Parabolic Geometries

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Preface

Since my lecture series on conformal Riemannian geometries at the University of Vienna in 1991/1992, I have been interested in a better understanding of this fascinating topic. This effort led quickly to a joint project with Andreas Čap (University of Vienna) and Vladimír Souček (Charles University, Prague), which resulted in a series of publications, see e.g. [CSS1, CSS2, CSS3, Slo1, Cap]. Our approach was based on consistent usage of the Lie algebra language and the principal fiber bundle framework. This allowed us to deal in a unified way with a whole class of geometries, the so called almost Hermitian symmetric structures (as introduced in [Bas]) and all their real forms. Thus we had got a sort of universal 'calculus' for all these geometries and we were able to deduce new results even for the best known example, the conformal Riemannian structures. At the same time, it was more and more clear that the methods had to admit a generalization which should lead to a similar calculus for a much wider class of geometries and that we should be able to discuss all of them in a nice and unified way.

The last mentioned ideas seem to have opened a new promising area of research and the main aim of this text is to summarize recent achievements, yet mostly unpublished. I have tried to present a clear and consistent description of a new general model, accompanied by a series of examples of particular geometries. I believe that each of these examples (and many similar ones) deserves a separate deep research and I hope these 'research lecture notes' will make the new area accessible.

The general theory of parabolic geometries is developed here along the lines of the special cases dealt with in [CSS1], however with special emphasis on various new ideas. Some inspiration comes from classical results on Weyl geometries and the papers [Gau] and [BaiE] have been most helpful.

The whole work has been, of course, influenced by fruitful contacts with many mathematicians. In particular, the long term cooperation with my collaborators in the project [CSS1–3] and many discussions with Michael Eastwood have been extremely useful. Furthermore, the whole research wouldn't be possible without the institutional support by the Erwin Schrödiger Institute in Vienna, the Grant Agency of Czech Republic, and first of all the Australian Research Council and University of Adelaide. Most of this research was done during my recent stay in Adelaide as ARC Senior Research Fellow in 1996/1997.

Brno, October 1997 Jan Slovák

1. Introduction

Roughly speaking, the geometries introduced by Cartan under the name 'espace generalisé' are curved deformations of homogeneous spaces G/P where P is a (closed) subgroup in a Lie group G. All such possibilities for G and P give the *flat models* $G \to G/P$ of the geometries in question. The properties of G are encoded in the (left) Maurer-Cartan form $\omega \in \Omega^1(G, \mathfrak{g})$ and the latter form is the subject of the deformations we have in mind. Thus instead of the principal P-bundle $G \to G/P$ we shall deal with a general principal P-bundle $\mathcal{G} \to M$, equipped with a one-form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, subject to the following properties (ζ_X denotes the fundamental vector field given by X):

Curved geometry $\mathcal{G} \to M$, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$	**Flat model** $G \to G/P$, Maurer-Cartan form ω
$\omega(\zeta_X) = X$ for all $X \in \mathfrak{p}$	$\omega(\zeta_X) = X$ for all $X \in \mathfrak{g}$
$(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega \ \forall b \in P$	$(r^b)^*\omega = \mathrm{Ad}(b^{-1}) \circ \omega \ \forall b \in G$
$\omega_{\mid T_u\mathcal{G}}: T_u\mathcal{G} \to \mathfrak{g}$ iso $\forall u \in \mathcal{G}$	$\omega_{\mid T_uG}: T_uG \to \mathfrak{g}$ iso $\forall u \in G$

A form $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ with the three properties listed above is called *Cartan connection (of the type G/P)*. Let us notice in particular the third property, which yields the *horizontal vector field* $\omega^{-1}(X)$ on $\mathcal{G}$ for each element $X \in \mathfrak{g}$. The first condition then tells that the latter fields are the fundamental fields ζ_X for all $X \in \mathfrak{p}$. The extent of the deformation is measured by the *curvature of the Cartan connection*, the two-form $\kappa \in \Omega^2(\mathcal{G}, \mathfrak{g})$ given by the *structure equation*

$$d\omega + \frac{1}{2}[\omega, \omega] = \kappa.$$

In particular, $(\mathcal{G}, \omega)$ is locally isomorphic to (G, ω) if and only if κ vanishes. It follows immediately from the definition, that κ is a horizontal form and due to the presence of the horizontal vector fields we can view κ as a function valued in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$, i.e. $\kappa \in C^\infty(\mathcal{G}, \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g})$. All these old ideas go back to E. Cartan and his concept of 'generalized spaces', but it is difficult to point back to a paper by Cartan and say 'there it is, look!'. In fact these beautiful concepts developed during Cartan's work on concrete examples of equivalence problems, see e.g. [Sha] for many illuminating comments.

We shall be interested in the special class of such geometries where either P is a parabolic subgroup in a (complex) semisimple Lie group G, or P and G represent a real form of such a situation. Following Fefferman and Graham, we are using the name *parabolic geometries* in this context, cf. [FefG]. As we shall see, this is justified by the nice explicit links of purely geometrical questions to the representation theory of parabolic subgroups.

More explicitly, we deal with a pair $(\mathfrak{g}, \mathfrak{p})$ where $\mathfrak{g}$ is a (real) semisimple Lie algebra of the Lie group G equipped with a finite grading $\mathfrak{g} = \mathfrak{g}_{-\ell} \oplus \cdots \oplus \mathfrak{g}_\ell$, $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\ell$. The group P is then the Lie subgroup corresponding to the subalgebra $\mathfrak{p}$. We also write $\mathfrak{g}_-$ for $\mathfrak{g}_{-\ell} \oplus \cdots \oplus \mathfrak{g}_{-1}$ and $\mathfrak{p}_+$ for $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$. Then $\mathfrak{g}_- \simeq (\mathfrak{g}/\mathfrak{p})$ and the Killing form yields $(\mathfrak{g}_-)^* \simeq \mathfrak{p}_+$. In the case of complex Lie groups $P \subset G$ this just means that P is a parabolic subgroup. See the beginning of Section6 for explicit definitions and more information on the Lie algebras.

In fact, the definition of a particular geometry on M in terms of such a 'mysterious bundle' $\mathcal{G}$ and the global parallelism ω seems to be quite unusual and we should rather prefer to have a theorem establishing their (unique) existence from some more familiar data. On the other hand, once we have a Cartan connection ω of type G/P on the manifold M, there is an extremely rich geometry hidden behind. The main source for various underlying concepts and their relations lies in the grading of the Lie algebra $\mathfrak{g}$. In particular, we can decompose the $\mathfrak{g}$-valued curvature form κ in two ways: according to values in the particular components of $\mathfrak{g}$ (denoted by subscripts like the components in $\mathfrak{g}$ themselves) and by the homogeneity degrees (denoted by superscripts). This means

$$\kappa = \sum_{i=-\ell}^{\ell} \kappa_i, \quad \kappa = \sum_{k=-\ell+2}^{3\ell} \kappa^k$$

where $\kappa^k(u)(X,Y) \in \mathfrak{g}_{i+j+k}$ for all $u \in \mathcal{G}$, $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$. Sometimes, we shall also decompose κ into $\kappa_- + \kappa_0 + \kappa_+$, according to the values. The negative part $\kappa_- = \kappa_{-\ell} + \dots + \kappa_{-1}$ will be called the *torsion part of the curvature*, the component κ_0 will be called the *Weyl part of the curvature.* If the length of the grading is $\ell = 1$ (the so called $|1|$-*graded* case), then both decompositions coincide ($\kappa_i = \kappa^{i+2}$).

We shall present a general model for all these geometries in Section 2 and the application to explicit examples will be indicated in Sections 4-5. However, in order to present some indication of what sort of concepts we are looking for, we first conclude this introduction with a review of the best known case. Whenever we shall not give explicit proofs of our claims and if no other source will be mentioned explicitly, the reader should consult the papers [CSS1, CSS2] or [Slo2] for further information. The whole lecture notes [Slo1] are devoted to the conformal Riemannian geometries.

Conformal Riemannian structures

Most easily, a *conformal Riemannian structure* on a manifold M is given by a choice of a Riemannian metric g on M. The metric g then defines the line bundle of metrics in S^2T^*M which are all *conformal*, i.e. each of them is given by $e^{2f}g$ for a unique smooth function f on M. Equivalently, we can define the structure by reducing the structure group of TM to the subgroup $G_0 = CO(m,\mathbb{R}) \subset GL(m,\mathbb{R})$, $m = \dim M$. None of the conformal metrics is privileged and a choice of one of them means a choice of a *scale* in each tangent space. It is well known, that this is a geometry of finite type and that all conformal isometries are fully determined by their 2-jets at a single point. In the flat case, the space of all (local) isomorphisms is parameterized by $\mathfrak{g} = \mathfrak{so}(m+1,1,\mathbb{R})$, while those keeping a given point fixed correspond to the subalgebra $\mathfrak{p}$. If $\mathfrak{g}$ is defined by the quadratic form

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{I}_m & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

then the grading can be described in block matrix form as follows

$$\mathfrak{g}_{-1} = \begin{pmatrix} 0 & 0 & 0 \\ X & 0 & 0 \\ 0 & -X^T & 0 \end{pmatrix}, \ \mathfrak{g}_0 = \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix}, \ \mathfrak{g}_1 = \begin{pmatrix} 0 & Z & 0 \\ 0 & 0 & -Z^T \\ 0 & 0 & 0 \end{pmatrix}.$$

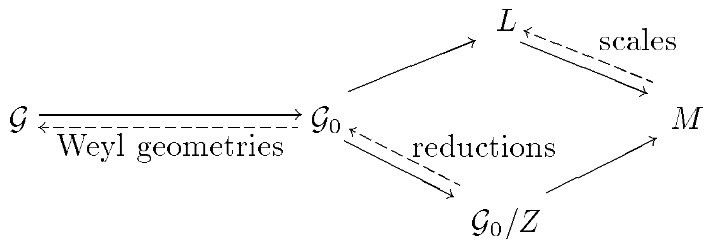
The Lie group P is the Poincaré conformal group.

The complete information on the structure is well encoded in the classical first prolongation of the defining G_0-structure on M, which can be found in a way producing the desired principal P-bundle $\mathcal{G} \to M$. Now, it is a (well known) *theorem* that the latter bundle comes equipped with the *normal Cartan connection* ω, which is normalized by the trace vanishing condition on its curvature (and vanishing of the whole torsion part). In fact, the construction of $\mathcal{G}$ fixes the $(\mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ component of ω in a way minimizing the torsion of the structure. The last component is then chosen to fit the normalization. Another class of well known objects related to conformal Riemannian structures are the *Weyl geometries* which are given by a choice of any torsion free linear connection γ on M preserving the conformal class of metrics.

The basic idea of our approach to parabolic geometries is that everything must be defined and expressed by means of the algebras in question. So let us try to recover all the above mentioned objects just from the pair $(\mathfrak{so}(m+1,1,\mathbb{R}), \mathfrak{p})$.

First, we have to fix the Lie groups G, P. Since $\mathfrak{g}_-$ is abelian, this choice does not influence much the general procedure (but it does effect the topological obstructions to the existence of the corresponding structures on particular manifolds). In general, we usually require G_0 to be the adjoint group acting on $\mathfrak{g}_-$. The geometric structure is then defined by requiring T_xM to be *isomorphic to* $\mathfrak{g}_-$ *up to elements in* G_0 for all $x \in M$. In our case, this means exactly a reduction of TM to G_0.) The standard prolongation construction provides the principal P-bundle $\mathcal{G}$ over M with the Cartan connection ω, see [CSS2] for more details.

