

BERNSTEIN–GELFAND–GELFAND SEQUENCES

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ABSTRACT. This survey follows the lecture presented at the conference “100 years after Sophus Lie”, RIMS Kyoto, December 12, 1999. The aim is to describe the recent geometric treatment of the distinguished complexes of invariant differential operators between homogeneous vector bundles, known under the name Bernstein–Gelfand–Gelfand resolutions, in the realm of the so called parabolic geometries. The basic reference for this paper is [12], the exposition has been influenced essentially by [14, 10].

The talk presents some results of a long time joint effort with Andreas Čap and Vladimír Souček. Further essential influence comes from the recent extensive cooperation with Michael Eastwood, Rod Gover, and Gerd Schmalz.

1. GENERAL BACKGROUND

1.1. **Klein’s geometries.** We shall deal with invariant operators for certain geometries. First we discuss such operators in the cases where the underlying geometry is that of a homogeneous space G/P for some Lie subgroup P in a Lie group G . This leads to problems studied for several decades in representation theory in terms of Verma module homomorphisms. Later on, we pass to the so called *parabolic geometries* and the homogeneous cases play then the rôles of the flat models. Our considerations apply to both smooth and holomorphic categories and we shall not distinguish these two cases explicitly. (The main difference is the local existence of the holomorphic sections.) On the other hand, we shall deal with complex representations only in order not to have to distinguish between many real forms of the complex groups.

In order to enjoy the general features in terms of explicit examples, we shall pay special attention to several flat models: four different geometries on the three–sphere (projective, conformal Riemannian, projective contact, and CR–contact), accomplished with the conformal Riemannian four–sphere. In the two projective cases, the sphere is considered as the space of the rays emanating from the origin, but with different group actions: $SL(4, \mathbb{R})$ and $Sp(4, \mathbb{R})$, respectively. The conformal spheres are regarded as projective quadrics in \mathbb{R}^{n+2} , $n = 3, 4$, and the corresponding symmetry groups are $O(n+1, 1)$. The CR–sphere is understood as the non–degenerate real quadric in \mathbb{C}^2 , and the symmetry group is $SU(2, 1)$. The isotropy groups of distinguished fixed points form the subgroups P in all cases.

For each Kleinian geometry G/P , there are the homogeneous vector bundles $\mathcal{E}(G/P)$ corresponding to P –modules \mathbb{E} . More explicitly, we consider $G \rightarrow G/P$ as the principal P –bundles and $\mathcal{E}(G/P)$ is the associated vector bundle $G \times_P \mathbb{E}$. This is a functorial construction and, in particular, the left action of G on the homogeneous space induces the action on the (sheaf of local) sections of $\mathcal{E}(G/P)$. Moreover, each (local) section $s : G/P \rightarrow \mathcal{E}(G/P)$ is expressed (in its *frame form*) as a P –equivariant function $G \rightarrow \mathbb{E}$ and, in this picture, the action of G on sections is given by the left shifts: $g \cdot s = s \circ \ell_{g^{-1}}$. The *invariant differential operators* are those operators between sections of homogeneous bundles which intertwine these actions.

1.2. Cartan's geometries. The curved version of these considerations was suggested by Cartan in connection with his exterior calculus. In this approach, the main object describing all features of the Kleinian geometry is the Maurer–Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ which is right-invariant (with respect to the whole G), reproduces the fundamental vector fields (even all left invariant fields), and offers an absolute parallelism (with vanishing curvature — the Maurer–Cartan equations). The curved geometry of type G/P (*generalized space* in Cartan's terminology) is then given by a principal fiber bundle $\mathcal{G} \rightarrow M$ with structure group P , and absolute parallelism $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ which is again right-invariant (with respect to P), and reproduces the fundamental vector fields. The structure equations

$$d\omega = -\frac{1}{2}[\omega, \omega] + K$$

define then the horizontal two-form $K \in \Omega^2(\mathcal{G}, \mathfrak{g})$, the *curvature*. By means of the absolute parallelism, the curvature is given by the *curvature function* $\kappa : \mathcal{G} \rightarrow \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$.

We talk about *Cartan geometries* (\mathcal{G}, ω) , and *Cartan connections* ω . Morphisms $\varphi : (\mathcal{G}, \omega) \rightarrow (\mathcal{G}', \omega')$ between Cartan geometries are those principal bundle morphisms (over identity on P) which preserve the Cartan connections, i.e. $\varphi^*\omega' = \omega$. In particular, the automorphisms of the flat model are just the left shifts by elements of G , cf. [25], Theorem 3.5.2.

