simple Lie groups. In this section, we shall prove some additional facts needed for a construction of P-homomorphisms and we shall deal with a general complex semi-simple Lie algebra $\mathfrak{g}$. Later on we shall use it for the semisimple part $\mathfrak{g}_0^s = [\mathfrak{g}_0, \mathfrak{g}_0]$ of $\mathfrak{g}_0$.

3.1 Notation. Let us consider a complex semi-simple Lie algebra $\mathfrak{g}$ with a Cartan subalgebra $\mathfrak{h}$, a set Δ^+ of positive roots and its subset $S = \{\alpha_1, \dots, \alpha_n\}$ of simple roots. Using the Killing form $(.,.)$, fundamental weights $\pi_1, \dots, \pi_n$ are defined by $(\alpha_i^\vee, \pi_j) = \delta_{ij}$, where $\alpha_i^\vee = 2\alpha_i/(\alpha_i, \alpha_i)$.

The (closed) dominant Weyl chamber $\overline{\mathcal{C}}$ is given by linear combinations of fundamental weights with nonnegative coefficients, let $\mathcal{C}$ denote its interior. Finite dimensional complex irreducible representations of $\mathfrak{g}$ are characterized by their highest weights λ, which lie in the weight lattice $\Lambda^+ = \{\sum \lambda_i \pi_i;\ \lambda_i \geq 0,\ \lambda_i \in \mathbb{Z}\}$. The corresponding representation will be denoted by $(\lambda, \mathbb{V}_\lambda)$ but the action $\lambda(X)v$, $X \in \mathfrak{g}$,

$v \in \mathbb{V}_\lambda$ will be often written simply as $X \cdot v$, if the representation is clear from the context. The set of all weights of $\mathbb{V}$ will be denoted by $\Pi(\mathbb{V})$.

Any weight $\lambda \in \mathfrak{h}^*$ can be characterized by its coefficients $\lambda_j = (\lambda, \alpha_j^\vee)$. In particular, the simple roots α_i have coefficients $a_{ij} = (\alpha_i, \alpha_j^\vee)$, where a_{ij} is the Cartan matrix of the Lie algebra $\mathfrak{g}$, which is encoded into its Dynkin diagram. Consequently, the reflection $\sigma_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$ with respect to a simple root α_i changes coefficients λ_j of λ into coefficients $\lambda_j - \lambda_i a_{ij}$. Due to properties of the Cartan matrix, the coefficient λ_i changes to $-\lambda_i$ and (if no multiple edges of the Dynkin diagram are involved), the coefficient λ_i adds to neighboring coefficients λ_j (for which $a_{ij} = -1$).

The reflections σ_i generate the Weyl group W. For $\rho = \sum_i \pi_i$, we shall denote by $\cdot$ the affine action of W on weights defined by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

In our applications of the theory, we shall mostly need the case of a simple Lie algebra $\mathfrak{g}$. The only exception will be the Grassmannian case, where our Lie algebra $\mathfrak{g}$ will have two simple parts $\mathfrak{g}^1 \oplus \mathfrak{g}^2$. Note that in this case, the Cartan subalgebra $\mathfrak{h}$ splits also into $\mathfrak{h}^1 \oplus \mathfrak{h}^2$, all weights can be written as couples $\lambda = (\lambda^1, \lambda^2)$ and the representation $\mathbb{V}_\lambda$ is the tensor product $\mathbb{V}_{\lambda^1} \otimes \mathbb{V}_{\lambda^2}$. The Killing form splits as well: $(\lambda, \mu) = (\lambda^1, \mu^1) + (\lambda^2, \mu^2)$. The Weyl group W is the direct product $W_1 \times W_2$ of the Weyl groups of $\mathfrak{g}_1$ and $\mathfrak{g}_2$.

3.2 Klimyk's algorithm. There is a useful and explicit algorithm for the decomposition of the tensor product of two irreducible representations of a simple Lie algebra $\mathfrak{g}$ into irreducible components, based on the Klimyk formula (see [H], Sec.24, Ex.9).

For any weight $\xi \in \mathfrak{h}^*$, let $\{\xi\}$ denote the dominant weight lying on the orbit of ξ under the Weyl group. If $\{\xi\} \in \mathcal{C}$, then there is the unique $w \in W$ such that $\{\xi\} = w\xi$. Let $t(\xi)$ be equal to the sign of w in this case and zero otherwise.

Suppose moreover that we know the list $\Pi(\mu)$ of all weights of the irreducible representation V_μ with the highest weight μ, including their multiplicities $m_\mu(\nu)$, for $\nu \in \Pi(\mu)$. Let $\mathbb{V}_\lambda$ denote the irreducible representation of $\mathfrak{g}$ with the highest weight λ. Then the Klimyk formula implies that it is sufficient to go through the list $\Pi(\mu)$, write a formal sum

$$\sum_{\nu \in \Pi(\mu)} m_\mu(\nu) t(\lambda + \rho + \nu) \mathbb{V}_{\{\lambda+\rho+\nu\}-\rho}$$

of irreducible representations and to add together coefficients at representations with the same highest weight. The resulting coefficients are always non-negative and give the multiplicity of the corresponding representation in the decomposition. Note that some cancelations happen often.

3.3 The decomposition of a tensor product of representations. There are certain facts known for a general case of a tensor product of two irreducible representations $\mathbb{V}_\lambda$ and $\mathbb{V}_\mu$ with highest weights λ and μ. For example, the highest weight ξ of an irreducible piece in the decomposition of the product $\mathbb{V}_\lambda \otimes \mathbb{V}_\mu$ has always form $\xi = \lambda + \nu, \nu \in \Pi(\mu)$ (see [FH], p.425). But in general, we know nothing about its multiplicity, it can be zero, one or bigger.

In the product $\mathbb{V}_\lambda \otimes \mathbb{V}_\mu$, there is always an irreducible piece with the highest weight $\lambda + \mu$ and it appears with multiplicity one. This special irreducible component is standardly denoted by $\mathbb{V}_\lambda \boxtimes \mathbb{V}_\mu$, and called the *Cartan product* of $\mathbb{V}_\lambda$ and $\mathbb{V}_\mu$. If e_λ, resp. e_μ, are weight vectors for highest weights λ, resp. μ, then $e_\lambda \otimes e_\mu$ is a weight vector with the weight $\lambda + \mu$. Consequently, $\boxtimes^k \mathbb{V} \subset \odot^k \mathbb{V}$.

The following general fact is much more difficult to verify. The Parthasarathy–Rao–Varadarajan (PRV) conjecture proved recently (see [Ku]) claims that for any $w \in W$, the module $\mathbb{V}_{\{\lambda + w\mu\}}$ with the extremal weight $\lambda + w\mu$ occurs in $\mathbb{V}_\lambda \otimes \mathbb{V}_\mu$ with multiplicity at least one.

In the case that one representation in a tensor product is in a suitable sense small, we can say more about the decomposition. In particular, there will be no multiplicities in the product for such cases. This is a substantial information needed in applications below. The simplest case is the following theorem.

Theorem. *Let μ be such that all weights $\nu \in \Pi(\mu)$ have multiplicity one. Let us suppose moreover that the coefficients of all weights $\nu \in \Pi(\mu)$ with respect to fundamental weights are ≥ -1. Then for any $\lambda \in \Lambda^+$, we have*

$$\mathbb{V}_\mu \otimes \mathbb{V}_\lambda = \sum_{\tau \in A} \mathbb{V}_\tau$$

where A is the set of all weights of the form $\tau = \lambda + \nu, \nu \in \Pi(\mu)$, which belong to the dominant Weyl chamber $\overline{\mathcal{C}}$. There are no multiplicities in the decomposition.

Proof. The coefficients in the decomposition of any weight $\lambda \in \Lambda^+$ into fundamental weights are, by definition, all nonnegative. The weight ρ has all coefficients equal to 1. Our assumptions above imply that for all weights $\nu \in \Pi(\mu)$, the sum $\rho + \nu$ belongs to $\overline{\mathcal{C}}$, hence $\lambda + \rho + \nu \in \overline{\mathcal{C}}$ as well. So no action of elements $w \in W$ is needed, $\{\lambda + \rho + \nu\} - \rho = \lambda + \nu$ for all $\nu \in \Pi(\mathbb{V}_\mu)$ and no cancelations or multiplicities in the decomposition of the tensor product can occur. The weight $\lambda + \nu$ appears in the decomposition (with nonzero coefficient) if and only if $\lambda + \rho + \nu$ belongs to the interior $\mathcal{C}$ i.e. if and only if $\lambda + \nu \in \overline{\mathcal{C}}$. □

The theorem just proved will be sufficient in most cases needed below. In two of them, we shall however need a case when some of components of weights will be equal to -2. We are going to prove the multiplicity one result for this case under a suitable additional assumption. In some particular cases (e.g. in two cases needed below, see Appendix A), it is possible to describe the set A in the decomposition more precisely, but we shall not need to formulate such results in general.

Theorem'. *Suppose that μ is such that all weights $\nu \in \Pi(\mu)$ have multiplicity one. Let us suppose moreover that for all weights $\nu \in \Pi(\mu)$, $\nu = \sum_i \nu_i \pi_i$, the following conditions are satisfied:*

(1) *$\nu_i \geq -2$ for all i;*
(2) *there exists at most one index i such that $\nu_i = -2$ and if it happens, we suppose moreover that for all $j \neq i$, $\nu_j \geq 0$ and $a_{ij} \geq -1$ (the last condition means that the ith node of the corresponding Dynkin diagram is not at the foot point of a double arrow).*

Then for any $\lambda \in \Lambda^+$, we have

$$\mathbb{V}_\mu \otimes \mathbb{V}_\lambda = \sum_{\tau \in A} V_\tau$$

where $A \subset (\{\lambda + \nu | \nu \in \Pi(\mu)\}) \cap \overline{\mathcal{C}}$ is some subset and there are no multiplicities in the decomposition.

Proof. For all weights ν with the property $\nu_j \geq -1$ for all j we get as above that $\lambda + \nu + \rho \in \overline{\mathcal{C}}$, hence no reflections are needed and $V_{\lambda+\nu}$ appears in the formal sum coming from the Klimyk formula if and only if $\lambda + \nu \in \overline{\mathcal{C}}$.

Let us consider a weight ν with the property that $\nu_i = -2$. The assumptions of the theorem imply that $(\lambda+\nu+\rho)_j \geq 1$, $j \neq i$, and $(\lambda+\nu+\rho)_i = \lambda_i - 1$. If $\lambda_i > 0$, then again $\lambda + \nu + \rho \in \overline{\mathcal{C}}$ and no reflection is needed.

If, however, $\lambda_i = 0$ then the weight $\lambda + \nu + \rho$ is not in $\overline{\mathcal{C}}$. Let $w \in W$ is the simple reflection with respect to ith simple root, then $(\lambda + \nu + \rho)_i = -1$ and $(w\,(\lambda + \nu + \rho))_i = 1$. For $j \neq i$ such that $a_{ij} = 0$, the coefficient $(\lambda + \nu + \rho)_j$ is not changed under the reflection, hence is nonnegative. If $j \neq i$ such that $a_{ij} = -1$, then $(w\,(\lambda + \nu + \rho))_j = (\lambda + \nu + \rho)_j - 1 \geq \rho_j - 1 = 0$, hence also these coefficients are nonnegative. Consequently, $w\,(\lambda+\nu+\rho) \in \overline{\mathcal{C}}$ and the irreducible representation $\mathbb{V}_{w\,(\lambda+\nu+\rho)-\rho}$ will appear in Klimyk's formal sum with coefficient -1.

All terms in the formal sum coming from the weights ν with the property $\lambda + \nu + \rho \in \overline{C}$ are distinct and with multiplicity one. All others are coming with the coefficients -1, hence they are necessarily canceled by some of previous ones. Hence all terms in the result have multiplicity one and their highest weights are contained in $\{\mu = \lambda + \nu, \nu \in \Pi(\mu)\} \cap \overline{\mathcal{C}}$. □

3.4 Multiple decompositions. We shall also have to understand irreducible components of a more complicated tensor product $(\otimes^k \mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$. For $k > 1$, there is no hope to get a multiplicity one result as before. As a consequence, only isotypic components of the product will be unambiguously defined and the complete splitting into irreducible components will depend on arbitrary choices. We shall show now that the results of the previous paragraph can be used for a classification of the pieces in the decomposition and for a construction of a distinguished decomposition useful for more detailed computations in following sections.

Let $\mathfrak{g}$ is a semi-simple Lie algebra and $\mathbb{V}_\mu$ its irreducible representation having the following property: For all $\lambda \in \Lambda^+$, there exists a set A_λ such that $\mathbb{V}_\mu \otimes \mathbb{V}_\lambda = \sum_{\lambda_1 \in A_\lambda} \mathbb{V}_{\lambda_1}$ and there are no multiplicities in the decomposition.