The group P is the semi-direct product of its Levi part G_0 and the nilpotent Lie group $P_+ = \exp \mathfrak{g}_1$ and there is the *affine space of global* G_0*-equivariant sections* $\sigma \colon \mathcal{G}_0 \to \mathcal{G}$ of the quotient projection $\mathcal{G} \to \mathcal{G}_0 := \mathcal{G}/P_+$, *modeled over one-forms on* M (cf. [CSS1, Lemma 3.6]). The pullbacks of the $\mathfrak{g}_0$-component of ω by the sections σ are torsion free linear connections on M which preserve the conformal structure by definition. Thus they coincide with the Weyl geometries. The Levi part G_0 is the product of its semisimple part $G_0^s = O(m,\mathbb{R})$ and the one-dimensional center $Z = \{\exp tE\}$, where E is the unique element in $\mathfrak{g}$ such that $\mathrm{ad}_E(X) = j.X$ for all $X \in \mathfrak{g}_j$, $j = -1, 0, 1$. The quotient bundle $L := \mathcal{G}_0/G_0^s$ is isomorphic to the associated bundle $\mathcal{G}_0 \times_{G_0} \{\exp tE\}$ and it is easy to show that there is a *bijective correspondence* between the sections σ (i.e. the corresponding Weyl structures γ_σ) and the induced connections γ_σ^L on L. In particular, the sections of L correspond to trivial connections and they represent the metrics in the conformal class. Notice also that all connections γ_σ share the same 'minimal torsion' given by the $\mathfrak{g}_{-1}$-component κ_{-1} of the curvature κ. This is of course vanishing for conformal Riemannian structures.



Any Weyl geometry γ_σ (thus in particular any Levi-Civita connection from the class of metrics) defines, together with the canonical soldering form $\theta \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1})$, the Cartan connection $(\theta + \gamma^\sigma) \in \Omega^1(\mathcal{G}_0, \mathfrak{g}_{-1} \oplus \mathfrak{g}_0)$ and there is the unique σ-related Cartan connection ω^σ on $\mathcal{G}$. The forms ω and ω^σ differ only in the $\mathfrak{g}_1$-component and so there must be a mapping $\mathsf{P} \in C^\infty(\mathcal{G}, \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_1)$ such that $\omega = \omega^\sigma - \mathsf{P} \circ \omega_{-1}$. An easy check shows that P is in fact a tensor in $T^*M \otimes T^*M$, the so called Rho-tensor which is a trace adjusted Ricci part of the curvature of γ^σ (cf. [CSS1, Lemma 3.10 and formula 6.3.(1)]). By definition, the $\mathfrak{g}_1$-component κ_1^σ of the curvature of ω^σ vanishes at all frames $u \in \sigma(\mathcal{G}_0)$ and a straightforward computation yields the relation between κ and κ^σ on $\sigma(\mathcal{G}_0)$ (of course, all these results will also follow from the general formulae in Section 2):

j	$(\kappa_j^\sigma - \kappa_j)(u)(X,Y)$	what is κ_j?
-1	0	the fixed torsion of γ^σ
0	$[X, \mathsf{P}.Y] - [Y, \mathsf{P}.X]$	Weyl part of curvature of γ^σ
1	$\nabla_X^{\gamma^\sigma} \mathsf{P}.Y - \nabla_Y^{\gamma^\sigma} \mathsf{P}.X + \mathsf{P}.\kappa_{-1}(X,Y)$	Cotton-York tensor (torsion adjusted)

What we want to say is that the trace vanishing condition on κ forces the choice of P and then the curvature κ itself is described by the right hand column. In particular, a simple computation yields the Rho-tensor well known from the conformal Riemannian geometry.

The canonical Cartan connection can be also easily defined on vector bundles coming from representations of whole G by means of restriction to P. Indeed, there is the classical principal connection $\tilde{\omega}$ on the extension $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$ given by ω and therefore also the induced linear connections on all bundles coming from G-modules. Any choice of a Weyl geometry yields a reduction of the structure group P to G_0, thus also a decomposition of the latter bundles into G_0-invariant subbundles. This recovers easily the twistor connections and their explicit formulae by means of covariant derivatives and the tensor P. A choice of a Weyl geometry also provides decompositions of all bundles coming from P-modules, however there is no canonical linear connection on them.

2. The general theory

The aim of this chapter is to provide the general model and several basic results and formulae. At the end we try to give a sort of recipe, how to understand any particular case and we shall try to illustrate its usage in the next chapters. Most of the material has not been published yet, so we present many full proofs here.

Joint papers with Andreas Čap and Vladimir Souček covering these topics and some deeper applications are in preparation.

We mainly extend and generalize the development from [CSS1], but the existence results for the regular normal Cartan connections are taken from the recent paper [CSch]. The latter paper provides a much more general and complete version of Tanaka's results on differential systems, cf. also [Tan, Yam].

Recently, also the book by R.W. Sharpe appeared, see [Sha]. Although it has not influenced the development of the present exposition, a careful reading of that book will definitely help a lot to understand broader context of our theory. The book also provides links to the original ideas by E. Cartan and many other great mathematicians. On the other hand, Sharpe's setting is very general and, in particular, he does not touch the impact of the gradings of our Lie algebras and the rich representation theoretic tools, which are in the center of our attention. Also only the projective and conformal geometries among all our examples are mentioned explicitly there.

Basic definitions and existence results

Let us recall that for any $|\ell|$-graded Lie algebra $\mathfrak{g}$ we write $\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_\ell$, $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\ell$ and further $\mathfrak{p}_+^k = \mathfrak{g}_k \oplus \cdots \oplus \mathfrak{g}_\ell = [\mathfrak{p}_+, \mathfrak{p}_+^{k-1}]$. Thus $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and we also write $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$, P the subgroup corresponding to $\mathfrak{p}$, and G_0 the Levi part of P (with Lie algebra $\mathfrak{g}_0$), $P_+ \subset P$, etc. Let us also recall the existence of the Lie algebra cohomology differential ∂ and its adjoint codifferential ∂^* on the spaces $\Lambda^k(\mathfrak{g}_-)^* \otimes \mathfrak{g}$, see 6.8, 6.9.

2.1. Definition. A *parabolic geometry of type G/P* on a manifold M is given by the principal fiber bundle $\mathcal{G} \to M$ with structure group P equipped by the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. We say that the Cartan connection is *normal* if its curvature $\kappa \in C^\infty(\mathcal{G}, \mathfrak{g}_-^* \wedge \mathfrak{g}_-^* \otimes \mathfrak{g})$ is co-closed, i.e. $\partial^* \circ \kappa = 0$. The Cartan connection is said to be *regular* if all non-positive homogeneous components κ^j, $j \leq 0$ vanish, and it is called *torsion-free* if the whole component κ_- vanishes.

2.2. The induced filtrations. As already mentioned, we would like to understand (and define) the parabolic geometries in terms of some objects more intrinsic to the underlying manifold M. First of all, the quotients of $\mathcal{G}$ by the actions of the closed subgroups $P_+^k \subset P$ are principal fiber bundles

$$\mathcal{G}_k = \mathcal{G}/P_+^{k+1}, \qquad k = 0, \dots, \ell.$$

The global parallelism ω transfers the filtration of the $\mathfrak{p}$-module $\mathfrak{g}$ (with respect to the adjoint representation) into the filtration of the tangent bundle $T\mathcal{G} = T^{-\ell}\mathcal{G} \supset \cdots \supset T^\ell\mathcal{G}$, which is P-invariant. Thus there also is the induced invariant filtration of the tangent spaces of all $\mathcal{G}_k$ and also of the underlying tangent bundle TM, where $T^iM = Tp(T^i\mathcal{G})$, $i = -\ell, \dots, -1$. Moreover, each choice of a frame $u \in \mathcal{G}$ provides the identification of T_xM with the filtered P-module $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}_-$. On the level of the associated graded vector spaces we then obtain $\operatorname{Gr} T_xM \simeq (\mathfrak{g}_{-\ell} \oplus \cdots \oplus \mathfrak{g}_{-1})$

as graded G_0-modules. This in turn provides a reduction of the structure group of $\operatorname{Gr} TM$ to G_0.

Assume now we are given a filtration $T^\ell M \supset \cdots \supset T^{-1}M$ on TM and a reduction of $\operatorname{Gr} TM$ to the structure group G_0. Since the Lie bracket on $\mathfrak{g}_-$ is G_0-equivariant, the choice of the reduction of $\operatorname{Gr} TM$ to structure group G_0 transfers the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}_-$ to an algebraic bracket $\{\ ,\ \}_0$ on $\operatorname{Gr} TM$. More explicitly, choosing a frame $u\colon \operatorname{Gr} T_xM \to \mathfrak{g}_-$ we define for all $\xi_x, \eta_x \in T_xM$ the bracket by the formula

$$\{\xi_x, \eta_x\}_0 = u^{-1}([u(\xi_x), u(\eta_x)]).$$

A replacement of u by $u.g = \operatorname{Ad}_{g^{-1}} \circ u$ leads to the same value

$$\{\xi_x, \eta_x\}_0 = u^{-1} \circ \operatorname{Ad}_g([\operatorname{Ad}_{g^{-1}} u(\xi_x), \operatorname{Ad}_{g^{-1}} u(\eta_x)]) = u^{-1}([u(\xi_x), u(\eta_x)]).$$

If our reduction of $\operatorname{Gr} TM$ to G_0 comes from a Cartan connection ω on $\mathcal{G}$ as above, then clearly the bracket is given by

$$\{\xi_x, \eta_x\}_0 = \pi(\omega^{-1}([\omega(\xi), \omega(\eta)])(u)) \tag{1}$$

where π is the obvious projection $T\mathcal{G} \to TM \to \operatorname{Gr} TM$ and ξ, η are any vectors in $T\mathcal{G}$ covering ξ_x, η_x.

On the other hand, for all vector fields $\xi \in T^iM$, $\eta \in T^jM$ and functions f, g on M we obtain

$$[f\xi, g\eta] = fg[\xi, \eta] \operatorname{mod} T^kM, \quad \text{where } k = \min\{i, j\}.$$

Thus the induced brackets $T^iM \times T^jM \to TM/T^kM$ with k as above are algebraic as well.

Assume that our filtration of TM comes from a Cartan connection ω. Vanishing of homogeneous components of the curvature κ then imposes restrictions on the non-integrability of the subspaces T^iM. Let us formulate this claim more explicitly:

Proposition. *If $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is a Cartan connection on M and $\kappa^i = 0$ for all $i < 0$, then the induced filtration of TM satisfies $[T^iM, T^jM] \subset T^{i+j}M$ and the Lie bracket of vector fields defines an algebraic bracket $\{\ ,\ \}_{\mathrm{Lie}}$ on the graded vector bundle $\operatorname{Gr} TM$. Moreover, if κ^0 vanishes too, then the latter bracket coincides with the algebraic bracket $\{\ ,\ \}_0$ on $\operatorname{Gr} TM$.*

Proof. The filtration is defined by

$$T^iM = \pi(\omega^{-1}(\mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_{-1})).$$

The defining equation for $\kappa^k(u)(X, Y)$, $u \in \mathcal{G}$, $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$ is

$$\kappa^k(u)(X, Y) = \begin{cases} [X, Y] - \omega_{i+j}(u)([\omega^{-1}(X)(u), \omega^{-1}(Y)(u)]) & \text{if } k = 0 \\ -\omega_{i+j+k}(u)([\omega^{-1}(X)(u), \omega^{-1}(Y)(u)]) & \text{if } k \neq 0. \end{cases} \tag{2}$$

Now, consider vector fields ξ in T^iM, η in T^jM and let us choose elements X_r, Y_s in $\mathfrak{g}_-$ such that $\xi = \pi \circ \sum_r f^r\omega^{-1}(X_r)$, $\eta = \pi \circ \sum_s g^s\omega^{-1}(Y_s)$ with suitable functions f^r, g^s on $\mathcal{G}$. Then

$$[\xi, \eta] = \pi \circ \sum_{r,s} f^r g^s[\omega^{-1}(X_r), \omega^{-1}(Y_s)] \mod T^{i+j}M.$$

Since all $X_r \in \mathfrak{g}_i \oplus \cdots \oplus g_{-1}$ and $Y_s \in \mathfrak{g}_j \oplus \cdots \oplus \mathfrak{g}_{-1}$, equality (2) with $k < 0$ implies $[\xi, \eta] \subset T^{i+j}M$.