Each P -module \mathbb{E} defines a functor on the category of Cartan geometries of type G/P , $(\mathcal{G} \rightarrow M, \omega) \mapsto \mathcal{G} \times_P \mathbb{E} =: \mathcal{E}(M)$ with the obvious action of morphisms. These functors are called *natural vector bundles* and the *invariant operators* are those systems of differential operators $D_{\mathcal{G}} : \Gamma(\mathcal{E}(M)) \rightarrow \Gamma(\mathcal{F}(M))$ which intertwine the action of morphisms.

The Cartan geometry (\mathcal{G}, ω) is locally isomorphic to its *flat model* G/P if and only if the curvature K vanishes. In particular, there is the full subcategory of locally flat Cartan geometries of type G/P .

A readable modern introduction to this approach to differential geometry is offered in [25].

2. BERNSTEIN–GELFAND–GELFAND RESOLUTIONS

2.1. $|k|$ -graded Lie algebras. In the rest of the paper, we shall assume that G is a semi-simple Lie group (real or complex) and P its parabolic subgroup. In particular this implies that \mathfrak{g} comes equipped by the grading

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k,$$

$k > 0$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$, the reductive part of \mathfrak{p} is \mathfrak{g}_0 and the nilpotent part is $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$. We also write \mathfrak{g}_- for the negative components and we identify this space with the P -module $\mathfrak{g}/\mathfrak{p}$. We say that \mathfrak{g} is *$|k|$ -graded*.

The Killing form provides the isomorphisms $\mathfrak{g}_i^* \simeq \mathfrak{g}_{-i}$ for all components of the $|k|$ -graded semisimple Lie algebra \mathfrak{g} , $i = -k, \dots, k$. In particular, its restrictions to the center \mathfrak{z} and the semisimple part \mathfrak{g}_0^{ss} of \mathfrak{g}_0 are non-degenerate. Now, for each Lie group G with the $|k|$ -graded Lie algebra \mathfrak{g} , there is the closed subgroup $P \subset G$ of all elements whose adjoint actions leave the \mathfrak{p} -submodules $\mathfrak{g}^j = \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k$ invariant, $j = -k, \dots, k$. The Lie algebra of P is just \mathfrak{p} and there is the subgroup $G_0 \subset P$ of elements whose adjoint action leaves invariant the grading by \mathfrak{g}_0 -modules \mathfrak{g}_i , $i = -k, \dots, k$. This is the reductive part of the parabolic Lie subgroup P , with Lie algebra \mathfrak{g}_0 . We also define subgroups $P_+^j = \exp(\mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_k)$, $j = 1, \dots, k$, and we write P_+ instead of P_+^1 . Obviously $P/P_+ = G_0$ and P_+ is nilpotent. Thus P is the semisimple product of G_0 and the nilpotent part P_+ . More explicitly (cf. [8,

Proposition 2.10], or [27, 29]), each element $g \in P$ is expressed in the unique way as $g = g_0 \exp Z_1 \exp Z_2 \dots \exp Z_k$, with $g_0 \in G_0$ and $Z_i \in \mathfrak{g}_i$, $i = 1, \dots, k$.

2.2. Jet-modules. In this section, we shall deal with operators between homogeneous vector bundles and we shall write briefly \mathcal{F} instead of $\mathcal{F}(G/P)$, for any P -module \mathbb{F} . The next step in our exposition consists in a few standard observations.

First, each k th order differential operator is given as a mapping $J^k E \rightarrow \mathcal{F}$ on the jet prolongation and the action of G on sections of \mathcal{F} induces an action on $J^k E(G/P)$. Moreover, there is the obvious identification $J^k \mathcal{E} \simeq G \times_P J^k \mathbb{E}$ where the P -module $J^k \mathbb{E}$ is the fiber over the origin of G/P with the induced action of P . Thus, the invariant operators are given by P -module homomorphisms $J^k \mathbb{E} \rightarrow \mathbb{F}$.