Then the decomposition can be iterated as follows. The product $\otimes^2(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda = \mathbb{V}_\mu \otimes (\sum_{\lambda_1 \in A_\lambda} \mathbb{V}_{\lambda_1})$ can be again decomposed in the same way as

$$\sum_{\lambda_1 \in A_\lambda} \sum_{\lambda_2 \in A_{\lambda_1}} \mathbb{V}_{\lambda_2, \lambda_1},$$

where the double index of $\mathbb{V}_{\lambda_2,\lambda_1}$ indicates how this particular component was obtained in the decomposition. By repeating this process, it is clear that the product $\otimes^k(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$ can be completely decomposed into irreducible components, each one being labeled by a sequence $\underline{\lambda} = (\lambda_k, \lambda_{k-1}, \dots, \lambda_1, \lambda)$ which records the way how this component was obtained through the process of successive decompositions. The final highest weight λ_k may appear many times and its precise position in the isotypic component is fixed by the whole sequence recording its history. Hence for a fixed λ, we shall define the set $A_k(\lambda)$ of all such sequences, i.e.

$$A_k(\lambda) = \{\underline{\lambda} = (\lambda_k, \lambda_{k-1}, \dots, \lambda_1, \lambda_0) \mid \lambda_0 = \lambda,\ \lambda_j \in A_{\lambda_{j-1}};\ j = 1, \dots, k\}.$$

Then

$$\otimes^k(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda = \sum_{\underline{\lambda} \in A_k(\lambda)} \mathbb{V}_{\underline{\lambda}}.$$

Together with the final irreducible component $\mathbb{V}_{\underline{\lambda}}$, we shall use also for computations all intermediate components given by $\mathbb{V}_{\underline{\lambda}_j}, \underline{\lambda}_j = (\lambda_j, \dots, \lambda_0)$ in $\otimes^j(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$, together with the corresponding invariant projections π_{λ_j}.

There is one important question connected with such a decomposition, namely to find a position of the above mentioned components with respect to the splitting of $\otimes^j(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$ into a direct sum of $\odot^j(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$ and its invariant complement. Such a knowledge would help to decide whether invariant operators obtained by the projection to the corresponding components in the decomposition will have nontrivial symbol or not. We shall answer this question in the case we need in the next paragraph.

3.5 Multiplicity one components. There are special pieces in the decomposition of $\otimes^j(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$ which always appear with multiplicity one. Even more, we shall be able to show that they must be included in $\boxtimes^j(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$, where $\boxtimes$ denotes the Cartan product of irreducible representations (cf. 3.3), hence their symbol will be nontrivial.

Theorem. *Let $\lambda, \mu \in \Lambda^+$. Let ν be an extremal weight of $\mathbb{V}_\mu$ (i.e. it belongs to the Weyl orbit of the highest weight μ). Let k be a positive integer such that $\lambda + k\nu$ is dominant.*

Then there is a unique irreducible component in $\otimes^k(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$ with highest weight $\tau = \lambda + k\nu$. Moreover, the component $\mathbb{V}_\tau$ is contained in $\boxtimes^k(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$.

Proof. The product $\otimes^k(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$ can be decomposed into the sum of $V_{\underline{\lambda}}$ as described above. All these chains $\underline{\lambda}$ can be considered as piecewise linear paths in the dominant Weyl chamber composed from the straight segment with directions given by weights of $\mathbb{V}_\mu$. If we are going straight on k times in the same direction given by an extremal weight of $\mathbb{V}_\mu$, no other path can reach the same point $\tau = \lambda + k\nu$ (extremal weights have extremal lengths). This implies the unicity of the component.

To prove the existence, note that the weight $k\nu$ is an extremal weight of $\boxtimes^k(\mathbb{V}_\mu)$. Hence we can use the PRV conjecture to show that $\mathbb{V}_\tau$ appears in the decomposition of $\boxtimes^k(\mathbb{V}_\mu) \otimes \mathbb{V}_\lambda$. $\square$

3.6 Partial projections. Let us recall that we always have $\boxtimes^k(\mathbb{V}) \subset \odot^k(\mathbb{V})$ and that $\boxtimes^k(\mathbb{V})$ coincides with

$$[\boxtimes^2(\mathbb{V})] \boxtimes [\boxtimes^{k-2}(\mathbb{V})] \subset [\boxtimes^2(\mathbb{V})] \otimes [\boxtimes^{k-2}(\mathbb{V})].$$

As a corollary we get

Lemma. *Denote by π the projection of $\otimes^k(\mathbb{V})$ onto $\boxtimes^k(\mathbb{V})$. Suppose that A is the invariant complement of $\boxtimes^2(\mathbb{V})$ in $\otimes^2(\mathbb{V})$ and π_A is the corresponding projection. Then $\left[A\otimes(\otimes^{k-2}(\mathbb{V})\right]\cap\left[\boxtimes^k(\mathbb{V})\right]=\emptyset$, or equivalently*

$$\pi\circ(\pi_A\otimes Id^{\,k-2})=0.$$

3.7. The results above will be applied below in the following special case. Let $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}\oplus\mathfrak{g}_1$ be a complex $|1|$-graded Lie algebra, cf. 2.1. The space $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0^s$-module which is 'small' enough, i.e. it satisfies assumptions of one of the Theorems in 3.3. To check it, it is necessary to inspect algebras $\mathfrak{g}$ case by case. The list of them together with details needed for the verification are collected in Appendix A.

Consequently, for any irreducible $\mathfrak{g}_0$-module $\mathbb{V}$, the tensor product $\mathfrak{g}_1\otimes\mathbb{V}$ decomposes into irreducible components without multiplicities and results of 3.5 and 3.6 can be used for decompositions of the product $\otimes^k(\mathfrak{g}_1)\otimes\mathbb{V}$.

4. Casimir computations

4.1 Notation. For this section, we shall suppose that $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$ is a complex $|1|$-graded simple Lie algebra. In general, a choice of $|k|$-graded structure on a complex simple Lie algebra $\mathfrak{g}$ is the same as a choice of its parabolic subalgebra. Any parabolic subalgebra is conjugated to a standard one (i.e. one containing a chosen Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$). There is one to one correspondence between standard parabolic subalgebras of $\mathfrak{g}$ and subsets of the set S of simple roots of $\mathfrak{g}$.

The $|1|$-graded structures on $\mathfrak{g}$ exist only for four classical series and for E_6 and E_7 cases and they are given by certain one-point subsets of S (Dynkin diagrams with the corresponding simple root crossed are often used to denote the chosen parabolic subalgebra). We shall choose numbering of the set S of simple roots so that the first simple root α_0 is the crossed one (for more information on $|k|$-graded Lie algebras see [BasE, Y]).

There is a unique grading element $E\in\mathfrak{g}_0$ satisfying $[E,X]=\ell X$ for $X\in\mathfrak{g}_\ell$, $\ell=-1,0,1$. A Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ can be chosen in such a way that $E\in\mathfrak{h}$, then $\mathfrak{h}\subset\mathfrak{g}_0$. The set Δ^+ of positive roots for $\mathfrak{g}$ can be chosen so that all root spaces for positive roots are included in $\mathfrak{g}_0\oplus\mathfrak{g}_1$.

It is often useful to normalize an invariant form $(.,.)$ on $\mathfrak{g}$ by the requirement $(E,E)=1$ (see e.g. [BOO]). For the Killing form, we have $B(E,E)=2\dim\mathfrak{g}_1$, hence $(X,Y)=(2\dim\mathfrak{g}_1)^{-1}B(X,Y)$. This normalized form $(.,.)$ induces nondegenerate invariant bilinear forms on $\mathfrak{g}_0$ and $\mathfrak{g}_{-1}\oplus\mathfrak{g}_1$, and it identifies $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ as dual spaces. Orthonormal bases and Casimir operators for $\mathfrak{g}_0$ will be computed using this normalized form.

The algebra $\mathfrak{g}_0$ splits into 1-dimensional center $\mathfrak{a}$ and a semisimple part $\mathfrak{g}_0^s=[\mathfrak{g}_0,\mathfrak{g}_0]$ which has $\mathfrak{h}_s=\mathfrak{h}\cap\mathfrak{g}_0^s$ as a Cartan subalgebra. Then $\mathfrak{h}=\mathfrak{a}\oplus\mathfrak{h}_s$. Irreducible representations of $\mathfrak{p}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ are trivial on $\mathfrak{g}_1$. Every such representation is a tensor product of a one-dimensional representation of $\mathfrak{a}$ and an irreducible representation of $\mathfrak{g}_0^s$, which can be characterized by its highest weight $\lambda\in(\mathfrak{h}^s)^*$. For convenience, we shall consider $(\mathfrak{h}^s)^*$ as a subset of $\mathfrak{h}^*$ of all elements, which restrict to zero on $\mathfrak{a}$. Representations of $\mathfrak{a}$ can be characterized by a (generalized) conformal weight

$w \in \mathbb{C}$. We shall say that a representation $\mathbb{U}$ of $\mathfrak{g}_0$ has a (generalized) *conformal weight* w, if $E \cdot v = wv$, $v \in \mathbb{U}$. The cotangent spaces of our manifolds are associated to the adjoint representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, hence 1-forms will have (generalized) conformal weight 1. An irreducible representation of $\mathfrak{g}_0$ with a conformal weight w and highest weight $\lambda \in (\mathfrak{h}^s)^*$ will be denoted by $\mathbb{V}_\lambda(w)$.

Let $\{Y_a\}$, $a = 0, 1, \dots$, be an orthonormal basis of $\mathfrak{g}_0$ with respect to the form $(.,.)$. We may choose it in such a way that $Y_0 = E \in \mathfrak{a}$ and $\{Y_{a'}\}$, $a' > 0$ is an orthonormal basis for $\mathfrak{g}_0^s$. For any representation $\mathbb{V}$ of $\mathfrak{g}_0^s$, the Casimir operator $C(\mathbb{V})$ is defined by $C(\mathbb{V}) = \sum_{a'>0} Y_{a'} \circ Y_{a'}$. It is well known (see [H]) that if $\mathbb{V}$ is an irreducible representation with a highest weight λ, then

$$C(\mathbb{V}) = (\lambda, \lambda + 2\rho); \quad \rho = 1/2 \sum_{\alpha \in \Delta^+(\mathfrak{g}_0^s)} \alpha.$$

As we have noticed already, our algebras $\mathfrak{g}_0^s$ are irreducible in all cases except the $\mathfrak{sl}(n, \mathbb{C})$ series, but even then the formula $C(\mathbb{V}_\lambda) = (\lambda, \lambda + 2\rho)$, $\rho = (\rho_1, \rho_2)$ is still valid, see 3.1 for the reasons.

4.2 Casimir computations. Suppose now that $X \in \mathfrak{g}_{-1}$, $Z \in \mathfrak{g}_1$ and let us consider an irreducible $\mathfrak{g}_0$-module $\mathbb{V}_\lambda(w)$, where $\lambda \in \mathfrak{h}^*$ is an integral dominant weight for $\mathfrak{g}_0^s$ and $w \in \mathbb{C}$. In the description of iterated invariant differentials, terms of type $[Z, X] \cdot s$, $s \in \mathbb{V}_\lambda(w)$, have appeared very often (the $\cdot$ means here the action of an element of $\mathfrak{g}_0$ under the representation characterized by λ and w), (see 2.4). It is hence important to understand them better.

Recall that we identify $\mathfrak{g}_1$ and $(\mathfrak{g}_{-1})^*$ using the scalar product $(.,.)$. The term $[Z, X] \cdot s$ defines a map from $\mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \otimes \mathbb{V}_\lambda(w)$ into $\mathbb{V}_\lambda(w)$, which can be interpreted also as a map $\Phi \colon \mathfrak{g}_1 \otimes \mathbb{V}_\lambda(w) \to \mathfrak{g}_1 \otimes \mathbb{V}_\lambda(w)$, defined by

$$\Phi(Z \otimes v)(X) := \lambda([Z, X])v; \quad Z \in \mathfrak{g}_1, s \in \mathbb{V}_\lambda(w), X \in \mathfrak{g}_{-1}.$$

Let us choose bases $\{\eta_\alpha\}$, resp. $\{\xi_\alpha\}$ of $\mathfrak{g}_{-1}$, resp. $\mathfrak{g}_1$, which are dual with respect to the scalar product $(.,.)$. Due to

$$[Z, X] \cdot s = \sum_\alpha [Z, (\eta_\alpha, X)\xi_\alpha] \cdot s = \Big(\sum_\alpha \eta_\alpha \otimes [Z, \xi_\alpha] \cdot s\Big)(X),$$

we get

$$\Phi(Z \otimes s) = \sum_\alpha \eta_\alpha \otimes [Z, \xi_\alpha] \cdot s.$$

The map Φ is a $\mathfrak{g}_0$-homomorphism (by direct computation or by the lemma below). Let $\mathfrak{g}_1 \otimes \mathbb{V}_\lambda(w) = \sum_\mu \mathbb{V}_\mu(w+1)$ be a decomposition of the product of $\mathfrak{g}_0$-modules into irreducible components and let $\pi_{\lambda\mu} : \mathfrak{g}_1 \otimes \mathbb{V}_\lambda(w) \to \mathbb{V}_\mu(w+1)$ be the corresponding projections. The $\mathfrak{g}_0$-homomorphism Φ acts as a multiple of the identity on each irreducible component, i.e. there are constants $\tilde{c}_{\lambda\mu} \in \mathbb{R}$ such that $\Phi = \sum_\mu \tilde{c}_{\lambda\mu} \pi_{\lambda\mu}$ and we are going to describe a formula expressing these constants in terms of the weights λ and μ.

