PARABOLIC GEOMETRIES AND BERNSTEIN–GELFAND–GELFAND SEQUENCES

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ABSTRACT. This is an enlarged abstract of the talk with the same title, presented at the Fourth International Workshop on Differential Geometry and its Applications in Brasov, 1999. The goal is to provide a brief survey on the background and recent achievements in this newly expanding area of differential geometry, as well as a guide to some of the old and new bibliography. More detailed information in all aspects can be found in [8, 9, 10, 11, 12, 14, 28, 30].

My talk aims to review the recent development of the general theory of geometrical structures modeled over the homogeneous spaces G/P with G complex semisimple and P parabolic, or any real form of this situation. The lecture is based on a long time project of the author joint with Andreas Čap and Vladimír Souček, and further joint papers with Mike Eastwood, Rod Gover, and Gerd Schmalz.

1. **Comments on the background.** Let us first say a few words about the history of the subject. The origin of all such geometries goes back to the Cartan's idea of generalized spaces, i.e. certain deformations of the homogeneous spaces G/P defined by means of an absolute parallelism on a principal P–bundle. These concepts are closely related to the Cartan's general method for the equivalence problem. Many well known geometries have been shown to allow a canonical object of such type with a suitable choice of simple G and parabolic P, cf. the theory of non–degenerate CR–structures of hypersurface type due to [32, 13], and the pioneering series of papers by Tanaka, see [33, 37, 26, 9] and references therein for more details and more recent results. Nowadays, the name *parabolic geometry* has been adopted, reflecting the relation to the parabolic invariants program initiated by Fefferman, [15]. Some years ago, the relation to the twistor theory caused the general interest in a new calculus for such geometries, with the aim to improve the techniques in conformal geometry and to extend them to the whole class of parabolic geometries, or at least to some of those. See e.g. the papers by Eastwood, Baston, Gover, Čap, and the author in the enclosed bibliography. The important source of inspiration was the classical theory of the conformal invariants, see e.g. [34, 35, 36] and also the differential geometry of conformal Riemannian manifolds (see e.g. [17, 16]). The most recent results are presented in [9, 10].

Here we shall discuss the first essential application of a new approach to this topic, combining the Lie algebraic tools with the frame bundle approach. This research started in [11] and the basic reference for this lecture is [12].

The Bernstein-Gelfand-Gelfand resolutions of arbitrary G–modules have been studied carefully in representation theory for many years, cf. [25] and references therein. An important feature of our approach is the exclusive usage of the elementary (finite dimensional) representation theory. One could even say that the representation theory enters rather as a language and the way of thinking. Some more involved representational theoretical aspects were indicated in [14].

The research supported by GACR, Grant Nr. 201/99/????

2. $|k|$**–graded Lie algebras.** Let G be a real or complex semisimple Lie group with a $|k|$-graded Lie algebra $\mathfrak{g}$, i.e.

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$$

and assume that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and that the (nilpotent) subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. We write $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. We also write $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$, and $\mathfrak{g}^j = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k$, $j = -k, \dots, k$. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and actually the grading is completely determined by this subalgebra, see e.g. [37, Section 3]. In particular, all complex simple $|k|$–graded $\mathfrak{g}$ are classified by subsets of simple roots of complex simple Lie algebras, up to conjugation. The real $|k|$–graded simple Lie algebras are classified easily by means of Satake diagrams, see [22] or [37] for more details. Very helpful notational conventions and computational recipes may be found in [4].