Once we know that the Lie bracket defines a mapping $T^iM \times T^jM \to T^{i+j}M$, then the algebraic bracket $\{\ ,\ \}_{\mathrm{Lie}}$ on $\operatorname{Gr} TM$ is clearly defined and the first claim of the Proposition has been proved.

Finally, equality (2) with $\kappa^0(u) = 0$ yields for all $X \in \mathfrak{g}_i$, $Y \in \mathfrak{g}_j$

$$\omega^{-1}([X, Y]) - \omega^{-1}(\omega_{i+j}([\omega^{-1}(X), \omega^{-1}(Y)])) = 0.$$

Thus, vanishing of $\kappa^{<0}$ implies for all $\xi_x = \pi(\omega^{-1}(X)(u))$ and $\eta_x = \pi(\omega^{-1}(Y)(u))$

$$\begin{aligned}
\{\xi_x, \eta_x\}_0 &= \pi(\omega^{-1}([X, Y])(u)) \\
&= \pi(\omega^{-1}(\omega_{i+j}([\omega^{-1}(X), \omega^{-1}(Y)](u)))) \\
&= \pi([\omega^{-1}(X), \omega^{-1}(Y)](u) \text{ mod } T^{i+j+1}\mathcal{G}) \\
&= [\pi \circ \omega^{-1}(X), \pi \circ \omega^{-1}(Y)](x) \text{ mod } T^{i+j+1}M \\
&= \{\xi_x, \eta_x\}_{\mathrm{Lie}}. \quad \square
\end{aligned}$$

2.3. Definition. Let TM be equipped with a filtration satisfying $[T^iM, T^jM] \subset T^{i+j}M$ for all i, j, and assume that a reduction of the structure group of $\operatorname{Gr} TM$ to G_0 is given. We say that the *structure equation* holds if the two algebraic brackets coincide, i.e $\{\ ,\ \}_0 = \{\ ,\ \}_{\mathrm{Lie}}$.

In fact, Proposition 2.2 shows that every regular Cartan connection induces a filtration on TM and a reduction of $\operatorname{Gr} TM$, such that the structure equation holds. It is remarkable that in nearly all cases these data are also sufficient to recover such a regular and normal Cartan connection ω, inclusive the construction of the principal fiber bundle $\mathcal{G}$. Let us make this statement more explicit. We say that two Cartan connections ω and $\bar{\omega}$ on principal P-bundles $\mathcal{G}$ and $\bar{\mathcal{G}}$ are isomorphic, if there is a principal fiber bundle isomorphism $\varphi \colon \mathcal{G} \to \bar{\mathcal{G}}$ such that $\varphi^*\bar{\omega} = \omega$.

2.4. Theorem. ([CSch]) *Let M be a manifold equipped with a filtration $TM = T^{-\ell}M \supset T^{-\ell+1}M \supset \cdots \supset T^{-1}M \supset \{0\}$ such that $[T^iM, T^jM] \subset T^{i+j}M$ for all i, j, and with a reduction of the associated graded vector bundle $\operatorname{Gr} TM = T^{-\ell}M/T^{-\ell+1}M \oplus \cdots \oplus T^{-2}M/T^{-1}M \oplus T^{-1}M$ to the structure group G_0. Let us further assume that the structure equation holds. If $H^1_k(\mathfrak{g}_-, \mathfrak{g})$ vanishes for all $k > 0$, then there is the unique Cartan connection ω on the unique principal P-bundle $\mathcal{G} \to M$ such that its curvature is ∂^*-closed, up to isomorphisms.*

2.5. The exceptional geometries. Up to completely degenerate cases like $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the Borel subalgebra, or semisimple algebras with simple components in $\mathfrak{g}_0$, there are just two series of pairs $(\mathfrak{g}, \mathfrak{p})$ which admit a first cohomology with non-vanishing components of positive homogeneity:

Exceptional pairs $(\mathfrak{g}, \mathfrak{p})$	**The obstructing cohomologies**
$\mathfrak{g} = \overset{1}{\times}\!\!-\!\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!\!-\!\!\overset{1}{\bullet}$	$\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\!\!-\!\!\overset{2}{\bullet} \in H^1_1(\mathfrak{g}_-, \mathfrak{g})$
$\mathfrak{g} = \overset{2}{\times}\!\!-\!\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\Leftarrow\overset{0}{\bullet}$	$\overset{-2}{\times}\!\!-\!\!\overset{3}{\bullet}\!\!-\!\!\overset{0}{\bullet} \cdots \overset{0}{\bullet}\Leftarrow\overset{0}{\bullet} \in H^1_1(\mathfrak{g}_-, \mathfrak{g})$

This is also shown in [CSch]. Let us notice, that in both cases the obstructions appear in homogeneity one. A more careful discussion then shows that we can still construct the canonical Cartan connections after making a further choice (see [CSch] again), and explicit computations of the second cohomologies then show that the principal bundles $\mathcal{G}$ defining the structures will be reductions of the second order frame bundle to the appropriate subgroup P, see Section 5 and [Slo3].

In fact, the C_ℓ cas
seems to be a usua
first order structu
well!

In fact, all real forms of these complex graded algebras lead to examples of such geometries. In particular, the first of these series involves the well known *projective geometries*. Since G_0 is the whole general linear group in this case, we have no structure on the $\mathcal{G}_0$ level. The flat models for some of the other structures are the Grassmannians of isotropic lines in $\mathbb{R}^{2n}$ with the standard symplectic structure, G_0 is the conformal symplectic linear group and the corresponding Lie algebra is $|2|$-graded (see 6.3 and Section 5 for more details).

would be nice to l
at them really!

2.6. Remark. In view of the latter Theorem, we could use the following alternative definition: A *(regular and normal) parabolic geometry of type G/P* on a manifold M is given by a filtration

$$TM = T^{-\ell}M \supset T^{-\ell+1}M \supset \cdots \supset T^{-1}M \supset \{0\}$$

with the property $[T^iM, T^jM] \subset T^{i+j}M$ for all i, j, and a reduction of the associated graded vector bundle

$$\operatorname{Gr} TM = T^{-\ell}M/T^{-\ell+1}M \oplus \cdots \oplus T^{-2}M/T^{-1}M \oplus T^{-1}M$$

to the structure group G_0, such that the structure equation holds.

We may also say that (regular and normal) parabolic geometries of type G/P on M are given by filtrations, subject to the right dimensions and quite subtle non-integrability conditions, which make $\operatorname{Gr} TM$ fiber-wise isomorphic to $\mathfrak{g}_-$.

Notice also that the structure equation gets void for all $|1|$-graded algebras $\mathfrak{g}$ while the filtrations are trivial in these cases. Then the whole definition reduces to the usual G_0-structures on the manifolds M.

2.7. Remark. Another description of the parabolic geometries follows rather the analogy to the classical first order G-structures and their soldering forms: Instead of a principal fiber bundle equipped with the soldering form, we have to define a principal G_0-bundle $p\colon \mathcal{G}_0 \to M$ over a filtered manifold M, equipped with the so called frame form of length one. The latter form is a sequence of G_0-equivariant (partially defined) forms $(\theta^{-\ell}, \dots, \theta^{-1})$, $\theta^i \in \Omega^1((Tp)^{-1}T^iM, \mathfrak{g}_i)$ such that $(Tp)^{-1}T^{i+1}M = \ker \theta^i$ and each θ^j induces at each frame u a linear isomorphism $T^iM/T^{i+1}M \to \mathfrak{g}_{-i}$. Moreover, the frame form has to satisfy the structure equation (the explicit meaning of which needs some further thoughts)

$$d\theta^{i+j} + [\theta^i, \theta^j] = 0 \mod T^{i+j+1}M$$

see [CSch] for a more explicit exposition. These frame forms are exactly what survives from a Cartan connection ω on a principal P-bundle after factoring out the action of P_+. On the other hand, given such a frame form, we can carefully extend the principal fiber bundle and construct longer frame forms, until we reach a Cartan connection after 2ℓ prolongation steps. Requiring suitable normalizing conditions, we end up with the normal Cartan connection ω on M. This is the beautiful procedure suggested and worked out in great detail in [CSch].

Generalized Weyl geometries

In the rest of the Section, we shall suppose that $\mathcal{G}$ is a principal P-bundle over M equipped with a fixed Cartan connection ω, i.e. a parabolic geometry on M in the most general sense.

2.8. Lemma. *On each principal P-bundle $\mathcal{G} \to M$, there is the affine space of global G_0-equivariant sections $\sigma\colon \mathcal{G}_0 = \mathcal{G}/P_+ \to \mathcal{G}$ of the quotient projection, modeled over the vector space of all one-forms on M. The sum of a global G_0-equivariant section σ and a 1-form $\Upsilon \in C^\infty(\mathcal{G}, \mathfrak{p}_+)$ is given by the formula*

$$(\sigma + \Upsilon)(u) = \sigma(u).\exp(\Upsilon(\sigma(u))). \tag{1}$$

Proof. Let us first show that there is at least one such section. We have to construct a global G_0-equivariant trivialization of $\mathcal{G} \to \mathcal{G}_0$. This can be achieved step by step, building G_0-equivariant trivializations of the principal bundles $\mathcal{G}_k = \mathcal{G}/P_+^{k+1} \to \mathcal{G}/P_+^k$ with abelian structure groups P_+^k/P_+^{k+1}, $k = 1, 2, \dots$. Recall that $P = G_0 \rtimes P_+$ and each $b \in P$ allows the unique expression in the form $b = b_0 \exp X_1 \dots \exp X_\ell$ with $b_0 \in G_0$, $X_i \in \mathfrak{g}_i$ (see e.g. [CSch, Proposition 2.17]). In particular, all bundles $\mathcal{G}_k$ are also equipped by the right action of G_0.

Via the exponential mapping, we can view the latter principal bundles as affine bundles modeled over $\mathfrak{g}_k$. So we can always choose a cocycle of local G_0-equivariant trivializations σ_α over a covering U_α of $\mathcal{G}_{k-1} = \mathcal{G}/P_+^k$ with transition functions $\chi_{\alpha\beta}\colon \mathcal{G}_{k-1} \to P_+^k/P_+^{k+1}$. Now, we can form the affine sum σ_k of all σ_α by means of a partition of unity subordinated to U_α. By the construction, σ_k will be G_0-equivariant as well. The composition of such σ^k, $k = 1, \dots, \ell$ provides the required section.

It remains to show that formula (1) defines another G_0-equivariant section and that given two such sections σ, σ', there is the uniquely defined 1-form ν such that $\sigma + \nu = \sigma'$. For each $b \in G_0$

$$\begin{aligned}(\sigma + \nu)(u.b) &= \sigma(u).b.\exp(\mathrm{Ad}_{b^{-1}}.\nu(\sigma(u))) \\ &= \sigma(u).b.b^{-1}.\exp(\nu(\sigma(u))).b = (\sigma + \nu)(u).b\end{aligned}$$

and the formula $\sigma'(u) = \sigma(u).\exp(\nu(\sigma(u)))$ defines the mapping ν on the image of σ. The previous computation, read backwards, shows that ν is G_0-equivariant on the image of σ. But since $P = G_0 \rtimes P_+$, there is the uniquely defined mapping $\mathcal{G} \to \mathfrak{p}_+$ coinciding with ν on $\sigma(\mathcal{G}_0)$ and satisfying $\nu(u.b) = \mathrm{Ad}_{b^{-1}}.\nu(u)$ for all $u \in \mathcal{G}$, $b \in P$. □

2.9. Definition. Let $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be a Cartan connection defining the parabolic geometry on M. A *(generalized) Weyl geometry* on M is given by a G_0-equivariant section σ of $\mathcal{G} \to \mathcal{G}_0$.

Let us notice that our definition of a (generalized) Weyl geometry does not use explicitly the one form ω but remember that the Weyl geometries on conformal Riemannian manifolds are linear connections. We observed in the Introduction that these connections are in bijective correspondence with G_0-equivariant sections of appropriate principal bundles. The next Lemma describes such a class of distinguished linear connections in our general situation which, of course, depend on the Cartan connection ω explicitly. Moreover, the relation between such connections and their defining sections will be shown to be bijective as well. In this context, we shall also use the name 'Weyl geometry' for each of these connections.