4.3 Lemma. *Let $\mathbb{V}_\lambda(w)$ be an irreducible representation of $\mathfrak{g}_0$ and let $\mathfrak{g}_1 \otimes \mathbb{V}_\lambda = \sum_\mu \mathbb{V}_\mu$ be a decomposition of the product into irreducible $\mathfrak{g}_0^s$-modules. Let α be the highest weight of $\mathfrak{g}_1$ and let ρ be the half sum of positive roots for $\mathfrak{g}_0^s$. Then for all $s \in \mathbb{V}_\lambda(w)$,*

$$\Phi(Z \otimes s)(X) = [Z, X] \cdot s = \sum_\mu (w - c_{\lambda\mu}) \pi_{\lambda\mu}(Z \otimes s)(X),$$

where $c_{\lambda\mu} = -\frac{1}{2}[(\mu, \mu + 2\rho) - (\lambda, \lambda + 2\rho) - (\alpha, \alpha + 2\rho)]$.

Proof. Let $\{\xi_\nu\}$, resp. $\{\eta_\nu\}$ be dual bases of $\mathfrak{g}_{-1}$, resp. $\mathfrak{g}_1$. The invariance of the scalar product implies

$$[Z, \xi_\nu] = \sum_a (Y_a, [Z, \xi_\nu]) Y_a = \sum_a ([Y_a, Z], \xi_\nu) Y_a$$

$$\Phi(Z \otimes s) = \sum_\nu \eta_\nu \otimes [Z, \xi_\nu] \cdot s = \sum_\nu \eta_\nu \otimes \left(\sum_a ([Y_a, Z], \xi_\nu) Y_a \right) \cdot s = \sum_a [Y_a, Z] \otimes Y_a \cdot s.$$

Since $Y_0 = E$, the first term in the sum is $[Y_0, Z] \otimes Y_0 \cdot s = w\, Z \otimes s$ and for the rest we can use the definition of the Casimir operator and its computation by means of highest weights, together with

$$\sum_{a'} Y_{a'} Y_{a'} \cdot (Z \otimes s) = \sum_{a'} (Y_{a'} Y_{a'} \cdot Z) \otimes s + \sum_{a'} Z \otimes (Y_{a'} Y_{a'} \cdot s) + 2 \sum_{a'} (Y_{a'} \cdot Z) \otimes (Y_{a'} \cdot s)$$

(notice $\cdot$ means the actions on different modules used in the formula) □

4.4 Example. Let us compute now a simple case of the formula above which will be needed below. The special double commutator terms $[[X, \tau], \tau]$ from 2.5 are appearing often in the algorithm mentioned in 2.6. We want to decompose them into irreducible pieces.

Againk, let α be the highest weight of $\mathfrak{g}_1$ considered as $\mathfrak{g}_0^s$-module. By our conventions, it has the conformal weight 1. The tensor square $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ decomposes always into symmetric and antisymmetric parts. But the symmetric square decomposes in all but one cases into two components (an exceptional case being projective structures, where is does not decompose). For our purposes, it is sufficient to know that there is always a piece in the decomposition with the highest weight 2α (the Cartan product of $\mathfrak{g}_1$ with itself), denoted by $\mathfrak{g}_1 \boxtimes \mathfrak{g}_1$.

Lemma. *Let $\mathfrak{g}_1 \otimes \mathfrak{g}_1 = \oplus_{i=1}^3 \mathbb{V}_{\alpha_i}$ be the decomposition into irreducible components with $\mathbb{V}_{\alpha_1} \simeq \boxtimes^2(\mathfrak{g}_1)$ and $\mathbb{V}_{\alpha_3} \simeq \Lambda^2(\mathfrak{g}_1)$ ($\mathbb{V}_{\alpha_2}$ is trivial in the projective case). Hence $\alpha_1 = 2\alpha$. Then there exist real numbers $A_i, i = 1, 2, 3$, such that*

$$-\frac{1}{2}[[X, \tau], \tau](Y) = \sum_{i=1}^3 A_i \pi_i [\tau \otimes \tau](X, Y),$$

where $X, Y \in \mathfrak{g}_{-1}$; $\tau \in \mathfrak{g}_1$, and π_i is the projection onto $\mathbb{V}_{\alpha_i}$. For A_1, we have $A_1 = \frac{1}{2}(|\alpha|^2 + 1)$.

Proof. This is the case $\mathbb{V}_\lambda = \mathfrak{g}_1$ of lemma 4.3, so the numbers A_i are given by

$$A_i = -\frac{1}{2}[c_{\alpha\alpha_i} - 1], \quad i = 1, 2, 3.$$

In particular, $c_{\alpha,2\alpha} = -\frac{1}{2}[(2\alpha, 2\alpha+2\rho) - 2(\alpha,\alpha+2\rho)] = -|\alpha|^2$. □

In computations below, we shall use often the constant A_1 but we shall see that its actual value does not influence the explicit formula for standard operators, because the constant A_1 will be absorbed by a renormalization of the deformation tensor Γ.

5. P-module homomorphisms

Let us suppose, as in the previous section, that $\mathfrak{g}$ is a complex $|1|$-graded Lie algebra, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and $\mathbb{V}$ is a (complex) irreducible $\mathfrak{p}$-module. The algebra $\mathfrak{g}_0$ splits into the sum of the commutative 1-dimensional ideal $\mathfrak{a}$ and the semisimple part $\mathfrak{g}_0^s$.

Using results obtained in the last two sections, it is possible to construct a broad class of $\mathfrak{p}$-homomorphisms Φ from $\bar{J}^k\mathbb{V}$ to a P-module $\mathbb{V}'$, where $\mathbb{V}'$ is a suitable irreducible component of the $\mathfrak{g}_0$-module $\otimes^k(\mathfrak{g}_1)\otimes\mathbb{V}$. Let us recall that there is a unique grading element $E \in \mathfrak{a}$ for $\mathfrak{g}$ and an invariant scalar product $(.,.)$ on $\mathfrak{g}$ is normalized by the condition $(E,E)=1$.

Before stating the corresponding result, we shall prove a simple auxiliary Lemma. A surprising and important fact coming from it is the independence of the constants $c_{j+1} - c_j$ of the chosen representations.

5.1 Lemma. *Let α be the highest weight of the $\mathfrak{g}_0^s$-module $\mathfrak{g}_1$ and θ one of its extremal weights. For any weight λ, let us define weights $\lambda_j = \lambda + j\theta$, $j \in \mathbb{N}$, and numbers*

$$c_j = c_{\lambda_j\lambda_{j+1}} = -\frac{1}{2}\big[(\lambda_{j+1},\lambda_{j+1}+2\rho) - (\lambda_j,\lambda_j+2\rho) - (\alpha,\alpha+2\rho)\big].$$

Then we have

(1) $c_0 = (\alpha,\rho) - (\theta,\lambda+\rho)$;
(2) $c_j - c_{j-1} = -|\alpha|^2$;
(3) $\sum_{j=0}^{k-1} c_j = k\,[(\alpha,\rho) - (\theta,\lambda+\rho) - \frac{k-1}{2}|\alpha|^2]$.

Proof. By definition

$$\begin{aligned} c_0 &= -\frac{1}{2}\big((\lambda+\theta,\lambda+\theta+2\rho) - (\lambda,\lambda+2\rho) - (\alpha,\alpha+2\rho)\big) = \\ &= (\alpha,\rho) - (\theta,\lambda+\rho) - \frac{1}{2}(|\theta|^2 - |\alpha|^2). \end{aligned}$$

The weight θ lies in the W-orbit of α, so they have the same norm, and (1) follows. Substituting λ_j instead of λ, we get

$$c_j = (\alpha,\rho) - (\theta,\lambda+\rho) - j|\theta|^2$$

as well as the formula (2). Using $c_j = c_0 - j|\alpha|^2$, we get

$$\sum_{j=0}^{k-1} c_j = \sum_{j=0}^{k-1}(c_0 - j|\alpha|^2) = k\,c_0 - \frac{k(k-1)}{2}|\alpha|^2. \qquad \square$$

5.2 The algebraic criterion. We want now to prove that certain G_0-homomorphisms are in fact P-homomorphisms. In [CSS1], the following algebraic condition for it was proved, but in the case when the invariant scalar product $(.,.)$ was equal to the Killing form $B(.,.)$. If the normalization of $(.,.)$ is different and if κ is a number such that $B(.,.) = \kappa(.,.)$, then it is easy to check that all terms in the Lemma below are scaled uniformly by the constant κ^k, hence the condition does not change.

Lemma. *Let $\mathbb{V}$ and $\mathbb{V}'$ be irreducible P-modules and $\Phi\colon \bar{J}^k(\mathbb{V}) \to \mathbb{V}'$ be a $\mathfrak{g}_0$-module homomorphism whose restriction to $\otimes^k(\mathfrak{g}_{-1}^*)\otimes\mathbb{V} \subset \bar{J}^k(\mathbb{V})$ does not vanish. Let us choose any invariant scalar product $(.,.)$ on $\mathfrak{g}$ and let us use it to identify $\mathfrak{g}_1$ with $\mathfrak{g}_{-1}^*$. Then Φ is a P-module homomorphism if and only if:*

(1) *It factors through the projection $\pi\colon \bar{J}^k(\mathbb{V}) \to \otimes^k(\mathfrak{g}_{-1}^*)\otimes\mathbb{V}$;*
(2) *Φ vanishes on the image of $\otimes^{k-1}(\mathfrak{g}_{-1}^*)\otimes\mathbb{V}$ in $\bar{J}^k(\mathbb{V})$ under the action of $\mathfrak{g}_1$, i.e. for all $Z, Y_1, \dots, Y_{k-1} \in \mathfrak{g}_1$, $v\in\mathbb{V}$ we have*

$$\Phi\Big(\sum_{i=0}^{k-1}(\sum_\beta Y_1\otimes\dots\otimes Y_i\otimes\eta_\beta\otimes\big([Z,\xi_\beta].(Y_{i+1}\otimes\dots\otimes Y_{k-1}\otimes v)\big)\Big) = 0,$$

where η_β and ξ_β are dual bases of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ with respect to the scalar product $(.,.)$ and the dot means the standard action of an element in $\mathfrak{g}_0$ on the argument.

This criterion looks quite complicated. Using results of Section 4, we can use it to prove easily the existence of a broad class of P-modules homomorphisms.

5.3 Corollary. *Let $\mathbb{V}_\lambda$ be an irreducible $\mathfrak{g}_0^s$-module and let α be the highest weight of the irreducible $\mathfrak{g}_0^s$-module $\mathfrak{g}_1$.*

Let us suppose that an extremal weight θ of $\mathfrak{g}_1$ and an positive integer k is chosen in such a way that $\mu = \lambda + k\theta$ is dominant. Let $\pi : \otimes^k\mathfrak{g}_1\otimes\mathbb{V}_\lambda \to \mathbb{V}_\mu$ be the projection on the unique irreducible component of the product with highest weight μ (see Theorem 3.5).

Then there is a unique value for the generalized conformal weight w such that π defines a P-homomorphism from $\bar{J}^k(\mathbb{V}_\lambda(w))$ to $\mathbb{V}_\mu(w+k)$. The value of that conformal weight is given by

$$w = (\alpha-\theta,\rho) - \frac{k-1}{2}(|\alpha|^2+1) - (\theta,\lambda),$$

where ρ is half the sum of positive roots for $\mathfrak{g}_0^s$.

Proof. Let us first recall the construction of the projection π. If $\lambda_{k'} = \lambda + k'\theta$, $k' = 0,\dots,k$, the projections $\pi_{k'}$, $k' = 1,\dots,k$, are defined inductively as the projections from $\mathfrak{g}_1\otimes\mathbb{V}_{\lambda_{k'-1}}$ onto the unique irreducible component $\mathbb{V}_{\lambda_{k'}}$ with highest weight $\lambda_{k'}$. The projection π is given by the formula

$$\pi(Z_1\otimes\dots\otimes Z_k\otimes v) = \pi_k(Z_1\otimes\pi_{k-1}(Z_2\otimes\dots\pi_1(Z_k\otimes v)\dots)),$$

where $Z_1,\dots,Z_k\in\mathfrak{g}_1$, $v\in\mathbb{V}_\lambda$.

To prove the theorem, we have to verify that with the choice of the weight w above, the condition in Lemma 5.2 is satisfied. So we want to find w in such a way that for all $Z, Z_1, \dots Z_{k-1} \in \mathfrak{g}_1$, $v \in \mathbb{V}_\lambda$,

$$\pi\Big(\sum_{i=0}^{k-1}\sum_\beta Z_1\otimes\dots\otimes Z_i\otimes\eta_\beta\otimes\big([Z,\xi_\beta].(Z_{i+1}\otimes\dots\otimes Z_{k-1}\otimes v)\big)\Big)=0,$$

where η_β and ξ_β are dual bases of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ with respect to the product $(.,.)$. Let us recall the notation $c_j = c_{\lambda_j,\lambda_{j+1}}$ from Lemma 5.1.