For each our $|k|$–graded Lie algebra $\mathfrak{g}$, there is the unique element $E \in \mathfrak{g}_0$ with the property $[E, Y] = jY$ for all $Y \in \mathfrak{g}_j$, $j = -k, \dots, k$, the *grading element*. Of course, E belongs to the center $\mathfrak{z}$ of the reductive part $\mathfrak{g}_0$ of $\mathfrak{p} \subset \mathfrak{g}$. Moreover, the Killing form provides isomorphisms $\mathfrak{g}_i^* \simeq \mathfrak{g}_{-i}$ for all $i = -k, \dots, k$ and, in particular, its restrictions to the center $\mathfrak{z}$ and the semisimple part $\mathfrak{g}_0^{ss}$ of $\mathfrak{g}_0$ are non–degenerate. Now, for each Lie group G with the $|k|$–graded Lie algebra $\mathfrak{g}$, there is the closed subgroup $P \subset G$ of all elements whose adjoint actions leave the $\mathfrak{p}$–submodules $\mathfrak{g}^j = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k$ invariant, $j = -k, \dots, k$. The Lie algebra of P is just $\mathfrak{p}$ and there is the subgroup $G_0 \subset P$ of elements whose adjoint action leaves invariant the grading by $\mathfrak{g}_0$–modules $\mathfrak{g}_i$, $i = -k, \dots, k$. This is the reductive part of the parabolic Lie subgroup P, with Lie algebra $\mathfrak{g}_0$. We also define subgroups $P_+^j = \exp(\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k)$, $j = 1, \dots, k$, and we write P_+ instead of P_+^1. Obviously $P/P_+ = G_0$ and P_+ is nilpotent. Thus P is the semisimple product of G_0 and the nilpotent part P_+. More explicitly (cf. [9, Proposition 2.10], or [33, 37]), each element $g \in P$ is expressed in the unique way as $g = g_0 \exp Z_1 \exp Z_2 \dots \exp Z_k$, with $g_0 \in G_0$ and $Z_i \in \mathfrak{g}_i$, $i = 1, \dots, k$.

3. **Parabolic geometries.** The (real or complex) *parabolic geometry* $(\mathcal{G}, \omega)$ *of type* G/P is a (smooth or holomorphic) principal fiber bundle $\mathcal{G}$ with structure group P, equipped by a (smooth or holomorphic) one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying

(1) $\omega(\zeta_Z)(u) = Z$ for all $u \in \mathcal{G}$ and fundamental fields ζ_Z, $Z \in \mathfrak{p}$
(2) $(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in P$
(3) $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

In particular, each $X \in \mathfrak{g}$ defines the *constant vector field* $\omega^{-1}(X)$ defined by $\omega(\omega^{-1}(X)(u)) = X$, $u \in \mathcal{G}$. The one–forms with properties (1)–(3) are called (smooth or holomorphic) *Cartan connections*, cf. [29].

The morphisms between parabolic geometries $(\mathcal{G}, \omega)$ and $(\mathcal{G}', \omega')$ are principal fiber bundle morphisms φ (over the identity on P) which preserve the Cartan connections, i.e. $\varphi : \mathcal{G} \to \mathcal{G}'$ and $\varphi^*\omega' = \omega$.

The structure equations define the horizontal smooth form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ called the *curvature* of the Cartan connection ω:

$$d\omega + \frac{1}{2}[\omega, \omega] = K.$$

The *curvature function* $\kappa : \mathcal{G} \to \wedge^2\mathfrak{g}_-^* \otimes \mathfrak{g}$ is then defined by means of the parallelism

$$\kappa(u)(X, Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) = [X, Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)]).$$

In particular, the curvature function is valued in the cochains for the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$. Moreover, there are two ways how to split κ. We may consider the target components κ_i according to the values in $\mathfrak{g}_i$. The whole $\mathfrak{g}_-$–component

κ_- is called the *torsion* of the Cartan connection ω. The other possibility is to consider the homogeneity of the two forms $\kappa(u)$, i.e.

$$\kappa = \sum_{\ell=-k+2}^{3k} \kappa^{(\ell)}, \quad \kappa^{(\ell)} : \mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j+\ell}.$$

Since we deal with semisimple algebras only, there is the adjoint codifferential ∂^* to the Lie algebra cohomology differential ∂, see e.g. [24]. Consequently, there is the Hodge theory on the cochains which allows to deal very effectively with the curvatures. In particular, we may use several restrictions on the values of the curvature which turn out to be quite useful: The parabolic geometry $(\mathcal{G}, \omega)$ with the curvature function κ is called *flat* if $\kappa = 0$, *torsion–free* if $\kappa_- = 0$, *normal* if $\partial^* \circ \kappa = 0$, and *regular* if normal and $\kappa^{(j)} = 0$ for all $j \leq 0$. In particular, the normality ensures that the curvature is in certain sense minimal and is governed by the Lie algebra cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ in a nice way, see [37, 9] for more details.