Second, seeking for P -module homomorphisms $J^k \mathbb{E} \rightarrow \mathbb{F}$ is equivalent to seeking for the dual homomorphisms $\mathbb{F}^* \rightarrow (J^k \mathbb{E})^*$, or better $\mathbb{F}^* \rightarrow (J^\infty \mathbb{E})^*$ where the latter module is the inverse limit of the k th order ones. For irreducible P -modules, these inverse limits are (\mathfrak{g}, P) -modules known (in representation theory) under the name generalized Verma modules. These modules are highest weight modules with the highest weights contained in \mathbb{F} . Thus we obtain the so called Frobenius reciprocity theorem claiming the bijective correspondence

$$\left\{ \begin{array}{c} P\text{-module homomorphisms} \\ J^k \mathbb{E} \rightarrow \mathbb{F} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{generalized Verma} \\ \text{module homomorphisms} \\ (J^\infty \mathbb{F})^* \rightarrow (J^\infty \mathbb{E})^* \end{array} \right\}.$$

2.3. Verma module homomorphisms. The homomorphisms of Verma modules have been studied for many years. The first breakthrough was achieved in [5]. It turned out, that for Borel subgroups P all homomorphisms group nicely into equal patterns, starting by a G -module \mathbb{V} and being described by suitable combinatorial properties of the Weyl group of \mathfrak{g} . In view of the Kostant's Bott-Borel-Weil theorem, we may state the final result roughly as follows: *Each P -module with a regular central character (i.e. sharing the central character with some G -module \mathbb{V}^*) appears in the Lie algebra cohomology $H^*(\mathfrak{p}_+, \mathbb{V}^*)$ with multiplicity one and all Verma module homomorphisms are then included in the pattern (including non-zero compositions)*

$$\mathbb{V}^* \longleftarrow H^0(\mathfrak{p}_+, \mathbb{V}^*) \longleftarrow \dots \longleftarrow H^{\max}(\mathfrak{p}_+, \mathbb{V}^*).$$

Moreover, the sequence always forms a complex which is called the Bernstein-Gelfand-Gelfand resolution of \mathbb{V}^* (shortened to BGG in what follows).

Let us remark that the cohomologies are always completely reducible and, of course, the non-zero compositions may appear only in the picture of the individual components of the horizontal arrows between the irreducible components of the cohomologies (and they have to cancel properly each other in the sum).

In terms of the homogeneous vector bundles and invariant operators, we obtain the resolution of the constant sheaf corresponding to \mathbb{V} :

$$(1) \quad \mathbb{V} \longrightarrow \Gamma(\mathcal{H}_{\mathbb{V}}^0) \longrightarrow \dots \longrightarrow \Gamma(\mathcal{H}_{\mathbb{V}}^{\max})$$

where $\mathcal{H}_{\mathbb{V}}^j$ are the homogeneous bundles corresponding to the P -modules $\mathbb{H}_{\mathbb{V}}^j = H^j(\mathfrak{g}/\mathfrak{p}, \mathbb{V})$. This resolution is called again the BGG resolution of \mathbb{V} .

Similar problems for arbitrary G -modules and parabolic subgroups P have been studied carefully in representation theory for many years, cf. [23] and the references therein. There are two types of homomorphisms in general, those coming as direct images of the Borel case, which create again resolutions of the constant sheaves, but also new ones appearing on places where the direct images vanish but non-zero

homomorphisms still exist. The former ones are called *standard homomorphisms*, the latter ones *non-standard*.

The general theorem due to [23] claims that all standard operators appear again in patterns (1), while the non-standard ones appear as additional arrows in the same patterns. The explicit form of these resolutions can be encoded nicely in terms of highest weights of the modules and Dynkin diagrams. For the relevant recipes, including the computation of the irreducible components in the cohomologies, see [3]. An algorithm for the determination of all non-zero homomorphisms is available in [6] (and the Brian Boe's computer implementation of this algorithm is very useful).

The highest weights of all complex irreducible representations of $\mathfrak{p} \subset \mathfrak{g}$ are described as integral linear combinations of the fundamental weights for \mathfrak{g} and their coefficients can be depicted as labels associated to the corresponding nodes in the Dynkin diagrams. The choice of the parabolic subalgebra is described by crossing out those nodes, which correspond to simple negative roots which are not in \mathfrak{p} . Finally, \mathfrak{p} -dominant weights are given by those labeled diagrams with non-negative coefficients over the uncrossed nodes.

2.4. Examples. Let us illustrate this notation on the adjoint representations of the symmetry groups of five geometries mentioned in the introductory part (projective 3-sphere, conformal 3-sphere, projective contact 3-sphere, CR-contact 3-sphere, and conformal 4-sphere). The adjoint representations \mathfrak{g} , viewed as P -modules, are never irreducible, and their highest weights generate the only irreducible components \mathfrak{g}_k (in the same order as above):

$$\begin{array}{ccccc} \overset{1}{\times} \text{---} \overset{0}{\bullet} \text{---} \overset{1}{\bullet} & \overset{0}{\rightleftarrows} \overset{2}{\bullet} & \overset{2}{\times} \text{---} \overset{0}{\bullet} & \overset{1}{\times} \text{---} \overset{1}{\times} & \overset{1}{\bullet} \text{---} \overset{0}{\times} \text{---} \overset{1}{\bullet} \end{array}$$