By Lemma 4.3, applied to elements from $\mathbb{V}_{\lambda_{k-1-i}}(w+k-1-i)$, we have

$$\pi_{k-i}\Big(\sum_\beta \eta_\beta\otimes\pi_{k-i-1}\big([Z,\xi_\beta].(Z_{i+1}\otimes\pi_{k-i-2}(\dots\otimes\pi_1(Z_{k-1}\otimes v)\dots))\big)\Big)=$$
$$\pi_{k-i}\Big(\sum_\beta \eta_\beta\otimes\big([Z,\xi_\beta].(\pi_{k-i-1}(Z_{i+1}\otimes\pi_{k-i-2}(\dots\otimes\pi_1(Z_{k-1}\otimes v)\dots)))\big)\Big)=$$
$$(w+k-1-i-c_{k-1-i})\,\pi_{k-i}\big(Z\otimes\pi_{k-i-1}(Z_{i+1}(\dots\otimes\pi_1(Z_{k-1}\otimes v)\dots))\big).$$

Due to the fact that all images of π^j belong to $\odot^j\mathfrak{g}_1\otimes\mathbb{V}_\lambda$, $j=1,\dots,k$, all elements

$$\pi(Z_1\otimes\ldots\otimes Z_i\otimes Z\otimes Z_{i+1}\otimes\dots\otimes Z_{k-1}\otimes v));\ i=0,\dots,k-1$$

coincide. It is hence sufficient to find w so that

$$kw+\frac{k(k-1)}{2}-\sum_{j=0}^{k-1}c_{k-1-j}=0.$$

To get the value for w, it is sufficient to use Lemma 5.1 (note that $|\alpha|=|\theta|$). □

6. Standard operators

6.1 A construction of invariant operators. As described in Section 2, the P-module homomorphisms constructed in the last Section define invariant differential operators. We can now summarize the whole construction and the data needed for it. Let us return to the situation of Section 2 with a given $|1|$-graded (real) simple Lie algebra $\mathfrak{g}$, the corresponding groups $P\subset G$, G_0, and a principal fiber bundle $\mathcal{G}$ over M with a given Cartan connection ω.

The complexification $\mathfrak{g}^{\mathbb{C}}$ is a complex semisimple $|1|$-graded Lie algebra and $\mathfrak{g}_j=\mathfrak{g}\cap\mathfrak{g}_j^{\mathbb{C}}$; $j=-1,0,1$. Any (complex) irreducible P-module $\mathbb{V}$ is an irreducible $\mathfrak{g}_0$-module as well as $\mathfrak{g}_0^{\mathbb{C}}$-module. They are characterized by an integral dominant weight for $(\mathfrak{g}_0^s)^{\mathbb{C}}$ and the (generalized) conformal weight w. The tensor product $\mathfrak{g}_1\otimes_{\mathbb{R}}\mathbb{V}$ is isomorphic to $\mathfrak{g}_1^{\mathbb{C}}\otimes_{\mathbb{C}}\mathbb{V}$, the same is true for iterated tensor products. The space $\mathfrak{g}_1^{\mathbb{C}}$ is an irreducible module for $\mathfrak{g}_0^s$ with a highest weight α.

Suppose that we have chosen the following data: An irreducible module $\mathbb{V}_\lambda$ for $\mathfrak{g}_0^s$, a 'direction' θ, which is an extremal weight of the $\mathfrak{g}_0^s$-module $\mathfrak{g}_1^{\mathbb{C}}$, and a positive integer k, such that $\mu=\lambda+k\theta\in\Lambda^+$.

Let π the projection to the unique irreducible component of the $\mathfrak{g}_0^s$-module $\otimes^k \mathfrak{g}_1 \otimes \mathbb{V}_\lambda$ with the highest weight $\mu = \lambda + k\theta$ (cf. Theorem 3.5), and let w be the corresponding (generalized) conformal weight from Corollary 5.3. Then the operator

$$D \equiv D(\lambda, \theta, k) = \pi \circ (\nabla^\omega)^k : \mathcal{C}^\infty(P, V_\lambda(w))^P \to \mathcal{C}^\infty(P, V_\mu(w+k))^P,$$

is an invariant differential operator of order k.

6.2 Standard operators. We have defined above a certain class of operators which were proved to be invariant. There is a traditional division of invariant operators into two classes — standard and nonstandard ones. We would like to show now that the operators constructed above include almost the whole class of so called standard operators.

(Fundamental) standard operators were originally defined in the homogeneous situation (on generalized flag manifolds G/P, with G complex simple and P parabolic). In the Borel case, the classification of all invariant differential operators was given (in the dual language of homomorphism between Verma modules) by Bernstein, Gelfand and Gelfand, see [BGG]. They are all defined uniquely by their source and target (up to a constant multiple) and they are precisely all operators forming the so called BGG resolutions. For a general parabolic, the BGG resolutions are also well known but the class of invariant operators corresponding to individual arrows in them — they are called (fundamental) standard operators — is no more the complete set of invariant operators. There exist also the so called non-standard operators. To show a relation of our invariant operators $D(\lambda, \theta, k)$ to the standard operators, we need just their following simple property (more details can be found e.g. in [BasE]).

Suppose that a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}^{\mathbb{C}}$ and the set of simple roots is chosen in such a way that $E \in \mathfrak{h}$ and that all positive spaces are contained in $\mathfrak{g}_0^{\mathbb{C}} \cap \mathfrak{g}_1^{\mathbb{C}}$. Irreducible representations of $\mathfrak{g}_0^{\mathbb{C}}$ can be characterized by their highest weight, considered as an element in $\mathfrak{h}^*$, such that its restriction to $(\mathfrak{h})^s = \mathfrak{h} \cap (\mathfrak{g}_0^{\mathbb{C}})^s$ is dominant. This carries information both on the highest weight for $(\mathfrak{g}_0^{\mathbb{C}})^s$ and on a generalized conformal weight. For any such $\Lambda \in \mathfrak{h}^*$, the symbol V_Λ denotes a homogeneous bundle given by the irreducible representation of $\mathfrak{g}_0^{\mathbb{C}}$, corresponding to this highest weight. The Weyl group W of $\mathfrak{g}^{\mathbb{C}}$ has a structure of a directed graph which is directly related to existence of invariant operators.

The property we need is the following. If $D : \Gamma(V_\Lambda) \to \Gamma(V_{\Lambda'})$ is a standard invariant operator, then there is a positive root Θ for $\mathfrak{g}^{\mathbb{C}}$ such that $\sigma_\Theta(\Lambda + \Delta) = \Lambda' + \Delta$, where σ_Θ is the reflection with respect to Θ and Δ is a half-sum of positive roots for $\mathfrak{g}^{\mathbb{C}}$. Consequently, we have also $|\Lambda + \Delta|^2 = |\Lambda' + \Delta|^2$. Before going further, we need two simple auxiliary lemmas.

6.3 Lemma. *Let $\mathfrak{g}$ be a complex $|1|$-graded Lie algebra, $S = \{\alpha_i\}_{i=0}^m$ the set of its simple roots with its numbering chosen in such a way that α_0 is the crossed simple root. Let $\{\pi_i\}$ be the corresponding set of fundamental weights.*

Then we have

(1) *If Λ is the highest weight of an irreducible $\mathfrak{g}_0$-module $\mathbb{V}$, then its conformal weight is equal to $w = \Lambda(E)$.*

(2) *The root space* $\mathfrak{g}_\alpha$ *belongs to* $\mathfrak{g}_j$, $j = -1, 0, 1$, *if and only if* $a_0 = j$, *where* a_i *are coefficients in the decomposition* $\alpha = \sum_{i=0}^m a_i \alpha_i$.

(3) *For any weight* $\Lambda \in \mathfrak{h}^*$, *we have* $(\pi_0, \Lambda) = \frac{|\alpha_0|^2}{2}\Lambda(E)$, *where* E *is the grading element.*

(4) *Let us consider two weights* Λ, Λ' *and a number* a *such that* $|\Lambda|^2 = |\Lambda'|^2$, $|\Lambda + a\pi_0|^2 = |\Lambda' + a\pi_0|^2$ *and* $(\Lambda - \Lambda', \pi_0) \neq 0$. *Then* $a = 0$.

Proof. (1) If v is a highest weight vector for $\mathbb{V}$, then $E \cdot v = \Lambda(E)v$, but by definition $E \cdot v = w\, v$.

(2) This is a special case of a simple general statement valid for all $|k|$-graded Lie algebras. The reason is that all simple roots but α_0 are in $\mathfrak{g}_0$, while α_0 generates $\mathfrak{g}_1$.

(3) There is an element $H \in \mathfrak{h}$ such that $(\pi_0, \Lambda) = \Lambda(H)$ for all $\Lambda \in \mathfrak{h}^*$. Then for all $j = 1, \dots, m$, we have $0 = (\pi_0, \alpha_j^\vee) = \alpha_j^\vee(H)$, where $\alpha_j^\vee = \frac{2\alpha_j}{|\alpha_j|^2}$. The element H is orthogonal to all roots of $\mathfrak{g}_0$, hence it is a multiple of E (which has the same property). To check the multiple, it is sufficient to note that $\alpha_0(E) = 1$, because the conformal weight for $\mathfrak{g}_1$ is 1.

4) The last property follows from

$$|\Lambda + a\pi_0|^2 - |\Lambda' + a\pi_0|^2 = 2a(\Lambda - \Lambda', \pi_0). \qquad \square$$

As a consequence, we get the following interesting fact.

6.4 Lemma. *In the setting of 6.1, let* λ, λ' *be two dominant integral weights for* $\mathfrak{g}_0^s$. *Suppose that there are two nontrivial standard invariant differential operators* $D, \tilde{D}$ *of order* $k > 0$ *such that*

$$D : \Gamma(\mathbb{V}_\lambda(w)) \to \Gamma(\mathbb{V}_{\lambda'}(w + k));\ \tilde{D} : \Gamma(\mathbb{V}_\lambda(\tilde{w})) \to \Gamma(\mathbb{V}_{\lambda'}(\tilde{w} + k)).$$

Then $w = \tilde{w}$.

Proof. Let Λ, Λ', $\tilde{\Lambda}$, $\tilde{\Lambda}'$ be in turn highest weights from $\mathfrak{h}^*$ for irreducible representations

$$\mathbb{V}_\lambda(w), \mathbb{V}_{\lambda'}(w + k), \mathbb{V}_\lambda(\tilde{w}), \mathbb{V}_{\lambda'}(\tilde{w} + k).$$

If Δ is the half-sum of positive roots for $\mathfrak{g}$, then existence of $D, \tilde{D}$ implies (see 6.2) that

$$|\Lambda + \Delta|^2 = |\Lambda' + \Delta|^2;\ |\tilde{\Lambda} + \Delta|^2 = |\tilde{\Lambda}' + \Delta|^2.$$

The differences $\tilde{\Lambda} - \Lambda$, $\tilde{\Lambda}' - \Lambda'$ annihilate $\mathfrak{h}^s$, hence there are numbers a, a' such that $\tilde{\Lambda} - \Lambda = a\pi_0$; $\tilde{\Lambda}' - \Lambda' = a'\pi_0$. But

$$a\pi_0(E) = (\tilde{\Lambda} - \Lambda)(E) = \tilde{w} - w = (\tilde{\Lambda}' - \Lambda')(E) = a'\pi_0(E),$$

hence $a = a'$. Moreover, $(\Lambda - \Lambda')(E) = k > 0$, hence $(\Lambda - \Lambda', \pi_0) \neq 0$. Now, Lemma 6.3 implies that $a = 0$. $\square$

6.5 Theorem. *Let D be a standard invariant differential operator acting between sections of V_Λ and $V_{\tilde\Lambda}$. Let $\Theta \in \mathfrak{h}^*$ be a positive root of $\mathfrak{g}$ such that $\tilde\Lambda + \Delta = \sigma_\Theta(\Lambda + \Delta)$. Denote by θ the restriction of Θ to $\mathfrak{h}^s$ and by λ the restriction of Λ.*

Then θ is a weight of $\mathfrak{g}_0^s$-module $\mathfrak{g}_1$ and the number $k = 2(\Lambda + \Delta, \Theta)/(\Theta, \Theta)$ is a positive integer.

If moreover the weight θ is an extremal weight of $\mathfrak{g}_1$, then the operator $D(\lambda, \theta, k)$ defined in 6.1 coincides (up to a multiple) with the operator D on sections of the homogeneous bundle V_Λ.

Proof. The root Θ is a positive root of $\mathfrak{g}$. Consequently, the value of $\Theta(E)$ is either 0 or 1. By the properties of standard operators (see 6.2), we have

$$\tilde\Lambda - \Lambda = k\Theta,$$

where $k = 2(\Lambda+\Delta, \Theta)/(\Theta, \Theta)$ must be an integer. Because any differential operator must increase (generalized) conformal weight (which is given by evaluation of the highest weight on E), the value $\Theta(E)$ cannot vanish. Hence $\Theta(E) = 1$ and $k > 0$.