4. **Flag structures.** The homogeneous models for parabolic geometries are the real generalized flag structures G/P and we still observe such structures in their infinitesimal forms on the curved structures. Indeed, the filtration of $\mathfrak{g}$ by the $\mathfrak{p}$–submodules $\mathfrak{g}^j$ is transferred to the right invariant filtration $T^j\mathcal{G}$ on the tangent space $T\mathcal{G}$ by the parallelism ω. The tangent projection $Tp : T\mathcal{G} \to TM$ then provides the filtration

$$TM = T^{-k}M \supset T^{-k+1}M \supset \cdots \supset T^{-1}M$$

of the tangent space of the underlying manifold M. Moreover, the structure group of the associated graded tangent space

$$\operatorname{Gr} TM = (T^{-k}M/T^{-k+1}M) \oplus \cdots \oplus (T^{-2}M/T^{-1}M) \oplus T^{-1}M$$

reduces automatically to G_0 since $\mathcal{G}_0 = \mathcal{G}/P_+$ clearly plays the role of its frame bundle. With some further simple conditions imposed, we talk about regular infinitesimal flag structures of type $\mathfrak{g}/\mathfrak{p}$ and then a universal construction recovers both the canonical Cartan bundle and the canonical normal Cartan connection, see [9, 10] for details, [28] for some new applications.

5. **Natural bundles.** Each P–module $\mathbb{V}$ defines for all parabolic geometries $(\mathcal{G} \to M, \omega)$ of type G/P over a manifold M the associated bundle $VM = \mathcal{G} \times_P \mathbb{V}$ over M. In fact, this is a functorial construction which may be restricted to all subcategories of parabolic geometries mentioned above and we call such bundles natural (vector) bundles. Similarly, we may treat bundles associated to any representation $P \to \operatorname{Diff}(\mathbb{S})$ on a manifold $\mathbb{S}$, the standard fiber for $SM = \mathcal{G} \times_P \mathbb{S}$. A special class of natural (vector) bundles defined by G–modules $\mathbb{W}$ is called *tractor bundles*, see [2, 8] for historical remarks. The remarkable feature of tractor bundles is that the extension of the Cartan connection ω to the principal connection form $\tilde{\omega}$ on the extended Cartan bundle $\tilde{\mathcal{G}}$ induces on them the canonical linear connections, see [8] for a powerful calculus for these objects.

6. **Semi–holonomic jet–modules.** While the standard jet prolongations of homogeneous vector bundles are again homogeneous vector bundles corresponding to certain jet–modules, this construction does not extend out of locally flat geometries, i.e. those without curvature. On the other hand, the defining absolute parallelism allows such a construction for one–jets and so we can go on to all orders with the semi–holonomic prolongations. This is the core of our approach to invariant operators in [12] and a straightforward iterative construction of suitable homomorphisms between semi–holonomic jet–modules provides all the distinguished operators in the BGG–sequences.

Let us consider a representation $\mathbb{V}$ of P, the corresponding homogeneous bundle $V(G/P) = G \times_P \mathbb{V}$ and its first jet prolongation $J^1(V(G/P)) \to G/P$. This is again a homogeneous bundle, and the corresponding action of P on its standard fiber

$$\mathcal{J}^1(\mathbb{V}) := J^1(V(G/P))_o = \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V})$$

is defined by means of the action of fundamental vector fields on the equivariant functions $s \in C^\infty(G, \mathbb{V})^P$. The formula for the action of $Z \in \mathfrak{p}_+$ on elements of $\mathcal{J}^1(\mathbb{V})$ viewed as pairs (v, φ), where $v \in \mathbb{V}$ and φ is a linear map from $\mathfrak{g}_-$ to $\mathbb{V}$, is given by

$$Z{\cdot}(v, \varphi) = (Z{\cdot}v, X \mapsto Z{\cdot}(\varphi(X)) - \varphi(\mathrm{ad}_-(Z)(X)) + \mathrm{ad}_{\mathfrak{p}}(Z)(X){\cdot}v),$$

i.e. we get the tensorial action plus one additional term mapping the value–part to the derivative–part.