All quotient projections $p^j_k : \mathcal{G}/P^{j+1}_+ \to \mathcal{G}/P^{k+1}_+$, $j > k$ are G_0-equivariant too. Thus each choice of a global G_0-equivariant section σ yields reductions of all the intermediate bundles $\mathcal{G}_k = \mathcal{G}/P^{k+1}_+$, $k = 0, \dots, \ell$ to the structure group G_0. In particular, each principal connection γ on $\mathcal{G}_0$ defines principal connections on all bundles $\mathcal{G}_k$ (their connection forms coincide with γ on the image of $\mathcal{G}_0$ and are defined uniquely by the equivariance elsewhere). We shall keep the same symbol γ for all of them.

2.10. Lemma. *For each Cartan connection $\omega = \omega_- \oplus \omega_0 \oplus \omega_+ \in \Omega^1(\mathcal{G}, \mathfrak{g})$ and each G_0-equivariant global section σ, the pullback $\gamma^\sigma := \sigma^*(\omega_0)$ is a principal connection on $\mathcal{G}_0$.*

In particular, the choice of a Weyl geometry σ defines also the principal connection γ^σ on $\mathcal{G}_{\ell-1}$, i.e. a linear connection on M.

Proof. The equivariance of ω, restricted to G_0, ensures the required equivariance of γ^σ. At the same time, $T\sigma.\zeta_Y(u) = \zeta_Y(\sigma(u))$ for each $Y \in \mathfrak{g}_0$ and so the fundamental fields are recovered as well. □

2.11. The tangent bundle. Obviously, the tangent bundle TM on a manifold with a parabolic geometry of the type G/P is associated to $\mathcal{G}$ via the adjoint representation of P on $\mathfrak{g}_- \simeq \mathfrak{g}/\mathfrak{p}$. The subgroup P^ℓ_+ acts trivially and the effective structure group of TM is always P/P^ℓ_+. The component ω_- of the canonical Cartan connection ω survives on $\mathcal{G}_{\ell-1} = \mathcal{G}/P^\ell_+$ as the soldering form for TM.

Now, each choice of a generalized Weyl geometry σ on M provides further reduction of the structure group of TM to G_0. In fact, we obtain explicit splittings of the projections $T^iM \to T^iM/T^{i+1}M \subset \mathrm{Gr}\,TM$. Moreover, the induced linear connections γ^σ always belong to the G_0 structure on M and so they have to preserve all G_0-invariant subspaces. In particular all the subspaces corresponding to $\mathfrak{g}_i$ are preserved.

2.12. Similarly, each P-module V enjoys a filtration by P-submodules, which gives rise to filtrations of the associated vector bundles $\mathcal{V} := \mathcal{G} \times_P V$, cf. 6.7. Each choice of a Weyl geometry then defines splittings of the quotient projections and yields a decomposition of $\mathcal{V}$ into irreducible associated bundles corresponding to G_0-modules. Of course, the linear connection γ^σ then induces linear connections on all these subbundles.

Each G-module can be considered as P-module by restriction and all the above considerations apply. On the other hand, there is the extension $\tilde{\mathcal{G}} = \mathcal{G} \times_P G$ endowed with the unique principal connection form $\tilde{\omega} \in \Omega^1(\tilde{\mathcal{G}}, \mathfrak{g})$ which coincides with the Cartan connection ω on the image of $\mathcal{G} \subset \tilde{\mathcal{G}}$. Thus there are the induced linear connections on all bundles coming from G-representations. Of course, we shall be able to express them explicitly by means of any of the Weyl geometries σ, i.e. by means of the induced connections γ^σ, and decompositions of the bundles into G_0-invariant subbundles. These procedures will recover analogies to twistor bundles, and more general objects in conformal Riemannian geometries, cf. [Eas].

2.13. The invariant differential. For each Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$, there is the obvious operator defined by Lie derivative of functions in the directions of the horizontal vector fields: For each P-module $\mathbb{E}$ we define

$$\boxed{\begin{gathered}\nabla^\omega \colon C^\infty(\mathcal{G}, \mathbb{E}) \to C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{E}) \\ \nabla^\omega s(u)(X) = \mathcal{L}_{\omega^{-1}(X)} s(u), \qquad \text{for all } X \in \mathfrak{g}_-,\ u \in \mathcal{G}\end{gathered}}$$

We call this operation the *invariant derivative* (with respect to the Cartan connection ω). We also write $\nabla^\omega_X s$ for the values on a fixed vector $X \in \mathfrak{g}_-$. An easy computation yields the (generalized) *Ricci identity* for sections of $\mathcal{E} = \mathcal{G} \times_P \mathbb{E}$ (i.e. $s \in C^\infty(\mathcal{G}, \mathbb{E})^P$)

$$\boxed{(\nabla^\omega_X \circ \nabla^\omega_Y - \nabla^\omega_Y \circ \nabla^\omega_X)s = \nabla^\omega_{[X,Y]} s + \lambda(\kappa_p(X,Y) \circ s) - \nabla^\omega_{\kappa_-(X,Y)} s}$$

Also the (generalized) *Bianchi identity* for the curvature κ is easily obtained:

$$\boxed{\sum_{\text{cycl}} \big([\kappa(X,Y), Z] - \kappa([X,Y], Z) - \kappa(\kappa_-(X,Y), Z) - \nabla^\omega_Z \kappa(X,Y)\big) = 0}$$

for all X, Y, $Z \in \mathfrak{g}_{-1}$, where $\sum_{\text{cycl}}$ denotes the sum over all cyclic permutations of the arguments. (The proofs of both claims in [CSS1] still apply.) In terms of the Lie algebra cohomology differential ∂ we may rewrite the latter formula as

$$-\partial\kappa(X,Y,Z) = \sum_{\text{cycl}} \big(\nabla^\omega_Z \kappa(X,Y) + \kappa(\kappa(X,Y), Z)\big).$$

In the very special case of affine connections (i.e. principal connections on the linear frame bundles together with the soldering forms), we recover exactly the classical identities.

Formulae in terms of Weyl geometries

In order to understand better our generalized Weyl geometries, we shall work out formulae relating the covariant differential ∇^{γ^σ} of the induced linear connections γ^σ and the invariant differential ∇^ω of the canonical Cartan connection ω. We shall also discuss the transformation of the covariant differentials under the change of the generalized Weyl geometry, i.e. in terms of the one-forms Υ, see 2.8. The resulting general formulae are somewhat messy, but quite simple. Since they get a bit more handy on the whole $\mathcal{G}$, we start with this case. As a straightforward consequence of our formulae, we shall be able to define distinguished Weyl geometries parameterized by closed forms. These are the analogues to metrics and Levi-Civita connections in conformal Riemannian geometries. The next Lemma also provides the analogy to the so called *Rho tensors* in conformal Riemannian geometry.

2.14. Lemma. *For each Cartan connection $\omega = \omega_- \oplus \omega_0 \oplus \omega_+ \in \Omega^1(\mathcal{G}, \mathfrak{g})$ and each G_0-equivariant global section σ, there is the unique Cartan connection ω^σ on $\mathcal{G}$ which is σ-related to $\sigma^*(\omega_- \oplus \omega_0)$. Then $\omega = \omega^\sigma - \mathsf{P} \circ \omega_-$ for a 2-tensor $\mathsf{P} \in C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathfrak{p}_+)^P \simeq C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathfrak{g}_-^*)^P$ on M.*

Proof. The Cartan connection ω^σ is defined by its restriction

$$\omega^\sigma_{|T\sigma(T\mathcal{G}_0)} = (\omega_- \oplus \omega_0)_{|T\sigma(T\mathcal{G}_0)}$$

and by the properties required for any Cartan connection. The easy check that the definition is consistent is left to the reader.

The Cartan connections ω and ω^σ differ only in the $\mathfrak{p}_+$-component and they both have to recover the fundamental vector fields on $\mathcal{G}$. Thus there must be a uniquely defined smooth function $\mathsf{P} \in C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathfrak{p}_+)$, such that the horizontal vector fields satisfy

$$\omega^{-1}(u)(X) = (\omega^\sigma)^{-1}(u)(X) + \zeta_{\mathsf{P}(u).X}(u)$$

for all $X \in \mathfrak{g}_-$ and $u \in \mathcal{G}$. Evaluation of ω^σ on $\xi = \omega^{-1}(u)(X)$ yields

$$\omega^\sigma(u) = \omega(u) - \mathsf{P}(u) \circ \omega_-(u), \quad \text{for all } u \in \mathcal{G}.$$

The equivariance of ω and ω^σ implies for all $b \in P$

$$(r^b)^*(\mathsf{P} \circ \omega_-) = (r^b)^*(\omega - \omega^\sigma) = \operatorname{Ad} b^{-1} \circ (\omega - \omega^\sigma) = \operatorname{Ad} b^{-1} \circ (\mathsf{P} \circ \omega_-).$$

On the other hand,

$$(r^b)^*(\mathsf{P} \circ \omega_-)(u)(\xi) = \mathsf{P}(u.b)(\omega_-(u.b)(Tr^b.\xi)) = \mathsf{P}(u.b)(\operatorname{Ad}_- b^{-1} \circ \omega_-(u)(\xi)).$$

Now, the choice $X = \operatorname{Ad}_- b^{-1} \circ \omega_-(u)(\xi)$ yields

$$\mathsf{P}(u.b) = \operatorname{Ad} b^{-1} \circ \mathsf{P}(u) \circ \operatorname{Ad}_- b.$$

The pairing $\mathfrak{p}_+ \times \mathfrak{g}_- \to \mathbb{R}$ defined by the Killing form then yields a mapping $(X, Y) \mapsto \langle \mathsf{P}(u).X, Y\rangle$ with the required equivariance

$$\langle \mathsf{P}(u.b).X, Y\rangle = \langle \operatorname{Ad}(b^{-1}).\mathsf{P}(u).\operatorname{Ad}_-(b)X, Y\rangle = \langle \mathsf{P}(u).\operatorname{Ad}_-(b)X, \operatorname{Ad}_-(b)Y\rangle.$$

Thus, we may view P as the frame form of a 2-tensor in this way. □

Following the tradition, the tensor P will be called the *Rho tensor* but also *deformation tensor* for σ (since P describes the deformation of ω^σ into ω). We shall also write P^σ whenever the specification of the chosen (generalized) Weyl geometry will be necessary.

Let us notice that the above proof works for all Cartan connections χ, ψ sharing the $\mathfrak{g}_-$ and $\mathfrak{g}_0$ components. The same computation shows that given a Cartan connection χ and a 2-tensor P on M, the 1-form $\psi = \chi - \mathsf{P} \circ \chi_-$ obeys the equivariance required for Cartan connections and the other two properties are clearly satisfied as well. Thus, all Cartan connection on a fixed principal bundle $\mathcal{G}$ which share the $\mathfrak{g}_-$ and $\mathfrak{g}_0$ components are sections of an affine bundle modeled over 2-tensors on the base manifold M.

The Rho tensors play a key role in conformal Riemannian geometry, but they proved to be useful also in projective and almost Grassmannian geometries, cf. [BaiE], [Eas], [Slo1], [Slo2]. Usually the authors introduced them because of their 'nice transformation properties' or via decompositions of the curvatures into irreducible components.

2.15. Notation. Given a G_0-equivariant section $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$, we define the mapping $\tau_\sigma \in C^\infty(\mathcal{G}, \mathfrak{p}_+)$ by

$$u = \sigma(p(u)).\exp \tau_\sigma(u).$$

We often omit the subscript σ if the dependence on σ is clear from the context.

Next we are going to compare the invariant differential ∇^ω with the covariant differential ∇^{γ^σ} for the principal connection γ^σ on $\mathcal{G}$, cf. 2.10. Recall that for each representation $\lambda_P\colon P \to GL(\mathbb{E}_\lambda)$, the sections of the associated bundle $\mathcal{G} \times_P \mathbb{E}_\lambda$ are viewed as equivariant mappings $s \in C^\infty(\mathcal{G}, \mathbb{E}_\lambda)^P$ and the covariant derivative ∇^{γ^σ} of s in the direction of a vector $\xi = \{u, X\} \in TM$ is given by the usual derivative of s in the direction of the horizontal lift of ξ with respect to γ^σ at u. This observation justifies also our notation $\nabla^{\gamma^\sigma}_X s(u)$ for the covariant derivative.