If we denote by λ, resp. $\tilde\lambda$, the restrictions of Λ, resp. $\tilde\Lambda$ to $\mathfrak{h}^s$, then we have also the relation

$$\tilde\lambda = \lambda + k\theta.$$

Hence the operators D and $D(\lambda, \theta, k)$ act between the same $\mathfrak{g}_0^s$ bundles and they are both invariant. By Lemma 6.4, their conformal weights coincide as well. Now, the standard operators are completely defined by their domains and targets up to multiples, see [BC], and D and $D(\lambda, \theta, k)$ differ at most by a constant multiple. □

6.6 Remark. We have just seen that our construction gives all standard invariant operators for those AHS structures, for which the set of weights of $\mathfrak{g}_1^{\mathbb{C}}$ is just one orbit of the Weyl group. This is true for all cases with two exceptions — the odd dimensional conformal case and the spinorial case.

There is indeed an exceptional set of standard operators for AHS structures which do not have a simple description of the form $D(\lambda, \theta, k)$ constructed above. A typical example is the case of odd conformal structures and the operators in the middle of the BGG resolution. These are operators acting between sections $\Gamma(\mathbb{V}_\lambda(w))$ and $\Gamma(\mathbb{V}_\lambda(w'))$. The representation $\mathbb{V}_\lambda$ of the semi-simple part of G_0 is the same for the source and the target, they differ only by their conformal weights. They correspond to the case of operators (λ, θ, k), where θ is the zero weight of $\mathfrak{g}_1$. In this case, however, the isotypic component $\mathbb{V}_\lambda$ appears in $\otimes^k(\mathfrak{g}_1) \otimes \mathbb{V}_\lambda$ with higher multiplicities.

In general, the BGG sequence of a representation $\mathbb{V}$ of $\mathfrak{g}$ can be realized using the twisted ($\mathbb{V}$-valued) de Rham sequence. In the particular case of the BGG sequence of the basic spinor representation $\mathbb{S}$ of $\mathfrak{g} = Spin(2n + 2, \mathbb{C})$, the middle operator corresponds to a second order operator D between $\Gamma(\mathbb{V}_\lambda(n - 1/2))$, and $\Gamma(\mathbb{V}_\lambda(n + 3/2))$, where $\lambda = (3/2, \dots, 3/2)$. There are 3 pieces in the decomposition of the tensor product $\otimes^2(\mathfrak{g}_1) \otimes \Gamma(\mathbb{V}_\lambda)$, corresponding to sequences of weights $(\lambda, \sigma, \lambda)$ with $\sigma_1 = (5/2, 3/2, \dots, 3/2)$; $\sigma_2 = (3/2, \dots, 3/2)$; $\sigma_3 = (3/2, \dots, 3/2, 1/2)$. It can be shown by methods described in [CSS4], [B], (see also [Sev]) that the corresponding standard operator is given by $\pi \circ (\nabla^\gamma)^2$, where the projection π is equal to $\pi =$

$\pi_2 + 1/4\pi_3$, where π_j are defined as projections to irreducible pieces corresponding to the sequences with σ_j. The form of the operator D is hence more complicated, it has the form

$$D\,t = \pi_2[(\nabla^\gamma)^2 t - (1/2)\Gamma \otimes t] + 1/4\pi_3[(\nabla^\gamma)^2 t - 2\Gamma \otimes t].$$

So it is clear that its formula has no more the simple universal form $D\,t = \pi((\nabla^\gamma)^2 t + \Gamma \otimes t])$ of the second order standard operators deduced below, see 7.11.

7. Explicit formulae for standard operators

7.1 Obstruction and correction terms. An algorithm for computation of $(\nabla^\omega)^k$ in terms of the principal connection ∇^γ and its deformation tensor Γ was given in [CSS1], Sec. 4. The formulae for obstruction terms (important for existence proofs) as well as for correction terms (important for explicit description of operators) become quickly very complicated. Using explicit description of the homomorphism Φ in Section 4 by means of Casimir operators, it is possible to simplify the algorithm substantially and to get quite explicit formulae for the coefficients in general correction terms for the invariant operators constructed in the previous section. It is quite remarkable that coefficients in the final formula for curvature correction terms do not depend on a choice of a representation V_λ as well as on a choice of a particular AHS structure! They depend only on the order of the operator.

Let us first simplify the algorithm given in [CSS1]. Let k be a fixed integer and let us consider an operator $D = \pi \circ (\nabla^\omega)^k$, where the projection π of $\otimes^k(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$ onto one of its irreducible components is determined by a chain of dominant weights, as described in Section 3. Knowing highest weights of all intermediate irreducible components in the chain of projections, Lemma 4.3 can be used to compute the values of the homomorphism Φ on all terms in the algorithm. The same is true for the action of the double commutator term $[[X, \tau], \tau]$ (see Example 4.4). This makes it possible to evaluate, in principle, all terms in the expansion. But the result is still quite complicated.

A considerable simplification in the algorithm can be achieved, if we restrict ourselves to the symmetric case, i.e. if the image of π is a subspace of $\odot^k(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$. Then many multiple tensor products contained in various terms of the formula may be reordered and combined together. Any term of the formula is then just a symmetric tensor product of a power of τ, suitable powers of Γ, its covariant derivatives and a covariant derivative of the section s. A problem to be solved is whether there is a way how to compute effectively coefficients in the corresponding linear combination of such terms.

An additional simplification can be achieved in the case, when we know which summand in the description of the action of the double commutator (Lemma 4.4) is really appearing in various terms. Such information is available in the case of the operators $D(\lambda, \theta, k)$ constructed above. In this case, we may use properties of the decomposition of the tensor product $\otimes^k(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$ proved in Section 3 to get an explicit form of the operator. Before tackling the main Theorems 7.4 and 7.9, we discuss the low order cases.

7.2 The first order operators. Using results from [CSS1], see 2.4, and Lemma 4.3, we get immediately the existence and an explicit form of the 1st order operators.

Corollary. *Let $\mathbb{V}_\lambda$ be an irreducible representation of $(\mathfrak{g}_0^{\mathbb{C}})^s$ and $\mathbb{V}_\mu$ be an irreducible component of the product $\mathfrak{g}_1 \otimes \mathbb{V}_\lambda$. Let $\pi = \pi_{\lambda\mu}$ be the corresponding projection. Then*

$$\pi(\nabla^\omega(p^*t)) = \pi[p^*(\nabla^\gamma t) + (c_0 - w)\tau \otimes t]$$

where $c_0 = c_{\lambda\mu}$ are the constants from 4.3.

In particular, there is the unique value $w = c_0$ of the conformal weight for which the projection defines a first order invariant operator $D\,t = \pi[p^*(\nabla^\gamma)t]$.

Operators of this type were introduced in conformal case in paper [SW] and are now standardly called generalized gradients or Stein–Weiss operators (see e.g. [Bra]). The result above was proved in the conformal case by Fegan (see [F]). He gave the first systematic classification of such operators. The theorem above treats completely all first order operators for all AHS structures (note that in odd conformal case, the class of them includes also certain exceptional standard operators of first order not covered by the class of operators $D(\lambda, \theta, k)$, e.g. the one in the middle in the de Rham resolution).

7.3 The second order operators. In a similar way, we can use the first order formula, the algorithm leading in [CSS1] to the formula in 2.5, and Lemma 4.2, in order to compute explicitly the form of the second order invariant differential projected to an irreducible component given by a sequence of dominant weights $\underline{\lambda} = (\lambda_0, \lambda_1, \lambda_2)$. Let π be the corresponding projection.

Corollary. *Using notation of Example 4.4 and Lemma 5.1, we have*

$$\begin{aligned}\pi\big[((\nabla^\omega)^2(p^*t))\big] = \pi[&p^*((\nabla^\gamma)^2 t) + (c_0 - w)\Gamma \otimes p^*t + \\ &(c_0 - w)\tau \otimes p^*(\nabla^\gamma t) + (c_1 - w - 1)p^*(\nabla^\gamma t) \otimes \tau + \\ &(c_0 - w)(c_1 - w - 1)\tau \otimes \tau \otimes t - \sum_{i=1}^{3} A_i \pi_i(\tau \otimes \tau \otimes t)\big].\end{aligned}$$

The most complicated term to compute is clearly the last one coming from the double commutator term. To understand that term, one has to understand well the relation among the chosen projection π defined by the chain of weights $\underline{\lambda}$ and the projections π_i coming from the splitting $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ into symmetric and antisymmetric parts. We shall see that for operators $D(\lambda, \theta, k)$, this relation can be understood and the formula above can be simplified further.

The operators $D(\lambda, \theta, 2)$ are invariant for a unique value for the (generalized) conformal weight, cf. 6.1. It is immediate to check that it is just given by the requirement that the sum of coefficients at terms linear in τ vanishes. It is also possible to verify directly that then the coefficient at the term of second order in τ vanishes as well.

We shall now follow line of reasoning suggested in 7.1 and we shall develop an effective procedure for explicit description of all operators $D(\lambda, \theta, k)$.

7.4 Theorem. *Let A_1 be the number defined in Example 4.4. The value of the operator $D(\lambda, \theta, k)t(u) = \pi_k \circ ((\nabla^\omega)^k(p^*t))(u)$ constructed in 6.1 expands into a sum of the form*

$$\sum a^{k,j}_{s_0,\dots,s_m} \pi_k[\tau^j \odot \Gamma^{s_0} \odot (\nabla\Gamma)^{s_1} \odot \ldots \odot (\nabla^m\Gamma)^{s_m} \odot \nabla^i t](u),$$

where the summation goes over

$$j, s_i \in \{0,1,2,\dots\} \text{ such that } j + \sum_{i'=0}^{m} s_{i'}(i'+2) + i = k,$$

$a^{k,j}_{s_0,\dots,s_m} \in \mathbb{R}$, $\tau(u) \in \mathfrak{g}_1^{\mathbb{C}}$, *and*

$$\tau^j = \odot^j \tau, \quad [\nabla^i t](X_1,\dots,X_i) = p^* \nabla^\gamma_{X_i} \dots \nabla^\gamma_{X_1} t,$$
$$[\nabla^\ell \Gamma](X,Y,X_1,\dots,X_\ell) = [p^* \circ \nabla^\gamma_{X_\ell} \dots \nabla^\gamma_{X_1}(\Gamma)](X,Y).$$

The expressions

$$F^k t(u) := \pi_k[(\nabla^\omega)^k (p^* t)](u) \in \odot^k(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$$

are given by recursive formulae

$$F^0 t(u) = p^* t(u)$$
$$F^{k+1} t(u) = [S_{\lambda+\tau}](F^k t(u)) + [S_\nabla](F^k t(u)) + [S_\Gamma](F^k t(u)).$$

The individual transformations $S_{\lambda+\tau}$, S_∇ and S_Γ act as follows:

$$S_{\lambda+\tau}[\pi_k(\tau^{j-1} \odot \omega_{k-j+1})] = (c_k - k + (j-1)A_1 - w)\pi_{k+1}(\tau^j \odot \omega_{k-j+1} \otimes t);$$
$$\text{where } \omega_{k-j+1} \in \odot^{k-j+1}(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda;\ c_k = c_{\lambda_k,\lambda_{k+1}};\ \lambda_k = \lambda + k\theta,\ j > 1.$$
$$S_\nabla[\pi_k(\tau^j \odot \Gamma^{s_0} \odot (\nabla\Gamma)^{s_1} \odot \dots \odot (\nabla^m \Gamma)^{s_m} \odot \nabla^i t)] =$$
$$= s_0[\pi_{k+1}(\tau^j \odot \Gamma^{s_0-1} \odot (\nabla\Gamma)^{s_1+1} \odot \dots \odot (\nabla^m\Gamma)^{s_m} \odot \nabla^i t)] +$$
$$+ \dots +$$
$$s_m[\pi_{k+1}(\tau^j \odot \Gamma^{s_0} \odot \dots \odot (\nabla^m\Gamma)^{s_m-1} \odot (\nabla^{m+1}\Gamma) \otimes \nabla^i t] +$$
$$[\pi_{k+1}(\tau^j \odot \Gamma^{s_0} \odot (\nabla\Gamma)^{s_1} \odot \dots \odot (\nabla^m\Gamma)^{s_m} \odot \nabla^{i+1} t)].$$
$$S_\Gamma[\pi_k(\tau^{j+1} \odot \omega_{k-j-1})] = (j+1)\pi_{k+1}(\tau^j \odot \Gamma \odot \omega_{k-j-1});$$
$$\text{where } \omega_{k-j-1} \in \odot^{k-j-1}(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda.$$

Proof. In [CSS1, 4.9], we have described an algorithm to inductively compute the difference $(\nabla^\omega)^k(p^*t) - p^*((\nabla^\gamma)^k t)$ as a sum of correction and obstruction terms. Computing instead of that difference the value of $F^k t(u) := (\nabla^\omega)^k(p^* t)$ inductively, the results of [CSS1, 4.9] read as follows: The expression $F^k t(u)$, evaluated at k arguments from $\mathfrak{g}_{-1}$, expands into a sum of terms of the form

$$a\lambda^{(t_1)}(\beta_1)\dots\lambda^{(t_i)}(\beta_i)p^*(\nabla^\gamma)^j t$$

where a is a scalar coefficient, the β_ℓ are iterated brackets involving some arguments $X_\ell \in \mathfrak{g}_{-1}$, the iterated covariant differentials $(\nabla^\gamma)^r\Gamma$ evaluated on some X's, and τ's. Exactly the first t_j arguments $X_1,\dots,X_{t_j}$ are evaluated after the action of

$\lambda^{(t_j)}(\beta_j)$, the other ones appearing on the right are evaluated before. For $k=1$, we have

$$F^1 t(u)(X_1) = p^*((\nabla^\gamma)t)(u)(X_1) + [X_1, \tau](p^* t)(u).$$

Inductively,

$$\begin{aligned} F^k t(u)(X_1, \dots, X_k) = \; & \big(\lambda^{(k-1)}([X_k, \tau(u)])F^{k-1}t(u)\big)(X_1, \dots, X_{k-1}) + \\ & \tilde S_\tau(F^{k-1}t(u))(X_1, \dots, X_k) + \\ & \tilde S_\nabla(F^{k-1}t(u))(X_1, \dots, X_k) + \\ & \tilde S_\Gamma(F^{k-1}t(u))(X_1, \dots, X_k). \end{aligned}$$

where $\lambda^{(k-1)}$ is the obvious tensor product representation on $\otimes^{k-1}\mathfrak{g} \otimes V_\lambda$ and the individual transformations $\tilde S_\tau$, $\tilde S_\nabla$, and $\tilde S_\Gamma$ act as follows.