For the sake of simplicity, the standard notational convention for the homogeneous bundles in the BGG-resolutions uses the dual modules (i.e. the highest weights for the corresponding Verma modules). A straightforward computation yields for all general complex \mathfrak{g} -modules \mathbb{V} (i.e. arbitrary integral coefficients $a, b, c \geq 0$) the sequences of invariant operators which are indicated by ∇^j where j refers to the order:

3-dimensional projective:

$$(2) \quad \overset{c}{\times} \text{---} \overset{b}{\bullet} \text{---} \overset{a}{\bullet} \xrightarrow{\nabla^{(a+1)}} \overset{-a-2}{\times} \text{---} \overset{c}{\bullet} \text{---} \overset{a+b+1}{\bullet} \xrightarrow{\nabla^{(b+1)}} \overset{-a-b-3}{\times} \text{---} \overset{b+c+1}{\bullet} \text{---} \overset{b}{\bullet} \xrightarrow{\nabla^{(c+1)}} \overset{a-b-c-4}{\times} \text{---} \overset{a}{\bullet} \text{---} \overset{b}{\bullet}$$

3-dimensional conformal Riemannian:

$$(3) \quad \overset{a}{\rightleftarrows} \overset{b}{\bullet} \xrightarrow{\nabla^{(a+1)}} \overset{-a-2}{\rightleftarrows} \overset{2a+b+2}{\bullet} \xrightarrow{\nabla^{(b+1)}} \overset{-a-b-3}{\rightleftarrows} \overset{2a+b+2}{\bullet} \xrightarrow{\nabla^{(a+1)}} \overset{-a-b-3}{\rightleftarrows} \overset{b}{\bullet}$$

3-dimensional projective contact:

$$(4) \quad \overset{a}{\leftarrow} \overset{b}{\bullet} \xrightarrow{\nabla^{(a+1)}} \overset{-a-2}{\leftarrow} \overset{a+b+1}{\bullet} \xrightarrow{\nabla^{(2b+2)}} \overset{-a-2b-4}{\leftarrow} \overset{a+b+1}{\bullet} \xrightarrow{\nabla^{(a+1)}} \overset{-a-2b-4}{\leftarrow} \overset{b}{\bullet}$$

3-dimensional CR-contact:

$$(5) \quad \begin{array}{ccccc} & & \overset{-a-2}{\times} \text{---} \overset{b}{\times} & & \\ & & \overset{a+b+1}{\times} & \xrightarrow{\nabla^{(a+b+2)}} & \overset{-a-b-3}{\times} \\ \nabla^{(a+1)} \nearrow & & & & \searrow \nabla^{(b+1)} \\ \overset{a}{\times} \text{---} \overset{b}{\times} & & & & \overset{-b-2}{\times} \text{---} \overset{-a-2}{\times} \\ \searrow \nabla^{(b+1)} & & \nabla^{(2a+2)} \nearrow & & \nearrow \nabla^{(a+1)} \\ & & \overset{a+b+1}{\times} \text{---} \overset{-b-2}{\times} & \xrightarrow{\nabla^{(2b+2)}} & \overset{-a-b-3}{\times} \text{---} \overset{a}{\times} \\ & & \xrightarrow{\nabla^{(a+b+1)}} & & \\ & & \overset{-a-b-3}{\times} \text{---} \overset{a}{\times} & & \end{array}$$

4-dimensional conformal:

$$(6) \quad \begin{array}{ccccc} & & \begin{array}{c} b \quad a+b+c+2 \\ \times \\ -a-b-3 \end{array} & & \\ & \nearrow \nabla^{(a+1)} & & \nwarrow \nabla^{(c+1)} & \\ \begin{array}{c} a \quad c \\ \times \\ b \end{array} & \xrightarrow{\nabla^{(b+1)}} & \begin{array}{c} a+b+1 \quad b+c+1 \\ \times \\ -b-2 \end{array} & & \begin{array}{c} b+c+1 \quad a+b+1 \\ \times \\ -a-b-c-4 \end{array} \xrightarrow{\nabla^{(b+1)}} \begin{array}{c} c \quad a \\ \times \\ -a-b-c-4 \end{array} \\ & \searrow \nabla^{(c+1)} & & \nearrow \nabla^{(a+1)} & \\ & & \begin{array}{c} a+b+c+2 \quad b \\ \times \\ -b-c-3 \end{array} & & \end{array}$$

2.5. de Rham complexes. The simplest examples are the trivial representations, i.e. the choice $a = b = c = 0$. For the $|1|$ -graded algebras, these are exactly the (complexified) de Rham complexes, see (2), (3), (6). Surprisingly enough, the remaining two sequences include bundles of lower dimensions. Indeed, instead of the standard one-forms the second column contains the dual space to the (complexified) contact distribution (which splits in the CR-case into the holomorphic and anti-holomorphic parts), etc. Another surprising fact is that the order of the operators is not always one. More generally, there is the so called twisted de Rham sequence corresponding to a G -module \mathbb{V} and the striking feature of the BGG-resolution is that they compute the same cohomology as the twisted de Rham complexes, but they have much smaller dimensions.