As usual, we shall write λ for the representation of $\mathfrak{p}$ on $\mathbb{E}_\lambda$ induced by λ_P. Since $\omega^{-1}(X) = (\omega^\sigma)^{-1}(X) + \zeta_{\mathsf{P}^\sigma.X}$, we have

$$(\nabla^\omega_X - \nabla^{\gamma^\sigma}_X)s(u) = (\nabla^{\omega^\sigma}_X - \nabla^{\gamma^\sigma}_X)s(u) - \lambda(\mathsf{P}^\sigma(u).X)(s(u))$$

for all representations $\lambda\colon \mathfrak{p} \to \mathfrak{gl}(\mathbb{E}_\lambda)$, $u \in \mathcal{G}$, and $s \in C^\infty(\mathcal{G}, \mathbb{E}_\lambda)^P$. Thus the main task will be to compare ∇^{ω^σ} and ∇^{γ^σ} on $\mathcal{G}$.

Let us notice that this comparison yields also formulae for the canonical covariant derivatives $\nabla^{\tilde{\omega}}$ on bundles coming from G-modules, cf. 2.12. More explicitly, the value of the horizontal lift of a tangent vector $\{u, X\} \in TM$ with respect to the principal connection $\tilde{\omega}$ at the frame $\{u,[e]\}\in\mathcal{G}\subset\tilde{\mathcal{G}}$ is $\omega^{-1}(X)(u)-\zeta_X(u)$. Thus the covariant differential $\nabla^{\tilde{\omega}}_X$ is given by the sum of our invariant differential ∇^{ω}_X and the action of X. In particular, we shall obtain the expression of the canonical convariant derivatives by means of the Weyl geometries and the induced decompositions into G_0-submodules, as promised in 2.12.

2.16. Lemma. *For each generalized Weyl geometry $\sigma\colon\mathcal{G}_0\to\mathcal{G}$, any section $s\in C^\infty(\mathcal{G},\mathbb{E}_\lambda)^P$, and $X\in\mathfrak{g}_-$ we have*

$$\begin{aligned}(\nabla^{\omega}_X-\nabla^{\gamma^\sigma}_X)s&=-\lambda\big(\mathsf{P}.X+\operatorname{Ad}\exp(-\tau).(\operatorname{Ad}\exp\tau.X)_{\mathfrak{p}}\big)\circ s\\&=-\lambda\Big(\mathsf{P}.X+\sum_{j=0}^{\ell}\tfrac{(-1)^j}{j!}\operatorname{ad}^j_\tau.\Big(\sum_{i=1}^{2\ell}\tfrac{1}{i!}(\operatorname{ad}^i_\tau.X)_{\mathfrak{p}}\Big)\Big)\circ s.\end{aligned}$$

In particular, if $X\in\mathfrak{g}_{-1}$ we obtain

$$(\nabla^{\omega}_X-\nabla^{\gamma^\sigma}_X)s=-\lambda([\tau,X]-\tfrac12[\tau,[\tau,X]]+\dots+\tfrac{(-1)^\ell}{(\ell+1)!}\operatorname{ad}^{\ell+1}_\tau.X+\mathsf{P}.X)\circ s.$$

Furthermore, if $\mathfrak{p}_+$ acts trivially on $\mathbb{E}_\lambda$, then

$$(\nabla^{\omega}_X-\nabla^{\gamma^\sigma}_X)s=-\lambda\Big(\sum_{i=1}^{\ell}\tfrac{1}{i!}(\operatorname{ad}^i_\tau.X)_{\mathfrak{g}_0}\Big)\circ s\qquad\textit{for all } X\in\mathfrak{g}_-,$$

$$(\nabla^{\omega}_X-\nabla^{\gamma^\sigma}_X)s(u)=\lambda([X,\tau(u)]_{\mathfrak{g}_0})(s(u))\qquad\textit{for all } u\in\mathcal{G},\ X\in\mathfrak{g}_{-1}.$$

Proof. For technical reason, let us simplify our notation for a moment. We fix the G_0-equivariant section $\sigma\colon\mathcal{G}_0\to\mathcal{G}$, we write γ for γ^σ and χ for ω^σ, and $\gamma^{-1}(X)(u)$ will denote the value of the horizontal vector field determined by the tangent vector $\{u,X\}\in\mathcal{G}\times_P\mathfrak{g}_-$.

The horizontal lifts of vectors $\xi\in TM$ with respect to γ are right-invariant vector fields on the fibers and $\{u,X\}=\{u.b,\operatorname{Ad}_-b^{-1}.X\}\in TM$. Thus

$$Tr^b.(\gamma^{-1}(u)(X))=\gamma^{-1}(\operatorname{Ad}_-b^{-1}.X)(u.b).$$

On the other hand, the equivariance of the Cartan connections yields for all $b\in P$, $u\in\mathcal{G}$, $X\in\mathfrak{g}_-$

$$Tr^b.(\chi^{-1}(u)(X))=\chi^{-1}(\operatorname{Ad}b^{-1}.X)(u.b).$$

Now, we insert $b=\exp\tau(u)$ and $\operatorname{Ad}_-b.X$ or $\operatorname{Ad}b.X$ instead of X, respectively, and compute

$$\begin{aligned}\nabla^\gamma s(u)(X)&=\nabla^\gamma s(\sigma(p(u)).\exp\tau(u))(X)\\&=Tr^{\exp\tau(u)}.\gamma^{-1}(\operatorname{Ad}_-(\exp\tau(u)).X)(\sigma(p(u))).s\\&=TsTr^{\exp\tau(u)}\gamma^{-1}(X+[\tau(u),X]_{\mathfrak{g}_-}+\dots+\tfrac{1}{2\ell!}(\operatorname{ad}^{2\ell}_{\tau(u)}.X)_{\mathfrak{g}_-})(\sigma(p(u)))\\\nabla^\chi s(u)(X)&=TsTr^{\exp\tau(u)}\chi^{-1}(X+[\tau(u),X]+\dots+\tfrac{1}{2\ell!}(\operatorname{ad}^{2\ell}_{\tau(u)}.X))(\sigma(p(u)))\end{aligned}$$

and since the values of $\chi^{-1}(X)$ and $\gamma^{-1}(X)$ coincide on the image of σ, the subtraction of these expressions yields

$$\begin{aligned}(\nabla^\chi - \nabla^\gamma)s(u)(X) &= Ts.Tr^{\exp\tau(u)}.\chi^{-1}(\sum_{i=1}^{2\ell}\tfrac{1}{i!}(\mathrm{ad}^i_{\tau(u)}.X)_{\mathfrak p})(\sigma(p(u))) \\ &= Ts.\chi^{-1}\Big(\mathrm{Ad}(\exp(-\tau(u))).\sum_{i=1}^{2\ell}\tfrac{1}{i!}(\mathrm{ad}^i_{\tau(u)}.X)_{\mathfrak p}\Big)(u)\end{aligned}$$

This implies already the first claim in the Lemma since the equivariance of s implies that the latter expression is

$$\begin{aligned}(\nabla^\chi - \nabla^\gamma)s(X) &= -\lambda\Big(\sum_{i=1}^{2\ell}\tfrac{1}{i!}(\mathrm{ad}^i_\tau.X)_{\mathfrak p} - \Big[\tau, \sum_{i=1}^{2\ell-1}\tfrac{1}{i!}(\mathrm{ad}^i_\tau.X)_{\mathfrak p}\Big] + \dots\Big)\circ s \\ &= -\lambda([\tau,X]_{\mathfrak p} + \tfrac{1}{2}[\tau,[\tau,X]]_{\mathfrak p} - [\tau,[\tau,X]_{\mathfrak p}] + \dots)\circ s.\end{aligned}$$

If $X \in \mathfrak g_{-1}$, then all brackets in this formula happen to be in $\mathfrak p$ and collecting all coefficients at ad^i_τ we obtain $\sum_{j=0}^{i-1}\frac{(-1)^j}{(i-j)!j!}$. Rewriting this sum by means of binomial coefficients, an easy computation yields the required coefficients $\frac{(-1)^{i+1}}{i!}$. The last claim is then obvious. □

2.17. Now, let us rewrite the formulae from the last lemma for fixed Weyl geometries. We shall write $\hat\gamma$ and γ for two choices of (generalized) Weyl geometries, Υ for the corresponding one-form, and $\hat\sigma$, σ for the G_0-equivariant sections of $\mathcal G \to \mathcal G_0$. In particular, $\hat\sigma(v) = \sigma(v).\Upsilon(v)$ for all $v \in \mathcal G_0$. Using the obvious equality

$$\nabla^{\hat\gamma}_X s - \nabla^\gamma_X s = (\nabla^\omega_X s - \nabla^\gamma_X s) - (\nabla^\omega_X s - \nabla^{\hat\gamma}_X s)$$

at the frame $u = \sigma(v)$ we obtain

$$\begin{aligned}\nabla^{\hat\gamma}_X s(\sigma(v)) &= \nabla^\gamma_X s(\sigma(v)) + \lambda\big((\hat{\mathsf P} - \mathsf P)(\sigma(v)).X\big)\big(s(\sigma(v))\big) \\ &\quad + \lambda\big(\mathrm{Ad}\exp\Upsilon(v).(\mathrm{Ad}\exp-\Upsilon(v).X)_{\mathfrak p})\big)\big(s(\sigma(v))\big)\end{aligned} \tag{1}$$

Next, observe

$$\nabla^{\hat\gamma}_X s(u) = \nabla^{\hat\gamma}_{\mathrm{Ad}_-\, b^{-1}.X} s(u.b).$$

Thus we can evaluate (1) with 'new coordinates' $\hat X = \mathrm{Ad}_-\exp(-\Upsilon(v)).X$ of the same tangent vector at the frame $\hat\sigma(v)$. The result can be easily interpreted in terms of the decompositions of the associated bundle $\mathcal E_\lambda = \mathcal G \times_P \mathbb E_\lambda$ determined by the choices of Weyl geometries $\hat\sigma$, σ. This means, we interpret s as the sections $\rho := s\circ\sigma = \rho_1 + \dots + \rho_k \in C^\infty(\mathcal G_0, V_1 \oplus \dots V_k)^{G_0}$ and $\hat\rho := s\circ\hat\sigma$, and we obtain the relation between the covariant derivatives of ρ and $\hat\rho$ with respect to the pricipal connections γ and $\hat\gamma$ on $\mathcal G_0$:

(2)

$$\nabla^{\hat\gamma}_{\hat X}\hat\rho(v) = \nabla^\gamma_X s_0(v) + \lambda\big((\hat{\mathsf P} - \mathsf P)(\sigma(v)).X + \mathrm{Ad}\exp\Upsilon(v).(\mathrm{Ad}\exp(-\Upsilon(v)).X)_{\mathfrak p}\big)(\rho(v))$$

In the very special case of an irreducible P-module $\mathbb{E}_\lambda$, there is no difference between $\hat\rho$ and ρ and we obtain for all $\rho \in C^\infty(\mathcal{G}_0, \mathbb{E}_\lambda)^{G_0}$, $X \in \mathfrak{g}_-$, $v \in \mathcal{G}_0$,

$$\nabla^{\hat\gamma}_X \rho(v) = \nabla^\gamma_{X+\sum_{i=1}^{\ell-1}\frac{1}{i!}(\mathrm{ad}^i_{\Upsilon(v)}.X)_{\mathfrak{g}_-}} \rho(v) - \lambda\Big(\sum_{i=1}^{\ell} \tfrac{1}{i!}(\mathrm{ad}^i_{\Upsilon(v)}.X)_{\mathfrak{g}_0}\Big)(\rho(v)) \tag{3}$$

These formulae reduce heavily if we restrict values of X and Υ suitably with respect to the grading. Let us also recall that the whole $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$ and this is reflected on the tangent bundles too, so that the case of $X \in \mathfrak{g}_{-1}$ is always of a particular interest. In particular, $\hat X = X$ in formula (2) while (3) reduces to the extremely simple equation

$$\nabla^{\hat\gamma}\rho(X) = \nabla^\gamma\rho(X) + \lambda([X,\Upsilon]) \circ \rho. \tag{4}$$

Thus, for all $|1|$-graded algebras we get exactly the formulae well known from conformal Riemannian geometries.