By iteration, we obtain the semi–holonomic jet modules

$$\mathcal{J}^k\mathbb{V} = \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V}) \oplus \cdots \oplus (\otimes^k \mathfrak{g}_-^* \otimes \mathbb{V})$$

with the appropriate action of P. Now, the semi–holonomic jet prolongations of natural bundles with standard fiber $\mathbb{V}$ turn out to be natural bundles corresponding to P–modules $\mathcal{J}^k\mathbb{V}$.

7. **The BGG–resolutions.** The straightforward consequence of the naturality of the semi–holonomic jet prolongation is that each P–module homomorphism

$$\Phi : \mathcal{J}^k\mathbb{V} \to \mathbb{W}$$

gives rise to a natural differential operator between the corresponding natural bundles. Unfortunately, this correspondence is not bijective as in the case of homogeneous bundles and standard jets, i.e. a non–zero homomorphism may lead to a trivial operator and there are natural operators which are not achieved in this way. However, there is an iterative construction of homomorphisms based on the relation between the Lie algebra cohomologies and the vector valued forms on manifolds equipped with parabolic geometry, which yields the so called BGG–sequences of differential operators, first constructed in [12].

In order to enjoy the flavor of the general result, let us look at the example of the most trivial BGG–sequence on 5–dimensional CR–geometries of hypersurface type, cf. [12, Theorem 5.2]. This example is a resolution of the sheaf of complex smooth functions by means of the holomorphic and anti–holomorphic duals $T^*_{1,0}M$, $T^*_{0,1}M$ to the complexified CR–subspace $T^{CR}_{\mathbb{C}}M$, and the quotient $Q = T_{\mathbb{C}}M/T^{CR}_{\mathbb{C}}$.

$$\begin{tikzcd}
 & & & \Lambda^2 T^*_{1,0} \arrow[r] \arrow[ddr] & Q^* \otimes \Lambda^2 T^*_{1,0} \arrow[dr] & & \\
 & & T^*_{1,0} \arrow[ur] \arrow[dr] & & & \otimes^2 Q^* \otimes T^*_{1,0} \arrow[dr] & \\
\mathbb{C} \arrow[r] & \mathcal{E} \arrow[ur] \arrow[dr] & & (\ker \mathcal{L})^* \arrow[r] \arrow[uur] \arrow[ddr] & Q^* \otimes (\ker \mathcal{L})^* \arrow[ur] \arrow[dr] & & \otimes^3 Q^* \\
 & & T^*_{0,1} \arrow[ur] \arrow[dr] & & & \otimes^2 Q^* \otimes T^*_{0,1} \arrow[ur] & \\
 & & & \Lambda^2 T^*_{0,1} \arrow[r] \arrow[uur] & Q^* \otimes \Lambda^2 T^*_{0,1} \arrow[ur] & &
\end{tikzcd}$$

This complex always computes the same cohomology as the De Rham complex, but it is of smaller dimension. The orders of all operators in the middle column are two, all the other operators are of first order. Here $G = SU(p+1, q+1)$, up to some covering fenomena, and for all G–modules $\mathbb{W}$, we obtain the nontrivial BGG–sequences of the same shape. Moreover, these compute the same cohomology as certain twisted De Rham complex on all locally flat manifolds. Let us also remark that each irreducible P–module with regular central character appears at exactly one position in exactly one such sequence.

All these operators belong to the class of *standard operators*. The name comes from representation theory and it is related to the fact that the corresponding Verma module homomorphisms for a general parabolics descend from the homomorphisms in the Borel case. There are also natural operators which are not standard, the so called non–standard ones, for which our methods have not been effective enough yet.

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