(1) The action of $\tilde S_\tau$ replaces each summand $a\lambda^{(t_1)}(\beta_1)\dots\lambda^{(t_i)}(\beta_i)p^*(\nabla^\gamma)^j t$ by a sum with just one term for each occurrence of τ where this τ is replaced by $[\tau, [\tau, X_k]]$ and the coefficient a is multiplied by $-1/2$.
(2) $\tilde S_\nabla$ replaces each summand in F^{k-1} by a sum with just one term for each occurrence of Γ and its differentials, where these arguments are replaced by their covariant derivatives $\nabla^\gamma_{X_k}$, and with one additional term where $(\nabla^\gamma)^j t$ is replaced by $\nabla^\gamma_{X_k}((\nabla^\gamma)^j t)$.
(3) $\tilde S_\Gamma$ replaces each summand by a sum with just one term for each occurrence of τ where this τ is replaced by $\Gamma(u).X_k$.

Now we are going to specialize these results to the case we are interested in here: Under the assumptions of the theorem, which we want to prove, the image of the projection π is included in $\odot^k(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$ hence order of factors in the multiple tensor product does not matter. Consequently all τ's can be shifted to the front of the product, derivatives of Γ can be reordered as indicated above, and all derivatives of t can be put to the end of the expression. Terms $\nabla^l\Gamma$ can be hence interpreted as elements of $\odot^{l+2}(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$ and $\nabla^i t$ can be substituted by its symmetrization in $\odot^i(\mathfrak{g}_1^{\mathbb{C}}) \otimes V_\lambda$. We have already seen that the expression $F^1 t$ has the required form (see 7.2). Using Casimir operators, we can now express the algorithm described above in the following way.

Suppose (by induction) that the term F^k has already been written in the form given in the theorem. The action of an element $[X_{k+1}, \tau(u)]$ on $F^k t(u)$ can be computed by Lemma 4.3, because we know that $F^k t(u)$ belongs to the image of π_k, which is, by assumption, an irreducible representation with the highest weight λ_k. The result is $(c_k - w - k)F^k t(u)$.

The action of $\tilde S_\tau$ was a replacement of τ at all $j-1$ places in the expression by $-1/2[\tau, [\tau, X_k]]$ Applying the projection π and using the result of Example 4.4 and 3.6, only the first part in the decomposition of $\tau \otimes \tau$ survives and the result is the same term containing one more τ multiplied by $(j-1)A_1$. Adding both contributions, we get the action of $S_{\lambda+\tau}$.

The action of $\tilde S_\nabla$ is just a derivation and action of $\tilde S_\Gamma$ is a substitution of Γ instead of τ, so we arrive directly at the description of S_∇ and S_Γ in the theorem.

The fact that F^k has the required form follows from the above description of the operators $S_{\lambda+\tau}$, S_∇, S_Γ by induction. □

Looking at the action of the individual transformations and at the form of the expansion, we get immediately the following algorithm for the unknown coefficients.

7.5 An algorithm for expansion coefficients. *The coefficients* $a^{k+1,j}_{s_0,\dots,s_m}$ *in theorem 7.4 satisfy the following recursive relations.*

$$\begin{aligned} a^{k+1,j}_{s_0,\dots,s_m} = {} & (1-\delta_{j,0})a^{k,j-1}_{s_0,\dots,s_m}(c_k - k + (j-1)A_1 - w) \\ & + a^{k,j}_{s_0,\dots,s_m} \\ & + (1-\delta_{s_0,0})(j+1)a^{k,j+1}_{s_0-1,s_1,\dots,s_m} + \\ & + (1-\delta_{s_1,0})(s_0+1)a^{k,j}_{s_0+1,s_1-1,\dots,s_m} + \\ & + \dots + \\ & + (1-\delta_{s_m,0})(s_{m-1}+1)a^{k,j}_{s_0,\dots,s_{m-2},s_{m-1}+1,s_m-1}. \end{aligned}$$

7.6 Constants $\tilde c_k$. In the algorithm above, the value $c_k - k + jA_1 - w$ has frequently appeared. It will be convenient to change the definition of constants c_j and to define new shifted constants $\tilde c_j$ instead. Let us define them by

$$\tilde c_j = c_0 - j\,A_1.$$

Then $c_k - k + j\,A_1 - w = c_0 - kA_1 - (k-j)A_1 - w = \tilde c_k - (k-j)A_1 - w$.

Note for future use that the differences $\tilde c_j - \tilde c_k = (k-j)A_1$ are always multiples of A_1.

7.7 Constants $B^m_{(s_0,\dots,s_m)}$. As the last item in the preparation of an explicit computation of the coefficients in the expansion, we are going to define inductively the following parametric system of constants B^n_s, where $n \geq 0$ is an integer, $s = (s_0, s_1, s_2, \dots)$ is a sequence of non-negative integers with a finite number of nonvanishing elements. We shall often write $s = (s_0 \dots s_m)$ by cutting the sequence at the last nontrivial entry; (0) will denote the sequence $(0, 0, \dots)$. For any finite sequence of integers s ,we shall use two integers $|s|$, $[s]$ associated with s, defined by

$$|s| = \sum_0^\infty s_i \text{ and } [s] = \sum_0^\infty s_i(i+1).$$

Symbols σ_i, $i = 0, 1, \dots$, will be used for special sequences of integers defined by

$$\sigma_0 = (1, 0, \dots);\ \sigma_1 = (-1, 1, 0, \dots);\ \sigma_2 = (0, -1, 1, 0, \dots);\ \dots$$

Definition. *Let* $\tilde c_0$, A_1, *and* w, *be any fixed real numbers and define* $\tilde c_j$, $j \in \mathbb{N}$, *by* $\tilde c_j = \tilde c_0 - j\,A_1$.

A system of real numbers B^n_s, *where* n *is a non-negative integer and* s *is a sequence of non-negative integers with finite number of nonzero terms, is defined by induction with respect to* $n + [s]$ *as follows*

$$B^0_0 = 1;$$

$$B^n_s = (1-\delta_{s_0,0})(n+|s|-1)(\tilde c_{n+|s|-2} - w)\left[\sum_{l=0}^{n-1} B^l_{s-\sigma_0}\right] + \sum_{i=1}^{\infty}(1-\delta_{s_i,0})(s_{i-1}+1)\sum_{l=0}^{n-1} B^l_{s-\sigma_i}.$$

In the formula above, we use the convention that any sum $\sum_a^b \dots$ *vanishes whenever* $a > b$.

In the sequel, we shall use the B's with the numbers A_1 and $\tilde{c}_0 = c_0$ chosen as in 4.4 and 5.1, respectively. Note that then the numbers B^n_s still depend implicitly on the value of the variable w which plays the role of the conformal weight.

The induction above works fine, because the smallest possible value of $n + [s]$ is achieved only for $n = 0$, $s = (0)$ and the value of B^0_0 is fixed as 1 in advance. The inductive formula for B^n_s clearly uses only B's with a smaller value of $n + [s]$.

Certain values of B's are immediately clear from definition: $B^n_{(0)} = 0$ for all $n \neq 0$ and $B^0_s = 0$ for all $s \neq (0)$. More generally, we get from the definition by induction (with respect to n) that $B^n_s = 0$ for all n, s such that $n < [s]$.

7.8 Basic properties of B^n_s. Before treating more complicated examples, we shall introduce one more piece of notation. For a positive integer n, the symbol $\{n\}$ will denote the number

$$\{n\} := n(\tilde{c}_{n-1} - w).$$

Later on, we shall consider values of these factors $\{n\}$ at special values of conformal weight $w = \tilde{c}_{k-1}$, $k \in \mathbb{N}$. Let us note already at this point that for this value of w the resulting number depends linearly on A_1 (see 7.6).

The case where $|s| = 1$. Using the shorthands $\{n\}$, we get immediately from the definition that

$$B^n_{(1)} = \{n\}, \quad \text{for all } n \geq 1,$$
$$B^n_{(2)} = \{n+1\} \sum_{l=1}^{n-1} \{l\}, \quad \text{for } n \geq 2,$$

while $B^1_{(2)} = 0$.

Similarly (by induction with respect to n), we get easily for any $n \geq m+1$

$$B^n_{(m+1)} = \{n+m\} \sum_{l_m=m}^{n-1} \{l_m + m - 1\} \sum_{l_{m-1}=m-1}^{l_m - 1} \{l_{m-1} + m - 2\} \sum_{l_{m-2}=m-2}^{l_{m-1}-1} \dots \sum_{l_1=1}^{l_2-1} \{l_1\},$$

and $B^n_{(m+1)} = 0$ for $n = 0, \dots, m$. Clearly, the numbers $B^n_{(m)}|_{w=\tilde{c}_{k-1}}$ are homogeneous of degree m in A_1 for each $k \in \mathbb{N}$.

The case where $|s| = 2$. To understand the definition of B^n_s better, let us also consider the numbers $B^n_{(ij)}$. Couples (ij) of non-negative integers can be considered as vertices of a graph in the plane. These vertices will be connected with arrows of length 1 going horizontally right and antidiagonal arrows of length $\sqrt{2}$ going up and left.

Any vertex in the lattice can be reached from (00) by one or more paths (lying completely in the first quadrant). For every path to a vertex (ij), it is possible to deduce a contribution to the value of $B^n_{(ij)}$ corresponding to this path from the algorithm defining B's. The actual value of $B^n_{(ij)}$ is then the sum of such contributions over all possible paths from (00) to (ij).

The situation for longer sequences s is similar. The numbers B^n_s play a principal role in the evaluation of coefficients for standard operators, so we shall study them in more details in Appendix B and we shall give an explicit formula for them there.

Using the very definition of B's and the simple relations $|s-\sigma_0| = |s|-1$, $|s-\sigma_i| = |s|$, for all $i>0$, we get immediately by induction with respect to values of n and $|s|$ the following important fact:

Lemma. *The numbers B^n_s evaluated at $w=\tilde{c}_{k-1}$ are homogeneous of degree $|s|$ in A_1.*

7.9 Formulae for expansion coefficients. Let $k\in\mathbb{N}$ be fixed. Suppose that $j\in\mathbb{N}$ and $s=(s_0,s_1,\dots,s_m)$ is a finite sequence of non-negative integers such that $j+[s]=j+\sum_{i=0}^m s_i(i+2)\le k$. Let $\tilde{c}_i$ be the real numbers defined in 7.6 and B^n_s the numbers defined in 7.7. Then we have the following theorem.

Theorem. *The coefficients $a^{k,j}_s$ in the expression for $D(\lambda,\theta,k)t$ in 7.4 are given by the formulae*

$$a^{k,j}_s := \binom{k}{j}\left[\prod_{i=k-j}^{k-1}(\tilde{c}_i-w)\right]\left[\sum_{l=0}^{k-j-|s|}B^l_s\right], \quad \textit{for all } j\ge 1 \tag{1}$$

$$a^{k,0}_s := \sum_{l=0}^{k-|s|}B^l_s. \tag{2}$$

Proof. The theorem will be proved by induction with respect to k, using the recursive relations from 7.5.

Let $k=1$. Then, according to Corollary 7.2, $F^1=\pi(\nabla t+(\tilde{c}_0-w)\tau\otimes t)$. The inequality $j+\sum_{i=0}^m s_i(i+2)\le 1$ is satisfied only for $s=(0)$ and $j=0,1$. The relations (1) and (2) read as $a^{1,0}_0=B^0_0+B^1_0$ and $a^{1,1}_0=(\tilde{c}_0-w)B^0_0$. The definition of B's yields $B^0_0=1, B^1_0=0$ which proves the claim in this case.