Our next aim is to express also the terms $\hat{\mathsf{P}} - \mathsf{P}$ via the one-forms Υ. This will fill the last gap in our understanding of the formulae.

2.18. Lemma. *Let $\mathcal{G}$ be a principal P-bundle, ω a Cartan connection on $\mathcal{G}$ and let σ_1, σ_2 be G_0-equivariant sections of $\mathcal{G} \to \mathcal{G}_0$, Υ the one-form satisfying $\sigma_2 = \sigma_1 . \exp \Upsilon$. Further, let γ_1 and γ_2 be the principal connections on $\mathcal{G}_0$ corresponding to σ_1 and σ_2. Then $\omega^{\sigma_2} = \omega^{\sigma_1} - \mathsf{P} \circ \omega_-$ with*

$$\begin{aligned}\mathsf{P}(\sigma_2(u))(X) &= \mathrm{Ad}(\exp -\Upsilon)\Big(\sum_{k\ge 0} \tfrac{1}{(k+1)!}\,\mathrm{ad}^k_\Upsilon .\nabla^{\gamma_2}_X \Upsilon - (\mathrm{Ad}\exp\Upsilon.X)_{\mathfrak{p}_+}\Big)(u)\\ &= \nabla^{\gamma_2}_X\Upsilon(u) - [\Upsilon(u),X]_{\mathfrak{p}_+} - \tfrac12[\Upsilon(u),\nabla^{\gamma_2}_X\Upsilon(u)] - \tfrac12[\Upsilon(u),[\Upsilon(u),X]]_{\mathfrak{p}_+}\\ &\quad + [\Upsilon(u),[\Upsilon(u),X]_{\mathfrak{p}_+}] + \dots\end{aligned}$$

Proof. We shall use the brief notation ω_i for the induced Cartan connections ω^{σ_i}, $i = 1, 2$. By definition, we have to compute

$$\zeta_{\mathsf{P}(\sigma_2(u))(X)}(\sigma_2(u)) = (\omega_2^{-1}(X) - \omega_1^{-1}(X))(\sigma_2(u)).$$

The horizontal lifts of the vector $\{u, X\} \in TM$ are given by

$$\gamma_i^{-1}(X)(u) = Tp_0^\ell.\omega^{-1}(X)(\sigma_i(u)) \in T\mathcal{G}_0.$$

In particular we obtain

$$\begin{aligned}\gamma_2^{-1}(X)(u) &= Tp_0^\ell.Tr^{\exp\Upsilon(u)}\omega^{-1}(\mathrm{Ad}\exp\Upsilon(u).X)(\sigma_1(u))\\ &= Tp_0^\ell\omega^{-1}(\mathrm{Ad}_-\exp\Upsilon(u).X)(\sigma_1(u)) + Tp_0^\ell\omega^{-1}((\mathrm{Ad}\exp\Upsilon(u).X)_{\mathfrak{g}_0})(\sigma_1(u))\\ &= \gamma_1^{-1}(\mathrm{Ad}_-\exp\Upsilon(u).X) + \zeta_{(\mathrm{Ad}\exp\Upsilon(u).X)_{\mathfrak{g}_0}}(u)\end{aligned}$$

Further, $\omega_i^{-1}(X)(\sigma_i(u)) = T\sigma_i(\gamma_i^{-1}(X)(u))$, $i = 1, 2$. Thus, the equivariance of the horizontal fields and the definition of Υ yield

$$\begin{aligned}\zeta_{\mathsf{P}(\sigma_2(u))(X)}(\sigma_2(u)) &= T\sigma_2.\gamma_2^{-1}(X)(u) - T\sigma_1.\gamma_1^{-1}(X)(u)\\ &= Tr.(T\sigma_1, T\exp\Upsilon).\gamma_2^{-1}(X)(u)\\ &\quad - Tr^{\exp(\Upsilon(u))}.\omega_1^{-1}(\mathrm{Ad}\exp\Upsilon(u).X)(\sigma_1(u))\end{aligned} \tag{1}$$

In order to resolve the first term in (1), let us choose a curve $c(t)$ in $\mathcal{G}_0$ such that $\frac{\partial}{\partial t}_{|0} c(t) = \gamma_2^{-1}(X)(u)$. Then

$$\begin{aligned}&Tr.(T\sigma_1, T\exp\Upsilon).\gamma_2^{-1}(X)(u) =\\ &\quad = \tfrac{\partial}{\partial t}_{|0}(\sigma_1(u).\exp\Upsilon(c(t))) + Tr^{\exp\Upsilon(u)}.T\sigma_1(\gamma_2^{-1}(X))(u)\\ &\quad = Tr^{\exp\Upsilon(u)}.\tfrac{\partial}{\partial t}_{|0}(\sigma_1(u).\exp\Upsilon(c(t)).\exp-\Upsilon(u)) +\\ &\quad\quad Tr^{\exp\Upsilon(u)}.T\sigma_1(\gamma_1^{-1}(\mathrm{Ad}_-\exp\Upsilon(u).X) + \zeta_{(\mathrm{Ad}\exp\Upsilon(u).X)_{\mathfrak{g}_0}})(u)\end{aligned}$$

Now, the first term is of the form $Tr^{\exp\Upsilon(u)}\zeta_A(\sigma_1(u))$ where A is the right logarithmic derivative δ of the function $\exp\Upsilon\colon \mathcal{G}_0 \to G$, evaluated on $\gamma_2^{-1}(X)$. Thus

$$\begin{aligned}A &= (Tr^{\exp-\Upsilon(u)}) \circ (T\exp) \circ (T\Upsilon)(\tfrac{\partial}{\partial t}_{|0} c(t)) = (\delta\exp)(\Upsilon(u)).(\nabla_X^{\gamma_2}\Upsilon(u))\\ &= \sum_{k\geq 0} \tfrac{1}{(k+1)!}\,\mathrm{ad}^k_{\Upsilon(u)}.\nabla_X^{\gamma_2}\Upsilon(u)\end{aligned}$$

see e.g. [KMS, p.39] for the formula for $\delta\exp$.

Next, the second term in (1) splits as

$$-Tr^{\exp(\Upsilon(u))}.\big(\omega_1^{-1}(\mathrm{Ad}_-\exp\Upsilon(u).X) + \zeta_{(\mathrm{Ad}\exp\Upsilon(u).X)_{\mathfrak{p}}}\big)(\sigma_1(u))$$

and we can collect easily all terms in (1). We obtain

$$\zeta_{\mathsf{P}(\sigma_2(u))(X)}(\sigma_2(u)) = Tr^{\exp\Upsilon(u)}\zeta_B(\sigma_1(u))$$

where

$$B = \sum_{k\geq 0} \tfrac{1}{(k+1)!}\,\mathrm{ad}^k_{\Upsilon(u)}.\nabla_X^{\gamma_2}\Upsilon(u) - (\mathrm{Ad}\exp\Upsilon(u).X)_{\mathfrak{p}_+}.$$

Finally, the equivariance of P yields the required formula

$$\mathsf{P}(\sigma_2(u))(X) = \mathrm{Ad}(\exp-\Upsilon(u))\Big(\sum_{k\geq 0} \tfrac{1}{(k+1)!}\,\mathrm{ad}^k_{\Upsilon(u)}.\nabla_X^{\gamma_2}\Upsilon(u) - (\mathrm{Ad}\exp\Upsilon(u).X)_{\mathfrak{p}_+}\Big).$$

□

Now, we can insert the formula from the last Lemma into 2.17.(1) and 2.17.(2).

2.19. Proposition. *Let $\hat{\gamma}$ and γ be two choices of (generalized) Weyl geometries on $\mathcal{G} \to M$ given by sections $\hat{\sigma}$, σ, and $\hat{\sigma}(v) = \sigma(v).\Upsilon(v)$ for all $v \in \mathcal{G}_0$. Then for each section $s \in C^\infty(\mathcal{G}, \mathbb{E}_\lambda)$, $v \in \mathcal{G}_0$, $X \in \mathfrak{g}_-$, $\hat{X} = \mathrm{Ad}_- \exp -\Upsilon(u).X$*

(1)
$$\begin{aligned}\nabla^{\hat{\gamma}}_{\hat{X}} s(\sigma_2(v)) = \nabla^{\gamma}_{X} s(\sigma_1(v)) + \lambda_P(\exp \Upsilon(v))\Big(&\lambda\big(\mathrm{Ad}\exp -\Upsilon(v).X\big)_{\mathfrak{g}_0} \\ &- \sum_{k\geq 0} \tfrac{1}{(k+1)!}\,\mathrm{ad}^k_{-\Upsilon(v)}\, \nabla^{\gamma}_{X}\Upsilon(v)\big)\big(s(\hat{\sigma}(v))\big)\Big)\end{aligned}$$

Proof. We have just to insert the expression for $\hat{\mathsf{P}} - \mathsf{P}$ into 2.17.(1). Since (in the notation of 2.18)
$$\omega = \omega^{\sigma_2} - \mathsf{P}^{\sigma_2} \circ \omega_- = \omega^{\sigma_1} - \mathsf{P}^{\sigma_1} \circ \omega_-$$
we have to replace γ_1, γ_2, σ_1, σ_2, Υ in 2.18 by $\hat{\gamma}$, γ, $\hat{\sigma}$, σ, $-\Upsilon$, respectively. Finally, we can use the relation $\lambda(\mathrm{Ad}\, b.Z)(s(v)) = \lambda_P(b)(\lambda(Z)(s(v.b)))$ to extract the common term $\mathrm{Ad}\exp\Upsilon(v)$. This yields our formula. □

2.20. Corollaries. *In the notation of 2.19 and 2.17, the formula 2.17.(2) gets*

(1)
$$\begin{aligned}\nabla^{\hat{\gamma}}_{\hat{X}} \hat{\rho} = \nabla^{\gamma}_{X} \rho + \lambda_P(\exp \Upsilon)\Big(&\lambda\big(\mathrm{Ad}\exp -\Upsilon(v).X\big)_{\mathfrak{g}_0} \\ &- \sum_{k\geq 0} \tfrac{1}{(k+1)!}\,\mathrm{ad}^k_{-\Upsilon}\, \nabla^{\gamma}_{X}\Upsilon\big)(\hat{\rho})\Big).\end{aligned}$$

The part of this formula up to linear terms in Υ is as follows

(2) $$\nabla^{\hat{\gamma}}_{\hat{X}} \hat{\rho} = \nabla^{\gamma}_{X} \rho + \lambda\big([X,\Upsilon]_{\mathfrak{g}_0} - \nabla^{\gamma}_{X}\Upsilon\big)(\rho) + \textit{higher order terms in } \Upsilon.$$

Let us also notice that two G_0-equivariant sections $\sigma_1, \sigma_2 \colon \mathcal{G}_0 \to \mathcal{G}$ define the same G_0-structure if and only if their compositions with $p^{\ell}_{\ell-1}$ coincide, which in turn means that the corresponding one-form $\Upsilon \colon \mathcal{G}_0 \to \mathfrak{p}_+$ has values in $\mathfrak{g}_\ell$ only. Also in this case, the formulae become particularly simple.

Scales and closed Weyl geometries

2.21. Definition. Let M be a manifold equipped with a parabolic geometry of type G/P. A (generalized) Weyl geometry $\sigma \colon \mathcal{G}_0 \to \mathcal{G}$ is called *closed* if the curvature of the corresponding principal connection γ^σ on $\mathcal{G}$ satisfies $\langle \kappa^\sigma(u)(X,Y), E\rangle = 0$ for all $u \in \mathcal{G}$, $X, Y \in \mathfrak{g}_-$.

