NON–STANDARD INVARIANT OPERATORS FOR QUATERNIONIC GEOMETRIES

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ABSTRACT. This is a survey along the lines of the talk at the conference Lie III, held in Clausthal in July 1999. The aim is to present amazing problems related to the invariant operators which are not curved analogues of the so called standard operators in the Bernstein–Gelfand–Gelfand resolutions. On the way, we provide some background and recent achievements, as well as a guide to some of the old and new bibliography.

The lecture is based on a long time project of the author joint with Andreas Čap and Vladimír Souček, and further joint papers with Mike Eastwood, Rod Gover, and Gerd Schmalz. The main reference is [22] and much more information can be found in the recent research papers [7, 8, 9, 10, 11, 14, 27] and expository works [13, 29, 30].

1. **Quaternionic geometry — an example of Parabolic geometries.** The origin of various types of geometries goes back to the Cartan's idea of generalized spaces, i.e. certain deformations of the homogeneous spaces G/P defined by means of an absolute parallelism on a principal P–bundle. Cartan developed these concepts in close relation to his general equivalence problem. The quaternionic geometry belongs to many geometries known to allow a canonical object of such type with a suitable choice of semisimple G and parabolic P, cf. the theory of non–degenerate CR–structures of hypersurface type due to [31, 12], and the pioneering series of papers by Tanaka, see [32, 33, 25, 8] and references therein for more details and more recent results. The current name *parabolic geometry* has been adopted in connection to the parabolic invariants program initiated by Fefferman, [15]. There is a striking relation to the twistor theory in the best known case of parabolic geometries, the conformal Riemannian ones. This relation suggested to seek for a new calculus for all similar geometries, with the aim to improve the techniques even in conformal geometry, see e.g. the papers by Eastwood, Baston, Bailey, Gover, Čap, and the author in the enclosed bibliography. For a different approach to similar questions see [16, 17].

As well known, the quaternionic geometry is defined as a classical torsion free G–structure on a $4m$–dimensional manifold with structure group

$$G_0 = S(GL(p/2, \mathbb{H}) \times GL(q/2, \mathbb{H})) \subset GL(4m, \mathbb{R})$$

where $4m = pq$, $2 = p \leq q$ even, cf. [26]. These geometries fit into a larger class of G–structures with quaternionic forms of the complexified group $G_0^{\mathbb{C}} = S(GL(p, \mathbb{C}) \times GL(q, \mathbb{C}))$ as structure groups and, more generally, other real forms with $1 \leq p \leq q$. On the level of the Lie algebras, these structure groups are distinguished by the requirement that $\mathfrak{g}_0$ is the component in a real graded Lie algebra $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where the complexification is $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}(p+q, \mathbb{C})$.

There is a nice geometric way to describe these structures, which mimics the situation in the four–dimensional conformal spin geometry: The complexified tangent

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bundle is identified as a tensor product of two auxiliary complex vector bundles of fiber dimensions p and q, in the realm of the Penrose's abstract index notation

$$(TM)^{\mathbb{C}} = \mathcal{E}^A \otimes \mathcal{E}_{A'},$$

together with the fixed identification of the top degree forms

$$\Lambda^p \mathcal{E}_{A'} \simeq \Lambda^q \mathcal{E}^A.$$

Of course, the way how this tensorial decomposition is reflected on the real level depends on the specific real form. In particular, we have such a real decomposition of TM for the so called *almost Grassmannian structures* corresponding to the real split form $SL(p+q,\mathbb{R})$ of $G^{\mathbb{C}}$.

In the general case, all these geometries (except $p=1$ and $p=q=2$) have two irreducible components of the total curvature and one of them is the canonical *torsion*. The *quaternionic geometries* are distinguished by the proper choice of G, as above, and the vanishing of the torsion. In a remarkable extent, they generalize the notion of the self dual four–dimensional conformal geometries.