Suppose now that the theorem holds for some fixed k. Let us first prove the relation (2), i.e. suppose first $j=0$. By inductive assumption and the recursive relations 7.5 for a's, we get

$$a^{k+1,0}_s = \left[\sum_{l=0}^{k-|s|}B^l_s\right] + (1-\delta_{s_0,0})\binom{k}{1}(\tilde{c}_{k-1}-w)\left[\sum_{l=0}^{k-|s|}B^l_{s-\sigma_0}\right] +$$
$$\sum_{i=1}^m(1-\delta_{s_i,0})(s_{i-1}+1)\left[\sum_{l=0}^{k-|s|}B^l_{s-\sigma_i}\right] = \sum_{l=0}^{k+1-|s|}B^l_s,$$

where we use

$$B_s^{k+1-|s|} = (1-\delta_{s_0,0})k(\tilde{c}_{k-1}-w)\left[\sum_{l=0}^{k-|s|} B^l_{s-\sigma_0}\right] + \sum_{i=1}^{m}(1-\delta_{s_i,0})(s_{i-1}+1)\left[\sum_{l=0}^{k-|s|} B^l_{s-\sigma_i}\right].$$

For positive j, we get

$$\begin{aligned}
a_s^{k+1,j} = & \binom{k}{j-1}\prod_{k-j+1}^{k-1}(\tilde{c}_i-w)\left[\sum_{l=0}^{k+1-j-|s|} B^l_s\right](\tilde{c}_k - w-(k-j+1)A_1)+\\
&+\binom{k}{j}\prod_{k-j}^{k-1}(\tilde{c}_i-w)\left[\sum_{l=0}^{k-j-|s|} B^l_s\right]+\\
&+(j+1)(1-\delta_{s_0,0})\binom{k}{j+1}\prod_{k-j-1}^{k-1}(\tilde{c}_i-w)\left[\sum_{l=0}^{k-j-|s|} B^l_{s-\sigma_0}\right]+\\
&+\sum_{i=1}^{m}(1-\delta_{s_i,0})(s_{i-1}+1)\binom{k}{j}\prod_{k-j}^{k-1}(\tilde{c}_i-w)\left[\sum_{l=0}^{k-j-|s|} B^l_{s-\sigma_i}\right]\\
=&\binom{k+1}{j}\prod_{k-j+1}^{k-1}(\tilde{c}_i-w)\left[\sum_{l=0}^{k-j-|s|} B^l_s\right]\cdot\\
&\cdot\left[\frac{j}{k+1}(\tilde{c}_k-w-(k-j+1)A_1)+\frac{k-j+1}{k+1}(\tilde{c}_{k-j}-w)\right]+\\
&+\binom{k+1}{j}\prod_{k-j+1}^{k-1}(\tilde{c}_i-w)\left[B_s^{k+1-j-|s|}\right]\cdot\\
&\cdot\left[\frac{j}{k+1}(\tilde{c}_k-w-(k-j+1)A_1)+\frac{k-j+1}{k+1}(\tilde{c}_{k-j}-w)\right]\\
=&\binom{k+1}{j}\prod_{k-j+1}^{k}(\tilde{c}_i-w)\left[\sum_{l=0}^{k+1-j-|s|} B^l_{s_1,\dots,s_m}\right],
\end{aligned}$$

where we have used the relations

$$B_s^{k+1-j-|s|} = (1-\delta_{s_0,0})(\tilde{c}_{k-j-1}-w)\sum_{l=0}^{k-j-|s|} B^l_{s-\sigma_0}(k-j)+ \sum_{i=1}^{m}(s_{i-1}+1)(1-\delta_{s_i,0})\sum_{l=0}^{k-j-|s|} B^l_{s-\sigma_i}. \qquad \square$$

7.10 Formulae for the operators $D(\lambda, \theta, k)$. Note that the form of the coefficients $a_s^{k,j}$ shows immediately that all obstruction terms vanish at once for the value $w = \tilde{c}_{k-1}$ of the (generalized) conformal weight. It confirms once more that the operators $D(\lambda, \theta, k)$ are invariant, independently of the algebraic proof worked out in Section 5. Theorem 7.9 gives at the same time the values of coefficients in the correction terms, i.e. the explicit form of the operators $D(\lambda, \theta, k)$. It is sufficient to use 7.9.(2) and to substitute there the corresponding value of w.

As a consequence of Lemma 7.8 and the definition of the constants $a_s^{k,0}$, it is clear that $a_s^{k,0}$ are homogeneous of degree $|s|$ in A_1. Hence the constants A_1 can be absorbed into the definition of the deformation tensor Γ by introducing news tensors $\tilde{\Gamma} := A_1\Gamma$ and the resulting formula is uniform and universal for all AHS structures (for conformal structures, the constant A_1 is equal to 1).

For practical calculations of curvature correction terms of standard operators, it is better to first write down formulas for coefficients B_s^n, because they have the same form for all k. Having k fixed, it is then easy to evaluate B_s^n at $w = \tilde{c}_{k-1}$ and to get the necessary coefficients $a_s^{k,0}$. Note, however, that for operators of order bigger than 10, it is better to implement the algorithm on a computer, since the list of correction terms is going quickly to be unmanageable. We have postponed the exposition of the general formulae for B_s^n to Appendix B, but let us illustrate the procedure by a few examples now.

In order to make the dependence on the order k and the corresponding fixed conformal weight w explicit, we shall use the notation $B_s^n(k)$, or $\{n\}(k)$, for the numbers B_s^n, or $\{n\}$, evaluated with $w = \tilde{c}_{k-1}$, respectively. Clearly $\{n\}(k) = n(k-n)A_1$. The numbers $B_s^n(k)$ are simplified considerably, because the term $\tilde{c}_{j-1} - w$ reduces to $k - j$. Note that after such substitution, 'symmetric' products $\{j\} = j(k-j)A_1$ are appearing repeatedly in formulas for $B_s^n(k)$. This leads to further simplifications of the formulae for some $B(k)$'s, for example $B_{(n)}^n(2n) = [(2n-1)!!]^2$.

7.11 Examples in low degrees. Let us recall that $B_s^n = 0$ for all n, s such that $n < [s]$ and $B_{(0)}^n = 0$ for all $n > 0$. We have already seen special cases of the previous general formulae:

$$B_{(1)}^n = \{n\},\ B_{(2)}^n = \{n+1\}\sum_{\ell-1}^{n-1}\{\ell\}.$$

The Example in Appendix B provides the coefficients

$$B_{(01)}^n = \sum_{l=1}^{n-1}\{l\};\quad B_{(001)}^n = \sum_{l'=2}^{n-1}\sum_{l=1}^{l'-1}\{l\}$$

$$B_{(11)}^n = 2\sum_{l'=2}^{n-1}\{l'+1\}\sum_{l=1}^{l'-1}\{l\} + \{n+1\}\sum_{l'=2}^{n-1}\sum_{l=1}^{l'-1}\{l\}.$$

We denote by $\tilde{\Gamma}$ here the corrected tensor $A_1\Gamma$ and we compute the universal formula for the operators $D(\lambda, \theta, k)$ independently of the choice of AHS structure and the data λ, θ for low values of k. The projection π denotes as before the projection onto the unique irreducible component $\mathbb{V}_\mu$ in $\otimes^k(\mathfrak{g}_1^{\mathbb{C}}) \otimes \mathbb{V}_\lambda$, the operator D is written using the conventions set up in Theorem 7.4, and we write a_s^k instead of $a_s^{k,0}$. Note that by formula (2) of theorem 7.9 we have $a^k_{(0)} = \sum_{l=0}^k B^l_{(0)} = B^0_{(0)} = 1$.

The case $k = 2$. Here we only need the coefficients $a^2_{(0)} = 1$ and

$$a^2_{(1)} = B^1_{(1)} = \{1\}(2) = 1.$$

Hence

$$D(\lambda, \theta, 2)t = \pi[\nabla^2 t + \tilde{\Gamma} \otimes t].$$

The case $k = 3$. We need the 3 coefficients $a^3_{(0)} = 1$,

$$a^3_{(1)} = B^1_{(1)} + B^2_{(1)} = \{1\} + \{2\} \text{ and } a^3_{(01)} = B^2_{(01)} = \{1\}.$$

Using $\{1\}(3) = 2$, $\{2\}(3) = 2$, we get

$$D(\lambda, \theta, 3)t = \pi[\nabla^3 t + 4\tilde{\Gamma} \otimes (\nabla t) + 2(\nabla\tilde{\Gamma}) \otimes t].$$

The case $k = 4$. Now, we need 5 coefficients: $a^4_{(0)} = 1$, and

$$a^4_{(1)} = B^1_{(1)} + B^2_{(1)} + B^3_{(1)} = \{1\} + \{2\} + \{3\} \qquad a^4_{(2)} = B^2_{(2)} = \{3\}\{1\}$$
$$a^4_{(01)} = B^2_{(01)} + B^3_{(01)} = 2\{1\} + \{2\} \qquad a^4_{(001)} = B^3_{(001)} = \{1\}.$$

Hence using again $\{n\}(k) = n(k - n)A_1$, we get

$$D(\lambda, \theta, 4)t = \pi[\nabla^4 t + 10\tilde{\Gamma} \otimes (\nabla^2 t) + 10(\nabla\tilde{\Gamma}) \otimes (\nabla t) + 9\tilde{\Gamma} \otimes \tilde{\Gamma} \otimes t + 3(\nabla^2\tilde{\Gamma}) \otimes t].$$

The case $k = 5$. Here we need 7 coefficients: $a^5_{(0)} = 1$, and

$$\begin{aligned}
a^5_{(1)} &= B^1_{(1)} + \ldots + B^4_{(1)} = \{1\} + \{2\} + \{3\} + \{4\}\\
a^5_{(2)} &= B^2_{(2)} + B^3_{(2)} = \{3\}\{1\} + \{4\}(\{1\} + \{2\})\\
a^5_{(01)} &= B^2_{(01)} + B^3_{(01)} + B^4_{(01)} = 3\{1\} + 2\{2\} + \{3\}\\
a^5_{(001)} &= B^3_{(001)} + B^4_{(001)} = \{1\} + (2\{1\} + \{2\})\\
a^5_{(0001)} &= B^4_{(0001)} = \{1\}\\
a^5_{(11)} &= B^3_{(11)} = 2\{3\}\{1\} + \{4\}\{1\}
\end{aligned}$$

Hence we get

$$\begin{aligned}
D(\lambda, \theta, 5)t = \pi[\nabla^5 t &+ 20\tilde{\Gamma} \otimes (\nabla^3 t) + 30(\nabla\tilde{\Gamma}) \otimes (\nabla^2 t) + 64\tilde{\Gamma} \otimes \tilde{\Gamma} \otimes (\nabla t) +\\
&18(\nabla^2\tilde{\Gamma}) \otimes (\nabla t) + 4(\nabla^3\tilde{\Gamma}) \otimes t + 64\tilde{\Gamma} \otimes (\nabla\tilde{\Gamma}) \otimes t].
\end{aligned}$$

As a further illustration we include the final formula in order seven. Here we use the concatenation of the symbols instead of the tensor products and we omit the projection π

$$\begin{aligned}
&\nabla^7 t + 56\tilde{\Gamma}\nabla^5 t + 140(\nabla\tilde{\Gamma})\nabla^4 t + 168(\nabla^2\tilde{\Gamma})\nabla^3 t + 784(\tilde{\Gamma})^2\nabla^3 t + 2352\tilde{\Gamma}(\nabla\tilde{\Gamma})\nabla^2 t +\\
&112(\nabla^3\tilde{\Gamma})\nabla^2 t + 2304(\tilde{\Gamma})^3\nabla t + 1180(\nabla\tilde{\Gamma})^2\nabla t + 1408\tilde{\Gamma}(\nabla^2\tilde{\Gamma})\nabla^t + 40(\nabla^4\tilde{\Gamma})\nabla t +\\
&708(\nabla\tilde{\Gamma})(\nabla^2\tilde{\Gamma})t + 312\tilde{\Gamma}(\nabla^3\tilde{\Gamma})t + 3456(\tilde{\Gamma})^2(\nabla\tilde{\Gamma})t + 6(\nabla^5\tilde{\Gamma})t
\end{aligned}$$

Appendix A.

For explicit description of all weights in the representation $\mathfrak{g}_1$ in individual cases, we shall use results gathered in [FH]. The facts which are not proved below can be found there.

A.1 Conformal case, even dimension. Here $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2n+2, \mathbb{C})$, $(\mathfrak{g}_0^{\mathbb{C}})_s = \mathfrak{so}(2n, \mathbb{C})$. Let $L_1, \dots, L_n$ be the standard basis for the dual of the Cartan subalgebra. The fundamental weights π_i, $i = 1, \dots, n$ are given by relations

$$\pi_i = L_1 + \ldots + L_i;\ i = 1, \dots, n-2;\ \pi_n + \pi_{n-1} = L_1 + \ldots + L_{n-1};\ \pi_n - \pi_{n-1} = L_n.$$

The dimension of $\mathfrak{g}_1$ is $2n$ and the list of all weights of $\mathfrak{g}_1$ (all with multiplicity one) is given by $\{\pm L_i;\ i = 1, \dots, n\}$. In terms of fundamental weights, we get

$$L_1 = \pi_1;\ L_i = \pi_i - \pi_{i-1},\ i = 2, \dots, n-2;$$
$$L_{n-1} = \pi_n + \pi_{n-1} - \pi_{n-2};\ L_n = \pi_n - \pi_{n-1}.$$

Hence all coefficients in the decompositions are in absolute values at most one. All weights of $\mathfrak{g}_1$ belong in this case to the same orbit of the Weyl group.

A.2 Conformal case, odd dimension.