We shall not pay any attention to the so called singular infinitesimal characters and the half-integral weights, although they involve many important operators, see e.g. [15] for a complete discussion in the special case of the conformal Riemannian geometries.

3. PARABOLIC GEOMETRIES

Even for the (curved) conformal Riemannian and projective geometries, the general discussion on the invariant operators occupies mathematicians for many decades. Since the beginning of the 20th century, a few similar geometrical structures were known to fit within the framework of the Cartan geometries, i.e. they were shown to allow a canonical Cartan connection under suitable normalizations. See e.g. Kobayashi's treatment of groups of geometric transformations in [21], the generalization of Cartan's description of 3-dimensional CR-geometry to all non-degenerate CR-structures of hypersurface type due to [26, 13], and the pioneering series of papers by Tanaka, cf. [27] and the references therein, as well as [29, 24, 8]. The name *parabolic geometry* has been commonly adopted for the general class of all Cartan geometries with G semisimple and P parabolic. There is also the closely related parabolic invariants program initiated by Fefferman, [16], see also [17, 4].

Tanaka's motivation came from pfaffian systems of PDE's, while the relation to twistor theory renewed the interest in a good calculus for such geometries, with the aim to improve the techniques in conformal geometry and to extend them to other geometries. See e.g. [3] for links to twistor theory and representation theory, [28] for classical methods in conformal geometry, and [1, 2, 4, 19] and references therein, for generalizations. One of the main objectives was the construction of invariant differential operators.

Motivated by the remarkable (but quite unclear) papers [1, 2], the systematic combination of Lie algebraic tools with the frame bundle approach was developed in [11] and the first strong applications for all parabolic geometries were given in [12]. The main aim of this lecture is to describe roughly the results of the latter paper. For further essential development of both the abstract calculus and the differential geometry in the general setting see [7, 9], and in particular [10].

3.1. Semi-holonomic jet-modules. The algebraic core of our approach are the semi-holonomic jet-modules. While the standard jet prolongations of homogeneous vector bundles are again homogeneous vector bundles corresponding to certain jet-modules, this construction does not extend out of locally flat geometries, i.e. those without curvature. On the other hand, the defining absolute parallelism allows such a construction for one-jets and a simple check shows that we can proceed to all orders with the semi-holonomic prolongations.

Let us consider a representation \mathbb{E} of P , the corresponding homogeneous bundle $\mathcal{E}(G/P) = G \times_P \mathbb{E}$ and its first jet prolongation $J^1(\mathcal{E}(G/P)) \rightarrow G/P$. The action of P on its standard fiber

$$\mathcal{J}^1(\mathbb{E}) := J^1(\mathcal{E}(G/P))_o = \mathbb{E} \oplus (\mathfrak{g}_-^* \otimes \mathbb{E})$$

is defined by means of the action of fundamental vector fields on the equivariant functions $s \in C^\infty(G, \mathbb{E})^P$. The formula for the action of $Z \in \mathfrak{p}_+$ on elements of $\mathcal{J}^1(\mathbb{E})$ viewed as pairs (v, φ) , where $v \in \mathbb{E}$ and φ is a linear map from \mathfrak{g}_- to \mathbb{E} , is given by

$$Z \cdot (v, \varphi) = (Z \cdot v, X \mapsto Z \cdot (\varphi(X)) - \varphi(\text{ad}_-(Z)(X)) + \text{ad}_{\mathfrak{p}}(Z)(X) \cdot v),$$

i.e. we get the tensorial action plus one additional term mapping the value-part to the derivative-part.