2. **General parabolic geometries.** Let G be a semisimple real Lie group and $P \subset G$ its parabolic subgroup. On the level of the Lie algebras, this amounts to the existence of the $|k|$–grading

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k.$$

We assume that no simple ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$ and that the (nilpotent) subalgebra $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is generated by $\mathfrak{g}_{-1}$. We write $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$. We also write $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$, and $\mathfrak{g}^j = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k$, $j = -k, \dots, k$. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and actually the grading is completely determined by this subalgebra, see e.g. [33, Section 3].

A (real) *parabolic geometry* $(\mathcal{G}, \omega)$ *of type* G/P is a smooth principal fiber bundle $\mathcal{G}$ with structure group P, equipped by a smooth one–form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ satisfying

(1) $\omega(\zeta_Z)(u) = Z$ for all $u \in \mathcal{G}$ and fundamental fields ζ_Z, $Z \in \mathfrak{p}$
(2) $(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega$ for all $b \in P$
(3) $\omega|_{T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$.

In particular, each $X \in \mathfrak{g}$ defines the *constant vector field* $\omega^{-1}(X)$ defined by $\omega(\omega^{-1}(X)(u)) = X$, $u \in \mathcal{G}$. The one–forms with properties (1)–(3) are called (smooth) *Cartan connections*, cf. [28]. The homogeneous space $G \to G/P$, together with the left Maurer–Cartan form is the *flat model* of the geometries of type G/P.

The morphisms between parabolic geometries $(\mathcal{G}, \omega)$ and $(\mathcal{G}', \omega')$ are principal fiber bundle morphisms φ (over the identity on P) which preserve the Cartan connections, i.e. $\varphi : \mathcal{G} \to \mathcal{G}'$ and $\varphi^*\omega' = \omega$.

The structure equations define the principal obstruction against the local flatness, the horizontal smooth form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$ called the *curvature* of the Cartan connection ω:

$$d\omega + \frac{1}{2}[\omega, \omega] = K.$$

The *curvature function* $\kappa : \mathcal{G} \to \wedge^2 \mathfrak{g}_-^* \otimes \mathfrak{g}$ is then defined by means of the parallelism

$$\kappa(u)(X,Y) = K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) = [X,Y] - \omega(u)([\omega^{-1}(X), \omega^{-1}(Y)]).$$

In particular, the curvature function is valued in the cochains for the second cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$. Moreover, there are two ways how to split κ. We may consider the target components κ_i according to the values in $\mathfrak{g}_i$. The whole $\mathfrak{g}_-$–component κ_- is called the *torsion* of the Cartan connection ω. The other possibility is to consider the homogeneity of the two forms $\kappa(u)$.

Since we deal with semisimple algebras only, there is the adjoint codifferential ∂^* to the Lie algebra cohomology differential ∂, see e.g. [24]. Consequently, there is the Hodge theory on the cochains which allows to deal very effectively with the curvatures. In particular, we may use several restrictions on the values of the curvature which turn out to be quite useful: The parabolic geometry $(\mathcal{G},\omega)$ with the curvature function κ is called *flat* if $\kappa=0$, *torsion–free* if $\kappa_-=0$, *normal* if $\partial^*\circ\kappa=0$, and *regular* if normal and $\kappa^{(j)}=0$ for all $j\le 0$. In particular, the normality ensures that the curvature is in certain sense minimal and is governed by the Lie algebra cohomology $H^2(\mathfrak{g}_-,\mathfrak{g})$ in a nice way, see [33, 8] for more details.

For each parabolic geometry $(\mathcal{G},\omega)$ over M, the filtration of $\mathfrak{g}$ by the $\mathfrak{p}$–submodules $\mathfrak{g}^j$ is transferred to the right invariant filtration $T^j\mathcal{G}$ on the tangent space $T\mathcal{G}$ by the parallelism ω. The tangent projection $Tp:T\mathcal{G}\to TM$ then provides the filtration

$$TM=T^{-k}M\supset T^{-k+1}M\supset\cdots\supset T^{-1}M$$

of the tangent space of the underlying manifold M. Moreover, the structure group of the associated graded tangent space

$$\operatorname{Gr}TM=(T^{-k}M/T^{-k+1}M)\oplus\cdots\oplus(T^{-2}M/T^{-1}M)\oplus T^{-1}M$$

reduces automatically to G_0 since $\mathcal{G}_0=\mathcal{G}/P_+$ clearly plays the role of its frame bundle. With some further simple conditions imposed, we talk about *regular infinitesimal flag structures of type* $\mathfrak{g}/\mathfrak{p}$ and then a universal construction recovers both the canonical Cartan bundle and the canonical normal Cartan connection, see [8, 9] for details, [27] for some new applications.