In conformal Riemannian geometry, the Weyl geometry γ^σ is closed if and only if it locally coincides with Levi-Civita connections. Our aim is to establish affine bundles similar to the line bundles of conformal metrics.

Let us observe that the defining condition for closed Weyl geometries is given by vanishing of the representation $\nu\colon \mathfrak{p} \to \mathbb{R}$, $\nu(X) = \langle X, E\rangle$. Similarly, we may start with an arbitrary $\nu\colon \mathfrak{p} \to \mathbb{R}$, i.e. with an arbitrary complement of E in the center $\mathfrak{z} \subset \mathfrak{g}_0$.

2.22. Distinguished line bundles. Consider a fixed representation $\nu_P\colon P \to GL^+(\mathbb{R})$ and the corresponding Lie algebra representation $\nu\colon \mathfrak{p} \to \mathbb{R}$, such that the grading element E acts by $\nu(E) = 1$. Clearly $P/\ker\nu \simeq GL^+(\mathbb{R}) = \mathbb{R}_+$. The nilpotent part P_+ of P and the semisimple part $G' \subset G_0$ are both in $\ker\nu_P$, and the factor group $P/\ker\nu$ is identified with the one-parametric subgroup $\exp tE$ generated by the grading element E.

The Lie algebra $\mathfrak{g}_0$ of the reductive part G_0 of P decomposes as the sum of two ideals, its center $\mathfrak{z}$ and its semisimple part $\mathfrak{g}_{0,s} = [\mathfrak{g}_0, \mathfrak{g}_0]$. Whenever we fix a complement $\mathfrak{z}'$ of the one-dimensional subspace $\mathfrak{z}_E = \langle E\rangle$ in $\mathfrak{z}$, there is the unique representation $\nu\colon \mathfrak{p} \to \mathbb{R}$ with $\nu(E) = 1$ and $\mathfrak{z}' \subset \ker\nu$, but the existence of ν_P to this data is not always available. In fact a very natural ν is defined by the Killing form, simply we define $\mathfrak{z}' = E^\perp$. Unfortunately, even such a natural choice does not always leed to integrable representations. We shall meet an example in Section 5. Thus we have to leave the subtle disscusion on the proper choices of ν_P to the study of the particular geometries, while working with a general ν_P here.

On the other hand, given a parabolic geometry on $\mathcal{G} \to M$ of type G/P and ν_P, there is the one-dimensional principal fiber bundle $\mathcal{S} = \mathcal{G}/\ker\nu$. The mapping $\mathcal{G} \ni u \mapsto (u, 1) \in \mathcal{G} \times \mathbb{R}_+$ yields the identification $\mathcal{S} = \mathcal{G} \times_\nu \mathbb{R}_+$. The exponential mapping $\mathfrak{z}_E \to \{\exp tE\}$ defines the structure of the affine bundle on $\mathcal{S}$ with the trivial modeling line bundle $M \times \mathfrak{z}_E$

We call $\mathcal{S}$ the affine *bundle of scales*. If necessary, we shall write $\mathcal{S}_{\nu_P}$ to indicate the chosen representation ν_P.

2.23. Lemma. *Let $\gamma^\sigma \in \Omega^1(\mathcal{G}, \mathfrak{p})$ be the principal connection on $\mathcal{G}$ defined by the choice of a G_0-equivariant section $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$. The induced connection on $\mathcal{S}$ is the principal connection $\gamma^\sigma_{\mathcal{S}}$ with connection form $\nu \circ \gamma^\sigma$. Its curvature $\kappa^\sigma_{\mathcal{S}}$ is given by the composition $\kappa^\sigma_{\mathcal{S}} = \nu \circ \kappa^\sigma$, where κ^σ is the curvature of γ^σ.*

If the one-dimensional othogonal complement to $\mathfrak{z} \cap \ker\nu$ in $\mathfrak{z}$ acts non-trivially on the whole $\mathfrak{p}_+$, then the correspondence $\sigma \mapsto \gamma^\sigma_{\mathcal{S}}$ between the (generalized) Weyl geometries and the induced connections on the affine bundle of scales is bijective.

Proof. Let us recall some general features of induced connections on associated bundles. If γ is a principal connection on $\mathcal{G}$, then its vertical projection is given by $\Phi = \zeta \circ \gamma$, i.e. by the composition of the fundamental vector field mapping with γ. The vertical projection of the connection induced on a associated bundle $\mathcal{G} \times_P V$ is then given by $\{\Phi, \mathrm{id}_V\}$. Moreover, if V is a vector bundle and the action of P on V is given by $\rho_P\colon P \to GL(V)$, then the curvature of the induced connection is defined by the composition of the corresponding representation $\rho\colon \mathfrak{p} \to \mathfrak{gl}(V)$ with the curvature form of γ (see e.g. [KMS, 11.8-11.16]).

In our case, V is the quotient of P by the normal subgroup $\ker\nu$ and the induced connections $\gamma^g_{\mathcal{S}}$ are again principal connections. Obviously, the above vertical projetion $\{\zeta\circ\gamma^\sigma, \mathrm{id}_{\mathbb{R}}\}$ is now given by the composition of the fundamental vector field mapping $\zeta^{\mathcal{S}}\colon \mathfrak{p}/\ker\nu \to \mathcal{X}(\mathcal{S})$ with the quotient of the original connection form $\gamma\in\Omega^1(\mathcal{G},\mathfrak{p})$. Now, due to our assumption $\nu(E)=1$, we additionally identify $\mathfrak{p}/\ker\nu$ with $\mathfrak{z}_E\simeq\mathbb{R}$ via ν itself.

Next, let us prove the injectivity of the correspondence $\sigma\mapsto\gamma^g_{\mathcal{S}}$. It suffices to study the horizontal lifts $\gamma^g_{\mathcal{S}}\xi$ of a fixed tangent vector $\xi=\{u,X\}\in TM$ and the key observation is that the formulae 2.17.(1) and 2.18 yield the necessary comparison of $\gamma^{\hat\sigma}\xi$ and $\gamma^\sigma\xi$ in terms of Υ (see also 2.19). After factoring out the kernel of ν_P, we obtain

$$\gamma^{\hat g}_{\mathcal{S}}\xi([u]) = \gamma^g_{\mathcal{S}}\xi([u]) - \zeta\big((\operatorname{Ad}\exp(-\Upsilon(u)).X)_{\mathfrak{p}/\ker\nu}\big)([u]).$$

Thus we have to show that for each fixed $\Upsilon(u)$, there always is some $X\in\mathfrak{g}_-$ with $\nu(\operatorname{Ad}\exp(-\Upsilon(u)).X)\neq 0$. Assume that $\mathfrak{g}_i\subset\mathfrak{p}_+$ is the first component with non-zero values of $\Upsilon(u)$ and take $X\in\mathfrak{g}_{-i}$. Then the only interesting contribution is $\nu([X,\Upsilon(u)])$. Let us assume that $\dim\mathfrak{z}=k\le\ell$ and choose an orthogonal basis $E_1,\dots,E_{k-1}$ of $\mathfrak{g}'=\mathfrak{g}\cap\ker\nu$ with respect to the Killing form. Completing the orthogonal basis of $\mathfrak{z}$ by $\tilde E = E+\varphi(E_1,\dots,E_k)$ with a suitable linear combination $\varphi(E_1,\dots,E_k)$, the action $\nu([X,\Upsilon(u)])\in\mathbb{R}$ equals

$$\tfrac{1}{\|\tilde E\|^2}\langle[X,\Upsilon(u)],\tilde E\rangle = \tfrac{1}{\|\tilde E\|^2}\langle\Upsilon(u),[\tilde E,X]\rangle.$$

Now, $\mathfrak{g}_-$ splits into irreducible components with respect to the adjoint action of $\mathfrak{g}_0$ and each element in $\mathfrak{z}$ acts by a scalar on each of those components. According to our assumption, the action of $\tilde E$ is non-zero everywhere. Thus all $X\in\mathfrak{g}_{-i}$, except the hyperplane $\Upsilon(u)(X)=0$, provide the required element.

Finally, let us observe that the connection forms on $\mathcal{S}$ have values in the one-dimensional Lie algebra $\mathbb{R}$. So they form an affine space modeled over one-forms on M. The same is true for the Weyl geometries σ. Thus the rest of our claims follows pointwise by dimension reasons. □

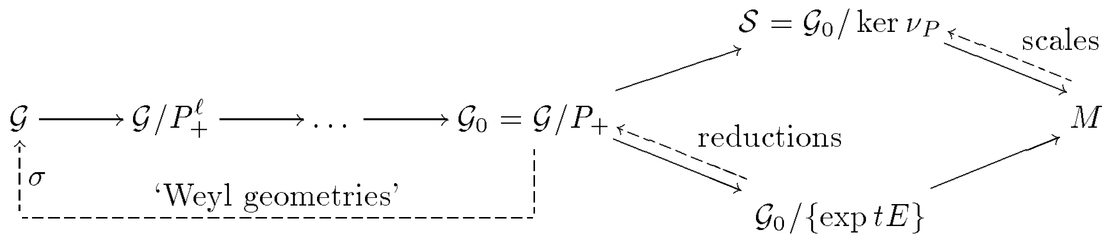
2.24. Under the technical assumption on the orthogonal complement to $\ker\nu$, the previous Lemma yields the promised generalization of the class of Levi-Civita connections in conformal geometries. Since the fibers of $\mathcal{S}$ are diffeomorpic to $\mathbb{R}$, this principal fiber bundle always admits global sections. Each section of $\mathcal{S}$ defines a trivial connection on $\mathcal{S}$, thus also a distinguished (generalized) Weyl geometry on $\mathcal{G}\to M$. Obviously, ν_P comes from a representation of $\mathcal{G}_0$, extended trivially to P and so $\mathcal{S}$ can be also viewed as the quotient of $\mathcal{G}_0/\ker\nu$. In particular, sections of $\mathcal{S}$ are in bijective correspondence with reductions of $\mathcal{G}_0$ to $\ker\nu_P\cap G_0$. The latter reduction can be identified with $\mathcal{G}_0/\{\exp tE\}$.

$$\begin{array}{ccc}
\{\text{smooth sections of } L\} & \longleftrightarrow & \{\text{trivial connections on } L\}\\
\updownarrow & & \updownarrow\\
\left\{\begin{array}{l}\text{distinguished class of reduc-}\\ \text{tions of } \mathcal{G}_0 \text{ to } G_0\cap\ker\nu_P\end{array}\right\} & \longleftrightarrow & \left\{\begin{array}{l}\text{distinguished subclass of}\\ \text{connections } \gamma^\sigma \text{ on } \mathcal{G}_0\end{array}\right\}
\end{array}$$

2.25. Definition. Let M be a manifold equipped with a parabolic geometry of type G/P. An *admissible bundle of scales* is the affine bundle S_{ν_P} such that the corresponding Lie algebra representation $\nu\colon \mathfrak{p} \to \mathfrak{gl}(\mathbb{R})$ has a kernel whose orthogonal complement in the center $\mathfrak{z}$ acts non-trivially on the whole $\mathfrak{p}_+$. Let $\mathcal{S}$ be such a bundle. A *scale* on M is a section of $\mathcal{S} \to M$. Weyl geometries γ^σ whose curvatures κ^σ satisfy $\nu \circ \kappa_\sigma = 0$ are called *closed with respect to* ν. Weyl geometries determined by scales are called *exact* (we shall often call them also *scales*).

2.26. Remarks. As already mentioned, the problems with the existence of all these objects are hidden in the existence of the representation ν_P itself. We shall see on examples that only some of the reasonable representations of the Lie algebra do integrate to the group level, but even then a particular choice of G may be necessary. This will of course put additional requirements on the geometries in question and the scales might then exist only locally. On the other hand, the definition of the closed Weyl geometries with respect to ν does depend on the Lie algebra representation ν only and could be used independently of the existence problems with the scales.

The whole situation is nicely visible on the following diagram. Recall that all compositions of a section σ with the quotient projections are also G_0-equivariant and we obtain principal connections γ^σ on all levels in the diagram. The last but one level $\mathcal{G}/P_+^\ell$ is of particular interest because its structure group P/P_+^ℓ is the effective transformation group of $\mathfrak{g}_-$.