By iteration, we obtain the semi-holonomic jet modules

$$\bar{J}^k \mathbb{E} = \mathbb{E} \oplus (\mathfrak{g}_-^* \otimes \mathbb{E}) \oplus \cdots \oplus (\otimes^k \mathfrak{g}_-^* \otimes \mathbb{E})$$

with the appropriate action of P as the equalizers of the natural projections

$$J^1(\bar{J}^{k-1} \mathbb{E}) \rightarrow \bar{J}^{k-1} \mathbb{E} \subset J^1(\bar{J}^{k-2} \mathbb{E}).$$

Now, the semi-holonomic jet prolongations of natural bundles with standard fiber \mathbb{E} turn out to be natural bundles corresponding to P -modules $\bar{J}^k \mathbb{E}$.

This has a striking consequence: *P -module homomorphisms $\Psi : \bar{J}^k \mathbb{E} \rightarrow \mathbb{F}$ give rise to invariant operators $D : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$.*

3.2. Setting of the problem. Still two essential questions are obviously left. First, how to recognize the non-zero operators? Second, are all invariant operators of this type? Unfortunately, the answer to the second question is *no*, while the first one provides an unpleasant challenge. We call the operators which come this way *strongly invariant* and the conformally invariant square of the Laplacian on function on four-dimensional conformal Riemannian manifolds (the so called Paneitz operator) is an example of an invariant operator which is not strongly invariant, cf. [20], or see [15] for more examples.

On the other hand, each invariant operator on the locally flat geometries has an invariant symbol. This is a tensor and thus it exists as an invariant on all curved geometries as well. Thus we have a simple problem to deal with: *Given invariant operator $D_{G/P}$ between homogeneous bundles, is there an invariant operator on all parabolic geometries which restricts to $D_{G/P}$?* We shall deal with this problem in the rest of the paper and we call such operators *curved versions* of the invariant operators on G/P . The first observation to make is that if an invariant operator on G/P is given by a homomorphism of the semi-holonomic jet-modules, then its symbol (i.e. the symmetrized part of the restriction to the highest order component) does not vanish and so the resulting strongly invariant operator definitely does not vanish too. Moreover, this operator clearly is the curved version of its restriction to G/P .

3.3. Remarks. Let us also notice that the defining Cartan connection ω determines a well defined differential operator on functions on \mathcal{G} . First, recall that ω is an absolute parallelism and so it defines the *constant vector fields* $\omega^{-1}(X)$ on \mathcal{G} for all $X \in \mathfrak{g}$, $\omega(\omega^{-1}(X)(u)) = X$, for all $u \in \mathcal{G}$. In particular, $\omega^{-1}(Z)$ is the fundamental vector field if $Z \in \mathfrak{p}$. The constant fields $\omega^{-1}(X)$ with $X \in \mathfrak{g}_-$ are called *horizontal*.

Now, let us consider any natural vector bundle $EM = \mathcal{G} \times_P \mathbb{E}$. Its sections are P -equivariant functions $s : \mathcal{G} \rightarrow \mathbb{E}$ and the Lie derivative of functions with respect to the constant horizontal vector fields defines the *invariant derivative*

$$\begin{aligned} \nabla^\omega : C^\infty(\mathcal{G}, \mathbb{E}) &\rightarrow C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{E}) \\ \nabla^\omega s(u)(X) &= \mathcal{L}_{\omega^{-1}(X)} s(u). \end{aligned}$$

We also write $\nabla_X^\omega s$ for values with the fixed argument $X \in \mathfrak{g}_-$.

The invariant derivative is a helpful substitute for the Levi-Civita connections in Riemannian geometry, but there is a problem: it does not produce P -equivariant functions even if restricted to equivariant $s \in C^\infty(\mathcal{G}, \mathbb{E})^P$. One good way how to avoid this drawback is to extend the derivative to all constant fields, i.e. to consider $\nabla : C^\infty(\mathcal{G}, \mathbb{E}) \rightarrow C^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{E})$ which preserves the equivariance. This is a helpful approach in the so called twistor and tractor calculus, see e.g. [7, 10]. An easy computation reveals also the (generalized) Ricci and Bianchi identities and a quite simple calculus is available, cf. [12, 9, 10]. Moreover, this calculus involves a class of distinguished connections underlying each parabolic geometry, always parametrized by one-forms. In the conformal case, these are the Weyl connections of the conformal Riemannian manifolds. The general theory extends surprisingly many features of the conformal geometry and it has been worked out recently in [9].

It is remarkable that the general calculus shows that each invariant operator is given by a uniform formula in terms of the (generalized) Weyl connections. Even in the locally flat cases, these formulae involve the curvatures of the Weyl connections. Their explicit and closed forms for the curved versions of all BGG-resolutions for $|1|$ -graded algebras have been computed in [11], Part III.