3. **Natural bundles and operators.** Each P–module $\mathbb{V}$ defines for all parabolic geometries $(\mathcal{G}\to M,\omega)$ of type G/P over a manifold M the associated bundle $VM=\mathcal{G}\times_P\mathbb{V}$ over M. In fact, this is a functorial construction which may be restricted to all subcategories of parabolic geometries mentioned above and we call such bundles natural (vector) bundles. Similarly, we may treat bundles associated to any representation $P\to\operatorname{Diff}(\mathbb{S})$ on a manifold $\mathbb{S}$, the standard fiber for $SM=\mathcal{G}\times_P\mathbb{S}$. The class of all natural (vector) bundles defined by G–modules $\mathbb{W}$ is called *tractor bundles*, see [2, 7] for historical remarks. The remarkable feature of tractor bundles is that the extension of the Cartan connection ω to the principal connection form $\tilde{\omega}$ on the extended Cartan bundle $\tilde{\mathcal{G}}$ induces on them the canonical linear connections, see [22, 7, 9] for much more information. The distinguished bundles $\mathcal{E}^A$, $\mathcal{E}_{A'}$, their duals $\mathcal{E}_A$, $\mathcal{E}^{A'}$, and invariant components of their tensor products are called, by analogy to the conformal geometry, the (generalized) *spinor bundles*.

The *natural operators*, for parabolic geometries of a fixed type G/P, are systems of differential operators $D_M:\Gamma(VM)\to\Gamma(V'M)$ between sections of the natural bundles, which intertwine the induced actions of the morphisms. Of course, we obtain exactly the (translational) invariant operators $D_{G/P}$ between the homogeneous bundles on the flat model. At the same time, each invariant operator $D_{G/P}$ extends uniquely to a natural operator on the full subcategory of locally flat parabolic geometries.

As we shall discuss below, the natural operators may be expressed by means of a universal operation, the *invariant differential* defined as the derivative of functions on $\mathcal{G}$ with respect to the constant vector fields $\omega^{-1}(X)$ on $\mathcal{G}$. Thus, for each section s of a natural bundle VM, i.e. $s\in C^\infty(\mathcal{G},\mathbb{V})^P$, there is the differential $\nabla^\omega s\in C^\infty(\mathcal{G},\mathfrak{g}_-^*\otimes\mathbb{V})$. Though $\nabla^\omega s$ is not P–equivariant, as a rule, it provides an extremely useful tool for the study of natural operators.

4. **Semi–holonomic jet–modules.** While the standard jet prolongations of homogeneous vector bundles are again homogeneous vector bundles corresponding to certain jet–modules, this construction does not extend out of locally flat geometries, i.e. those without curvature. On the other hand, the invariant differential yields such a construction for one–jets and so we can go on to all orders with

the semi–holonomic prolongations. This is the core of our approach to invariant operators in [10, 11] and a straightforward iterative construction of suitable homomorphisms between semi–holonomic jet–modules provides all the distinguished operators in the BGG–sequences.

Let us consider a representation $\mathbb{V}$ of P, the corresponding homogeneous bundle $V(G/P) = G \times_P \mathbb{V}$ and its first jet prolongation $J^1(V(G/P)) \to G/P$. This is again a homogeneous bundle, and the corresponding action of P on its standard fiber

$$\mathcal{J}^1(\mathbb{V}) := J^1(V(G/P))_o = \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V})$$

is defined by means of the action of fundamental vector fields on the equivariant functions $s \in C^\infty(G, \mathbb{V})^P$. The formula for the action of $Z \in \mathfrak{p}_+$ on elements of $\mathcal{J}^1(\mathbb{V})$ viewed as pairs (v, φ), where $v \in \mathbb{V}$ and φ is a linear map from $\mathfrak{g}_-$ to $\mathbb{V}$, is given by

$$Z\cdot(v,\varphi) = (Z\cdot v, X \mapsto Z\cdot(\varphi(X)) - \varphi(\mathrm{ad}_-(Z)(X)) + \mathrm{ad}_{\mathfrak{p}}(Z)(X)\cdot v),$$

i.e. we get the tensorial action plus one additional term mapping the value–part to the derivative–part.