Here $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2n+3, \mathbb{C})$, $(\mathfrak{g}_0^{\mathbb{C}})_s = \mathfrak{so}(2n+1, \mathbb{C})$. Let $L_1, \dots, L_n$ be the standard basis for the dual of the Cartan subalgebra. The fundamental weights π_i, $i = 1, \dots, n$ are given by relations

$$\pi_i = L_1 + \ldots + L_i;\ i = 1, \dots, n-1;\ \pi_n = (1/2)[L_1 + \ldots + L_{n-1}].$$

The dimension of $\mathfrak{g}_1$ is $2n+1$ and the list of all weights of $\mathfrak{g}_1$ (all with multiplicity one) is given by $\{0;\ \pm L_i;\ i = 1, \dots, n\}$. In terms of fundamental weights, we get

$$L_1 = \pi_1;\ L_i = \pi_i - \pi_{i-1},\ i = 2, \dots, n-1;\ L_n = 2\pi_n - \pi_{n-1}.$$

So it not true in this case that all weights of $\mathfrak{g}_1$ have coefficients (with respect to fundamental weights) in absolute value less or equal to 1. There are two orbits of the Weyl group in the set of all weights of $\mathfrak{g}_1$. All nonzero weights form the first orbit and the zero weight the second one.

A.3 Grassmannian case. Here $\mathfrak{g}^{\mathbb{C}} = A_{p+q+1}$, $(\mathfrak{g}_0^{\mathbb{C}})_s = A_p \times A_q$. This is the only case, where $(\mathfrak{g}_0^{\mathbb{C}})_s$ is not a simple Lie algebra. Irreducible representations $V_{\lambda,\lambda'}$ of $(\mathfrak{g}_0^{\mathbb{C}})_s$ are just tensor products $V_\lambda \otimes V_{\lambda'}$ of two irreducible representations V_λ, resp. $V_{\lambda'}$ of A_p, resp. A_q. To decompose the product $V_{\lambda,\lambda'} \otimes \mathfrak{g}_1$ means to decompose individual products $V_\lambda \otimes V$ and $V_{\lambda'} \otimes V'$, where V, resp. V' are defining representations of both parts of $(\mathfrak{g}_0^{\mathbb{C}})_s$ and then to multiply both decompositions.

So it is sufficient to study just the case A_n. Let us consider the algebra $A_n = \mathfrak{sl}(n+1, \mathbb{C})$. Let $L_1, \dots, L_{n+1}$ be the canonical basis for $\mathbb{C}^{n+1}$. The dual of the Cartan subalgebra can be identified with the quotient $\{(L_i) \in \mathbb{C}^{n+1}\}/\{\sum_{i=1}^{n+1} L_i = 0\}$. The fundamental weights π_i, $i = 1, \dots, n$ are given by relations

$$\pi_i = L_1 + \ldots + L_i;\ i = 1, \dots, n.$$

The dimension of the defining representation V of $\mathfrak{sl}(n+1,\mathbb{C})$ is $n+1$ and the list of all weights of $\mathfrak{g}_1$ (all with multiplicity 1) is given by $\{\pm L_i;\ i=1,\dots,n+1\}$. In terms of fundamental weights, we get

$$L_1 = \pi_1;\ L_i = \pi_i - \pi_{i-1},\ i=2,\dots,n;\ L_{n+1} = -\pi_n.$$

Hence all coefficients in the decompositions are in absolute values at most one. All weights of $\mathfrak{g}_1$ belong in this case to the same orbit of the Weyl group.

A.4 Symplectic case. Here $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}(2n,\mathbb{C})$, $(\mathfrak{g}_0^{\mathbb{C}})_s = \mathfrak{sl}(n-1,\mathbb{C})$, hence the algebra $(\mathfrak{g}_0^{\mathbb{C}})_s$ is again of type A_k. Let $L_1,\dots,L_n$ be the canonical basis for the defining representation $V=\mathbb{C}^n$. The dual of the Cartan subalgebra is again identified with the quotient $\{(L_i)\in\mathbb{C}^n\}/\{\sum_{i=1}^n L_i = 0\}$. The fundamental weights $\pi_i,\ i=1,\dots,n-1$ are given by relations

$$\pi_i = L_1 + \ldots + L_i;\ i=1,\dots,n-1.$$

In this case, the representation $\mathfrak{g}_1$ of $(\mathfrak{g}_0^{\mathbb{C}})_s$ is equivalent to $\Lambda^2(V)$ and its highest weight is equal to the second fundamental weight π_2. The dimension of $\mathfrak{g}_1$ is equal to $n(n-1)/2$ and the list of all weights of $\mathfrak{g}_1$ (all with multiplicity 1) is given by $\{e_{ij} = L_i + L_j;\ i,j=1,\dots,n;\ i<j\}$. Using conventions $\pi_0=\pi_n=0$, we can express e_{ij} using π_j by

$$e_{ij} = (\pi_i - \pi_{i-1}) + (\pi_j - \pi_{j+1}).$$

Hence all coefficients in the decompositions are in absolute values at most one. All weights of $\mathfrak{g}_1$ belong in this case to the same orbit of the Weyl group.

A.5 Spinorial case. Here $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}(2n,\mathbb{C})$, $(\mathfrak{g}_0^{\mathbb{C}})_s = \mathfrak{sl}(n-1,\mathbb{C})$ and the algebra $(\mathfrak{g}_0^{\mathbb{C}})_s$ is again of type A_k. In this case, the representation $\mathfrak{g}_1$ of $(\mathfrak{g}_0^{\mathbb{C}})_s$ is equivalent to $\odot^2(V)$ and its highest weight is equal to $2\pi_1$. The dimension of $\mathfrak{g}_1$ is equal to $(n+1)n/2$ and the list of all weights of $\mathfrak{g}_1$ (all with multiplicity 1) is given by

$$\{e_{ij} = L_i + L_j;\ i,j=1,\dots,n;\ i\le j\}.$$

Using the same conventions $\pi_0=\pi_n=0$, we can express e_{ij} using π_j by

$$e_{ij} = (\pi_i - \pi_{i-1}) + (\pi_j - \pi_{j+1});\ i\le j.$$

Hence $e_{ii} = 2\pi_i - 2\pi_{i-1}$ and the corresponding coefficients are ± 2. There are two orbits of the Weyl group — $\{e_{ii}\}$ and $\{e_{ij} | i<j\}$.

A.6 E_6 case. Here $\mathfrak{g}^{\mathbb{C}} = E_6$, $(\mathfrak{g}_0^{\mathbb{C}})_s = D_5$ and $\mathfrak{g}_1$ is one of the basic (half)-spinor representations. Its dimension is 16. All weights form one orbit of the Weyl group and all their coefficients with respect to the fundamental weights are in absolute value at most one. The structure of the orbit as well as all these coefficients can be found in [Kr].

A.7 E_7 case. Here $\mathfrak{g}^{\mathbb{C}} = E_7$ and $(\mathfrak{g}_0^{\mathbb{C}})_s = E_6$. All weights of $\mathfrak{g}_1$ form one orbit of the Weyl group and all their coefficients are in absolute value at most one (for details, see [Kr]).

Appendix B.

To understand the definition of B^n_s better, we discussed the case of numbers $B^n_{(ij)}$ already in 7.8. Couples (ij) of non-negative integers were considered as vertices of a graph in plane and these vertices were connected with arrows of length 1 going horizontally right and antidiagonal arrows of length $\sqrt{2}$ going up and left.

Any vertex in the lattice can be reached from (00) by one or more paths. For every path to a vertex (ij), it is possible to deduce its contribution to the value of $B^n_{(ij)}$ from the algorithm defining B's. The actual value of $B^n_{(ij)}$ is then the sum of such contributions over all possible paths from (0) to (ij). The situation for longer sequences s is similar. It would be possible to define a similar graph for all sequences s, but it is not possible to draw it in more general cases. We shall do the same in the language of sequences, which also makes possible to prove an explicit formula for the values of B^n_s, resp. $B^n_s(k)$.

Let us first introduce a few additional notations. Let $\mathcal{A}$ denote the set of all finite sequences (of a variable length) $J = (j_1, j_2, \dots, j_\alpha)$, where $j_1 = 0$ and $j_2, \dots, j_\alpha$ are non-negative integers and put $|J| := \alpha$. For a positive integer a and $J \in \mathcal{A}$, let us define the sequences s^J, s^J_a by

$$s^J := \sum_{a'=1}^{|J|} \sigma_{j_{a'}}; \qquad s^J_a := \sum_{a'=1}^{a} \sigma_{j_{a'}};\ a = 1, \dots, |J|-1; \qquad s^J_0 := (0)$$

where σ_i are the sequences from 7.7. The subset $\mathcal{A}_0$ of $\mathcal{A}$ is defined by

$$\mathcal{A}_0 := \{J \in \mathcal{A} \mid (s^J_a)_i \geq 0;\ a = 1, \dots, |J|,\ i = 0, 1, \dots\}.$$

We have the following simple properties

$$[\sigma_i] = 1 \text{ for all } i \text{ and } [\sigma_i] + [\sigma_j] = [\sigma_i + \sigma_j] \text{ for all } i, j$$
$$[s^J] = |J|.$$

In order to generalize formulas for $B^n_{(m)}$ deduced in Section 7, let us introduce for every sequence s of non-negative integers the set

$$\mathcal{A}^0_s := \{J \in \mathcal{A}^0 \mid s^J = s\}.$$

This set is a generalization of the set of all different paths from (0) to s discussed above in the case of sequences of length two.

We also need a generalization of the numbers $\{n\}$ from 7.8. Let us define the numbers $\{s, l, a\}$, where s is a finite sequence of integers and l, a are positive integers

$$\{s, l, a\} := \begin{cases} \{l + |s|\} & \text{if } a = 0 \\ s_{a-1} & \text{if } a \neq 0. \end{cases}$$

Using all this notation we obtain the following explicit formula for the numbers B^n_s:

Theorem. *The numbers B^n_s are given by the formula*

$$\sum_{J\in\mathcal{A}^0_s}\{s^J_{\alpha-1},n,j_\alpha\}\sum_{l_{\alpha-1}=\alpha-1}^{n-1}\{s^J_{\alpha-2},l_{\alpha-1},j_{\alpha-1}\}\sum_{l_{\alpha-2}=\alpha-2}^{l_{\alpha-1}-1}\dots\sum_{l_2=2}^{l_3-1}\{s^J_1,l_2,j_2\}\sum_{l_1=1}^{l_2-1}\{l_1\}$$

where $\alpha=[s]=|J|$.

Proof. We can use induction with respect to α. The case $\alpha=1$ means that $s=(1)$. This case was discussed in 7.8: $B^n_{(1)}=\{n\}$. But $s=\sigma_0$, there is just one element $J=(0)$ in $\mathcal{A}^0_s$ and the theorem holds.

Suppose now that the formula is valid for all s with $[s]\le k-1$ and consider a sequence s with $[s]=k$. The set $\mathcal{A}^0_s$ of sequences J can be split into a disjoint union of subsets by an additional condition $j_{[s]}=i, i=0,1,\dots,$ (all but a finite number of them being empty). Now, let us have a look at the algorithm defining B's. Using the induction assumption for terms $\sum_{l=0}^{n-1}B^l_{s-\sigma_i}$, $i=0,1,\dots$ and noticing that $n+|s|-1=n+|s-\sigma_0|$; $s_{i-1}+1=(s-\sigma_i)_{i-1}$, we get the correct value for B^n_s. □

Examples. Let us use the formula in a few cases. If $s=(01)$, then the set $\mathcal{A}^0_s$ is a one point set. It consists of $J=(0,1)$, $s=\sigma_0+\sigma_1$. Hence

$$B^n_{(01)}=\{(1),n,1\}\sum_{l=1}^{n-1}\{l\}=\sum_{l=1}^{n-1}\{l\}.$$

Similarly, for $s=(001)$, we have $\mathcal{A}^0_s=\{(0,1,2)\}$, $s=\sigma_0+\sigma_1+\sigma_2$. Hence

$$B^n_{(001)}=\{(01),n,2\}\sum_{l'=2}^{n-1}\{(1),l',1\}\sum_{l=1}^{l'-1}\{l\}=\sum_{l'=2}^{n-1}\sum_{l=1}^{l'-1}\{l\}.$$

If $s=(11)$, there are two elements in the set $\mathcal{A}^0_s$, namely $J=(0,0,1)$, $s=\sigma_0+\sigma_0+\sigma_1$ and $J=(0,1,0)$, $s=\sigma_0+\sigma_1+\sigma_0$. So

$$\begin{aligned}B^n_{(11)}&=\{(2),n,1\}\sum_{l'=2}^{n-1}\{(1),l',0\}\sum_{l=1}^{l'-1}\{l\}+\{(01),n,0\}\sum_{l'=2}^{n-1}\{(1),l',1\}\sum_{l=1}^{l'-1}\{l\}=\\&=2\sum_{l'=2}^{n-1}\{l'+1\}\sum_{l=1}^{l'-1}\{l\}+\{n+1\}\sum_{l'=2}^{n-1}\sum_{l=1}^{l'-1}\{l\}.\end{aligned}$$

A similar computation leads to the last constant $B^4_{(0001)}=\{1\}$ which we have used in 7.11.

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