Another essential part of the general theory is the construction of the normalized Cartan connection out of some more elementary underlying structures. We do not touch this problem here and refer the reader to [27, 29, 24, 8]. In fact, our constructions of the curved BGG-sequences work for all Cartan connections, without any normalization.

3.4. Twisted invariant operators. A useful observation reveals that for each P -module \mathbb{E} and each G -module \mathbb{V} , the mapping

$$s \otimes v \mapsto (g \mapsto s(g) \otimes g^{-1} \cdot v)$$

defines the identification $\Gamma(\mathcal{E}) \otimes \mathbb{V} \simeq \Gamma(\mathcal{E} \otimes \mathbb{V})$. This implies that for each invariant operator $D : \Gamma(\mathcal{E}) \rightarrow \Gamma(\mathcal{F})$ on the flat model, there is the twisted invariant operator

$$D_{\mathbb{V}} : \Gamma(\mathcal{E} \otimes \mathbb{V}) \simeq \Gamma(\mathcal{E}) \otimes \mathbb{V} \xrightarrow{D \otimes \text{id}_{\mathbb{V}}} \Gamma(\mathcal{F}) \otimes \mathbb{V} \simeq \Gamma(\mathcal{F} \otimes \mathbb{V}).$$

Now, reading off the information on level of the semi-holonomic jet modules, we conclude: *For each strongly invariant operator D and each G -module \mathbb{V} , there is the twisted strongly invariant operator $D_{\mathbb{V}}$.*

The easiest, but most important, example is the exterior differential $d : \Omega^j(M) \rightarrow \Omega^{j+1}(M)$ which is clearly strongly invariant. For each G -module \mathbb{V} , the twisted operator $d_{\mathbb{V}}$ is given by the homomorphism $J^1(\Lambda^j \mathfrak{p} \otimes \mathbb{V}) \rightarrow \Lambda^{j+1} \mathfrak{p}$

$$(v_0, Z \otimes v_1) \mapsto \partial v_0 + (j+1)(Z \wedge v_1)$$

where $Z \in \mathfrak{p}_+$ and $Z \wedge v_1$ means the obvious alternation and ∂ is the Lie algebra cohomology differential.

There are two crucial remarks in order: First, $\Omega^j(M; \mathcal{V}(M))$ splits into irreducible components once a reduction of the structure group P to its reductive part is chosen. The above formula shows, that only the differential ∂ preserves the homogeneity, while the rest increases the homogeneity. Second, the exterior covariant derivative d^ω with respect to the Cartan connection ω (which acts on \mathcal{V} -valued forms on \mathcal{G}), relates to $d_{\mathbb{V}}$ as $d_{\mathbb{V}}\varphi = d^\omega\varphi - i_\kappa\varphi$ where κ is the curvature function of (\mathcal{G}, ω) .

3.5. Main construction. Since our $|k|$ -graded \mathfrak{g} is semisimple, there is the adjoint codifferential ∂^* to the Lie algebra cohomology differential ∂ , see e.g. [22]. Consequently, there is the Hodge theory on the cochains which allows to deal very effectively with the curvatures. Moreover ∂^* is a P -module homomorphism and so there are the well defined projections

$$\pi : \Omega^j(M; \mathcal{V}M) \supset \ker \partial^* \rightarrow \mathcal{H}_{\mathbb{V}}^j M.$$

Next, consider an irreducible G_0 -component \mathbb{E}_0 of the P -module $\mathbb{H}_{\mathbb{V}}^j$. Of course, \mathbb{E}_0 is in the kernel of the algebraic Laplacian, but this is not P -invariant. Thus we consider the P -submodule \mathbb{E} generated by \mathbb{E}_0 and we try to define a suitable homomorphism $\bar{J}^\ell \mathbb{E}_0 \rightarrow \mathbb{E}$, i.e. a differential operator, splitting π . There is the surprising technical result:

Lemma. *There is a unique differential operator*

$$L : \Gamma(\mathcal{H}_{\mathbb{V}}^j \rightarrow \ker \partial^*) \subset \Omega^j(M; \mathcal{V}M)$$

such that $\pi \circ L(s) = s$ and $d_{\mathbb{V}} \circ L(s) \in \ker \partial^* \subset \Omega^{j+1}(M; \mathcal{V}M)$.