By iteration, we obtain the semi–holonomic jet modules

$$\mathcal{J}^k\mathbb{V} = \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V}) \oplus \cdots \oplus (\otimes^k \mathfrak{g}_-^* \otimes \mathbb{V})$$

with the appropriate action of P. Now, the semi–holonomic jet prolongations of natural bundles with standard fiber $\mathbb{V}$ turn out to be natural bundles corresponding to P–modules $\mathcal{J}^k\mathbb{V}$ and, moreover, the iterated invariant differential provides the natural operator $s \mapsto (s, \nabla^\omega s, \dots, (\nabla^\omega)^k s)$ valued in the semi–holonomic jet prolongation. This is just an explicit version of the embedding of holonomic jets into the semi–holonomic ones, which restricts to the the usual embedding in the flat case but involves the curvatures in general.

A straightforward consequence of the naturality of the semi–holonomic jet prolongation is that each P–module homomorphism

$$\Phi : \mathcal{J}^k\mathbb{V} \to \mathbb{W}$$

gives rise to a natural differential operator between the corresponding natural bundles. Of course, the invariant differential provides explicit formulae for such operators.

5. **The BGG–resolutions and the standard natural operators.** In general, the latter correspondence is not bijective as in the case of homogeneous bundles and standard jets, i.e. a non–zero homomorphism may lead to a trivial operator and there are natural operators which are not achieved in this way, cf. [23, 14]. However, if we manage to express an invariant operator between homogeneous vector bundles by means of a homomorphism Φ of semi–holonomic jet modules (instead of the holonomic ones as in the classical representation theoretical approach), then the symbol of the resulting operator will remain always the same. In particular, the whole operator is of the same order as in the flat case and we talk about the *curved analogue* of the given invariant operator on the flat model.

As well known, the invariant differential operators between the homogeneous vector bundles are described in terms of the homomorphisms between the (generalized) Verma modules (just dualization and the so called Frobenius reciprocity principle). In the case of the Borel subgroup $P \subset G$, all these operators are compositions of some basic ones, which form the so called Bernstein–Gelfand–Gelfand resolutions of constant sheaves corresponding to G–modules. A remarkable feature of more general parabolics appears: there are again some basic operators establishing the (generalized) Bernstein-Gelfand–Gelfand resolutions, the *standard operators*, but apart from these and their non–zero compositions which all come in certain sense from the Borel case, there are also the so–called *non–standard operators.* The latter operators appear in situations where the prospective compositions of the standard ones vanish and they have been studied in quite detail in conformal Riemannian geometries.

While *all* standard invariant operators admit a distinguished curved analogue, first constructed in full generality in [11], the problem of the existence of curved analogues of the non–standard ones still remains open, partly even in the conformal case, cf. [23, 14].

6. **Back to quaternionic geometries – local twistor calculus.** Let us fix one of the real forms of $G^{\mathbb{C}} = SL(p+q,\mathbb{C})$ mentioned in Section 1, for instance the quaternionic one. We shall follow the notation established in [22] which extends the standard conventions used in the twistor theory.

The standard representation of G on $\mathbb{C}^{p+q}$ yields the filtered P–modules (recall the notation used for the basic spinor bundles $\mathcal{E}^A$, $\mathcal{E}_{A'}$ in Section 1 and write V^A and $V_{A'}$, or V_A, $V^{A'}$ for the corresponding P–modules and their duals)

$$V^\alpha = V^A + V^{A'}, \quad V_\alpha = V_{A'} + V_A.$$

This notational convention means that the 'right ends' in the formal sums are submodules while the 'left ends' are quotients. These filtrations determine filtrations of the corresponding special case of tractor bundles, called *twistor bundles*

$$\mathcal{E}^\alpha = \mathcal{E}^A + \mathcal{E}^{A'}, \quad \mathcal{E}_\alpha = \mathcal{E}_{A'} + \mathcal{E}_A.$$

We also write $X^\alpha_{A'}$ for the canonical section of $\mathcal{E}^\alpha_{A'}$ which gives the injecting morphism $\mathcal{E}^{A'} \to \mathcal{E}^\alpha$ via $v^{A'} \mapsto X^\alpha_{A'}v^{A'}$, etc. More generally, tensor products of the twistor bundles and scalar densities (the latter ones are no more coming from G–modules) are called the *weighted twistor bundles.* We denote them by $\mathcal{E}^{\alpha\ldots\beta}_{\gamma\ldots\nu}[w]$ where w is the weight coming from the densities. Let us notice that the irreducible components at the right hand ends of the filtrations of weighted twistor bundles are invariant subbundles while those at the left hand ends are irreducible quotients. In particular, all irreducible natural bundles are easily accommodated as both subbundles and quotients of the weighted twistor bundles.

The invariant differential provides the formulae for the canonical linear connections on these bundles. For example on $\mathcal{E}^\alpha$ and $\mathcal{E}_\alpha$

$$\nabla^{P'}_A \begin{pmatrix} v^B \\ v^{B'} \end{pmatrix} = \begin{pmatrix} \nabla^{P'}_A v^B + \delta^B_A v^{P'} \\ \nabla^{P'}_A v^{B'} - \mathsf{P}^{P'B'}_{AB} v^B \end{pmatrix}$$
$$\nabla^{P'}_A (u_B \;\; u_{B'}) = (\nabla^{P'}_A u_B + \mathsf{P}^{P'B'}_{AB} u_{B'} \;\; \nabla^{P'}_A u_{B'} - \delta^{P'}_{B'} u_A),$$

where the nablas and P on the right hand side indicate the usual spinor connection and the so called Rho–tensor determined by a choice of (generalized) Weyl structure, i.e. a choice of a reduction of the structure group P to its reductive (or even semi–simple) part. Thus, we have extended the well known formulae from the four–dimensional conformal Riemannian geometry. Of course, the formulae are independent of any of these choices.

The definition of very important objects in our local twistor calculus, the *D-operators*, is based on the observation that the spinor–twistor object

$$D^{A'}_\beta f := (\nabla^{A'}_B f \;\; w\delta^{A'}_{B'} f)$$

is invariant for all weighted twistors $f \in \Gamma(\mathcal{E})[w]$. We regard this as an injecting part of the invariant twistor object $D^\alpha_\beta f := X^\alpha_{A'} D^{A'}_\beta f$. More generally, the operator D^α_β is well defined and invariant on sections of the weighted twistor bundles $\mathcal{E}^{\rho\cdots\mu}_{\alpha\cdots\gamma}[w]$ (here we exploit the canonical twistor connection ∇).