The proof consists of an iterative procedure and represents the technical core of [12]. At the same time, it provides an explicit construction of the operator L . On the level of the operators, we obtain the diagram

$$\begin{array}{ccccc} & & \ker(\partial^*) & & \ker(\partial^*) \\ & & \uparrow \downarrow & & \uparrow \downarrow \\ & & \pi \downarrow \uparrow L & & \pi \downarrow \uparrow L \\ d_{\mathbb{V}} \circ L \nearrow & & & & \nearrow d_{\mathbb{V}} \circ L \\ \dots \dashrightarrow & \Gamma(H_{\mathbb{V}}^i M) & \dashrightarrow & \Gamma(H_{\mathbb{V}}^{i+1} M) & \dashrightarrow \dots \end{array}$$

where the dotted horizontal arrows are the newly constructed operators $D^{\mathbb{V}}$.

In other words, the twisted exterior derivatives produce plenty of natural differential operators in a purely algebraic way.

A few further arguments lead in [12] to

3.6. Theorem. *Let (\mathcal{G}, ω) be a real parabolic geometry of the type (G, P) on a manifold M , \mathbb{V} be a G -module. If the twisted de Rham sequence*

$$0 \longrightarrow \Omega^0(M; \mathbb{V}M) \xrightarrow{d_{\mathbb{V}}} \Omega^1(M; \mathbb{V}M) \xrightarrow{d_{\mathbb{V}}} \dots \xrightarrow{d_{\mathbb{V}}} \Omega^{\dim(G/P)}(M; \mathbb{V}M) \longrightarrow 0.$$

is a complex, then also the Bernstein–Gelfand–Gelfand sequence

$$0 \longrightarrow \Gamma(H_{\mathbb{V}}^0 M) \xrightarrow{D^{\mathbb{V}}} \Gamma(H_{\mathbb{V}}^1 M) \xrightarrow{D^{\mathbb{V}}} \dots \xrightarrow{D^{\mathbb{V}}} \Gamma(H_{\mathbb{V}}^{\dim(G/P)} M) \longrightarrow 0$$

defined above is a complex, and they both compute the same cohomology.

The same statement is true for complex parabolic geometries (\mathcal{G}, ω) under the additional requirement that $\mathcal{G} \rightarrow \mathcal{G}/P_+$ admits a global holomorphic G_0 -equivariant section.

All these operators belong to the class of *standard operators*. The name comes from representation theory and it is related to the fact that the corresponding Verma module homomorphisms for a general parabolics descend from the homomorphisms in the Borel case. There are also natural operators which are not standard, the so called non-standard ones, for which our methods have not been effective enough yet.

An important feature of our theory is the exclusive usage of the elementary (finite dimensional) representation theory. With a bit of exaggeration we could say that the representation theory enters rather as a language and the way of thinking. On the other hand, there are also purely representation theoretical aspects of interest as indicated in [15].

3.7. Remarks. The complex version of the Theorem may be understood as: *If the twisted de Rham sequence induces a complex on the sheaf level, then the same is true for the Bernstein–Gelfand–Gelfand sequence.* In particular, if the twisted de Rham sequence induces a resolution of \mathbb{V} , then so does the BGG–sequence. Now, the original representation theoretical version of the (generalized) BGG–resolution follows immediately by duality.

The Theorem also claims that all the BGG–resolutions on homogeneous spaces admit canonical curved analogs. In particular, the examples (2), (3), (4), (5), and (6) make sense for all curved geometries of the corresponding types. Moreover, the powers of the nablas refer to the iteration of the invariant derivative and we may expand this derivative in terms of the underlying Weyl connections. Partial results in this direction were achieved earlier in [1, 18]

Next, consider any torsion free real parabolic geometry of type (G, P) on M . Then the de Rham cohomology of M with coefficients in $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is computed by the (much smaller) complex

$$0 \longrightarrow \Gamma(H_{\mathbb{K}}^0 M) \xrightarrow{D^{\mathbb{K}}} \Gamma(H_{\mathbb{K}}^1 M) \xrightarrow{D^{\mathbb{K}}} \dots \xrightarrow{D^{\mathbb{K}}} \Gamma(H_{\mathbb{K}}^{\dim(G/P)} M) \longrightarrow 0.$$

Similarly, if (\mathcal{G}, ω) is a flat real parabolic geometry, then for any representation \mathbb{V} of G the BGG–sequence is a complex, which computes the twisted de Rham cohomology of M with coefficients in the bundle VM , which is defined as the cohomology of the complex given by the covariant exterior derivative d^{ω} induced by the Cartan connection ω .

3.8. Further development. The theory is developing very quickly and we do not have place here to mention all main recent achievements. But we cannot miss the paper [10] which extends the definition of our operator L to the whole spaces of forms (and provides a nice alternative definition of L too). This enables the authors to work out differential pairings which restrict to cup product on the cohomologies in the homogeneous case. Some applications are included as well, in particular first steps toward the deformation theory and an interpretation in terms of linearized field theories.

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