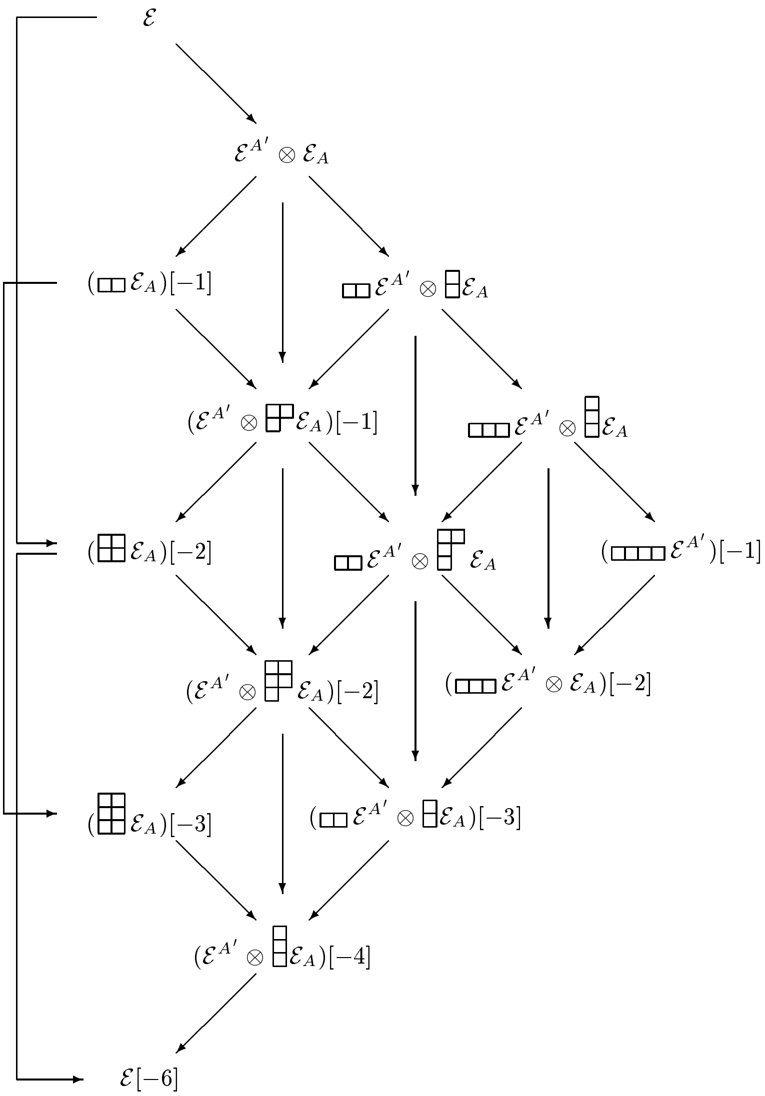

FIGURE 1

The latter construction yields natural operators which admit compositions, but they are highly redundant. The next step is to make their targets smaller and symmetrized enough to kill many contractions. This is essentially the core of the definition of the operators $\mathsf{D}^{\alpha\cdots\delta}_{\rho\cdots\nu}$ as a sort of symmetrized concatenations of D^{α}_{β} in [22].

7. **The non–standard operators.** The general results in [22] tell us that all natural operators can be obtained via the latter operators $\mathsf{D}^{\alpha\cdots\delta}_{\rho\cdots\nu}$ but a first good test of the actual power of this calculus is to try to construct some of the non–standard operators. This task seems to be quite hard and a straightforward use of representation theoretical tools for finding the homomorphisms of semi–holonomic jet–modules has not brought a reasonable understanding yet.

We shall use the Young symmetrizers in order to describe the individual irreducible components of the natural bundles. This notation should be clear from

Figure 1, where the short arrows show the decomposition of the de Rham resolution of (complex) functions into irreducible components, in the special case of 8–dimensional underlying manifolds M.

Now, the long arrows on the left–hand side describe the non–standard invariant operators on the flat model. A quite lengthy verification in [22] reveals that the operators $\mathsf{D}^{\alpha\cdots\delta}_{\rho\cdots\nu}$ give rise to the fourth order operators

$$\Box_{ABCD} : \Big(k\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\, \mathcal{E}_E\Big)[-k] \to \Big(k+2\updownarrow \begin{array}{|c|c|}\hline \ & \ \\ \hline \vdots & \vdots \\ \hline \ & \ \\ \hline \end{array}\, \mathcal{E}_E\Big)[-k-2],$$

for all quaternionic geometries, which are curved analogues of the above mentioned long arrows.

Let us observe that the operator emanating from $\mathcal{E}$ cannot come from a P–module homomorphism on the corresponding semi–holonomic jet modules. Indeed, this would imply the same property for the second power of the Laplacian in the four–dimensional conformal geometry, which is excluded in [14]. We do not know the answer for the rest of them, although this is a very appealing question. Indeed, those operators which are given by the P–module homomorphisms on semi–holonomic jets allow a curved version of the translation principle.

There is a good hope to extend the technique from [14] to all quaternionic geometries.

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