Let us come back to the natural candidate for ν, the representation given by the Killing form and recall that the evaluation of the one-form Υ on X is also given by the Killing form. Chosing two Weyl geometries σ_2, σ_1, with $\sigma_2(u) = \sigma_1(u).\Upsilon(u)$, the equivariance of the Killing form implies for all $X \in \mathfrak{g}_j$

$$\begin{aligned}\nu \circ (\gamma^{\sigma_2} - \gamma^{\sigma_1})(\xi_u) &= \tfrac{1}{\langle E,E\rangle}\langle \operatorname{Ad}_{\exp -\Upsilon(u)}.X, E\rangle = \tfrac{1}{\langle E,E\rangle}\langle X, \operatorname{Ad}_{\exp \Upsilon(u)}.E\rangle \\ &= \tfrac{1}{\langle E,E\rangle}\langle X, E + [\Upsilon(u), E] + \tfrac{1}{2}[\Upsilon(u),[\Upsilon(u),E]] + \dots\rangle \\ &= \tfrac{-1}{\langle E,E\rangle}\langle X, \textstyle\sum_{i=1}^{\ell} i\Upsilon_i(u) + \text{higher order terms in } \Upsilon\text{'s}\rangle \\ &= \tfrac{1}{\langle E,E\rangle} j\Upsilon(u)(X) + \text{higher order terms in } \Upsilon\text{'s}.\end{aligned}$$

Consequently, the variation of the connection forms on $\mathcal{S}$ is linked to the one forms Υ in this way. In particular, they coincide for all $|1|$-graded Lie algebras $\mathfrak{g}$.

If we choose two scales, i.e. two global sections $\hat{s}$, s of $\mathcal{S}$ and $\hat{s} = f.s$ for a positive smooth function f, then

$$\nabla^{\hat{\gamma}s}_{\xi} s = \nabla^{\hat{\gamma}s}_{\xi} f^{-1}\hat{s} = -f^{-1}df(\xi).s$$

Thus the one-form corresponding to f is always the closed form $-f^{-1}df$.

Curvatures and torsions

In order to complete the discussion of the basic objects along the lines of the conformal geometries, we should understand better the curvatures and torsions of the Weyl geometries. First step towards this task is the next Proposition.

Let us recall that given the Weyl geometry $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$, the induced Cartan connection ω^σ is σ-related to the principal connection γ^σ on $\mathcal{G}_0$. Thus the torsion and curvature of γ^σ are σ-related to the $\mathfrak{g}_- \oplus \mathfrak{g}_0$ component of the curvature κ^σ of ω^σ as well. The next Proposition describes the difference between κ^σ and the canonical curvature κ on $\mathcal{G}$. This is only one of the sources of the variation of torsions and curvatures of the Weyl geometries, in general. The other contribution comes then from the evaluations in different frames in $\mathcal{G}$. The results admit a nice presentation in terms of the Lie algebra differential ∂. In particular, the values $\mathsf{P}(u)$ can be viewed as cochains $\mathsf{P}(u) \in \mathfrak{g}_-^* \otimes \mathfrak{p}_+ \subset C^1(\mathfrak{g}_-, \mathfrak{g})$ and then $\partial\mathsf{P} \in C^\infty(\mathcal{G}, C^2(\mathfrak{g}_-, \mathfrak{g}))$,

$$\partial\mathsf{P}(u)(X,Y) = [X, \mathsf{P}(u).Y] - [Y, \mathsf{P}(u), X] - \mathsf{P}(u).[X,Y].$$

2.27. Proposition. *Let κ^σ be the curvature of ω^σ, let P be the corresponding deformation tensor. Then for all $u \in \sigma(\mathcal{G}_0)$*

$$\begin{aligned}(\kappa^\sigma - \kappa)(u)(X,Y) = \partial\mathsf{P}(u)(X,Y) &+ \nabla_X \mathsf{P}(u).Y - \nabla_Y \mathsf{P}(u).X \\ &- [\mathsf{P}(u).X, \mathsf{P}(u).Y] + \mathsf{P}(u) \circ \kappa^\sigma_-(u)(X,Y)\end{aligned}$$

Proof. We shall prove the Proposition for arbitrary two Cartan connections ψ and χ which differ only in the $\mathfrak{p}_+$-components. Let us write κ^ψ and κ^χ for their curvatures.

By definition

$$\begin{aligned}(\kappa^\chi - \kappa^\psi)(X,Y) &= \psi([\psi^{-1}(X), \psi^{-1}(Y)]) - [X,Y] - \chi([\chi^{-1}(X), \chi^{-1}(Y)]) + [X,Y] \\ &= (\chi - \mathsf{P} \circ \chi_-)([\chi^{-1}(X), \chi^{-1}(Y)] + [\zeta_{\mathsf{P}.X}, \chi^{-1}(Y)] + \\ &\qquad [\chi^{-1}(X), \zeta_{\mathsf{P}.Y}] + [\zeta_{\mathsf{P}.X}, \zeta_{\mathsf{P}.Y}]) - \chi([\chi^{-1}(X), \chi^{-1}(Y)])\end{aligned}$$

In order to resolve the individual terms, we are going to evaluate the structure equation on appropriate data.

$$\begin{aligned}&d\chi(\chi^{-1}(\mathsf{P}.X), \chi^{-1}(Y)) = -[\mathsf{P}.X, Y] + 0 \\ &\quad = \mathcal{L}_{\chi^{-1}(Y)}\chi(\chi^{-1}(\mathsf{P}.X)) - \chi([\chi^{-1}(\mathsf{P}.X), \chi^{-1}(Y)]) \\ &d\chi(\chi^{-1}(\mathsf{P}.X), \chi^{-1}(\mathsf{P}.Y)) = -[\mathsf{P}.X, \mathsf{P}.Y] + 0 \\ &\quad = \mathcal{L}_{\chi^{-1}(\mathsf{P}.X)}\chi(\chi^{-1}(\mathsf{P}.Y)) - \mathcal{L}_{\chi^{-1}(\mathsf{P}.Y)}\chi(\chi^{-1}(\mathsf{P}.X)) - \chi([\chi^{-1}(\mathsf{P}.X), \chi^{-1}(\mathsf{P}.Y)])\end{aligned}$$

Further, for any $Z \in \mathfrak{p}$ we have

$$\mathcal{L}_{\chi^{-1}(Z)}(\mathsf{P}.Y) = \frac{\partial}{\partial t}_{|0}\big(\mathrm{Ad}(\exp -tZ)(\mathsf{P}.(\mathrm{Ad}(\exp tZ).Y))\big) = \mathsf{P}.[Z,Y] - [Z,\mathsf{P}.Y]$$

and so the second equality above yields

$$\chi([\chi^{-1}(\mathsf{P}.X),\chi^{-1}(\mathsf{P}.Y)]) = \mathsf{P}.[\mathsf{P}.X,Y] - \mathsf{P}.[\mathsf{P}.Y,X] - [\mathsf{P}.X,\mathsf{P}.Y]$$

Finally, we can collect (and cancel) all terms

$$\begin{aligned}(\kappa^\chi - \kappa^\psi)(X,Y) &= \begin{cases} \mathsf{P}\circ\kappa^\chi_-(X,Y) - \mathsf{P}.[X,Y] + \\ [\mathsf{P}.X,Y] - \mathsf{P}.[\mathsf{P}.X,Y] - \nabla^\chi_Y \mathsf{P}.X - \\ [\mathsf{P}.Y,X] + \mathsf{P}.[\mathsf{P}.Y,X] + \nabla^\chi_X\mathsf{P}.Y + \\ \mathsf{P}.[\mathsf{P}.X,Y] - \mathsf{P}.[\mathsf{P}.Y,X] - [\mathsf{P}.X,\mathsf{P}.Y] \end{cases} \\ &= \partial\mathsf{P}(X,Y) + (\nabla^\chi_X\mathsf{P}.Y - \nabla^\chi_Y\mathsf{P}.X) \\ &\quad - [\mathsf{P}.X,\mathsf{P}.Y] + \mathsf{P}\circ\kappa^\chi_-(u)(X,Y) \qquad \square\end{aligned}$$

Let us collect some corollaries for the curvatures and torsions of the Weyl geometries which are σ-related to the curvature κ^σ.

2.28. Observations.

(1) *We meet exactly the same behavior as in the conformal Riemannian geometries for all $|1|$-graded algebras $\mathfrak{g}$.*

(2) *Only the Lie algebra cohomology differential of P^σ can be involved in the $\mathfrak{g}_-$ component (i.e. in the torsion part) and in the $\mathfrak{g}_0$ component (the Weyl curvature).*

(3) *We have rather to decompose κ^σ by the homogeneities than by components in $\mathfrak{g}$. The non-vanishing component κ^i of the lowest degree is constant on the fibers of $\mathcal{G}$.*

(4) *Clearly, $\partial P(u)(\mathfrak{g}_i \wedge \mathfrak{g}_j) \subset \mathfrak{g}_{\min\{i,j\}+1} \oplus \dots \oplus \mathfrak{g}_\ell$ and so all torsion components $\mathfrak{g}_i \wedge \mathfrak{g}_j \to \mathfrak{g}_{\leq \min\{i,j\}}$ of κ^σ are equal to those of κ. In particular, all κ^i with $i \leq 1$ are always shared by all κ^σ.*

(5) *The homogeneous components of κ^σ of degrees $k \geq 2$ depend explicitly on P^σ*

The next theorem, together with the Kostant's version of the Bott-Borel-Weil theorem, yields a very efficient tool for discussions on local invariants of normal parabolic geometries. The proof can be found in [CSch]. See also 6.8-6.10 for basic exposition of the Lie algebra cohomology $H^*_*(\mathfrak{g}_-,\mathfrak{g})$, its Hodge structure, and the BBW-theorem.

2.29. Theorem. *The curvature κ of a normal Cartan connection vanishes if and only if its harmonic part does. Moreover, if all homogeneous components of κ of degrees less than j vanish identically and there is no cohomology $H^2_j(\mathfrak{g}_-,\mathfrak{g})$, then also the curvature component of degree j vanishes.*

2.30. Consequences. The latter theorem shows that it is again the Lie algebra cohomology (this time $H^2(\mathfrak{g}_-, \mathfrak{g})$) which causes vanishing of many parts of κ. In fact the behavior of regular and normal parabolic geometries is always similar to conformal Riemannian geometries. There, in dimensions $m > 3$, κ_+ is essentially the Cotton-York tensor, which is non-zero in general but vanishes automatically whenever the Weyl curvature does. On three-dimensional manifolds, κ_1 is the first non-vanishing part of the curvature, thus it is constant along the fibers of $\mathcal{G}$ and yields an invariant, the Cotton-York tensor.

As an immediate corollary we see that the first possibly non-vanishing homogeneity in the curvature κ of a normal parabolic geometry is that one corresponding to the first non-vanishing cohomology. If the geometry is regular, this must be of positive homogeneous degree. Unfortunately, even for regular and normal geometries only the homogeneous component of degree one is shared by all torsions of the Weyl geometries, since the values of the deformation tensors P^σ enter otherwise, together with the effect of the evaluation in different frames. Thus even if there is no cohomology in the torsion part, this could only imply that some parts of the torsions of the Weyl geometries will vanish in general (analogously to the behavior of the curvature in the conformal Riemannian case).

2.31. Remark. Proposition 2.27 also helps a bit to understand the curvatures of the connections on the affine bundles of scales $\mathcal{S}_{\nu_P}$ in terms of the deformation tensors of the corresponding Weyl geometries. In fact, we have to study $\nu \circ \kappa^\sigma$, evaluated at $\sigma(v)$, $v \in \mathcal{G}_0$.

For the sake of simplicity, let us deal directly with the $\mathfrak{z}_E$ component of the curvature κ^σ for closed Weyl geometries. This will also give the required information of the curvature $\kappa^{\mathfrak{g}}_{\mathcal{S}}$ with ν defined by means of the Killing form, whenever this bundle exists. Only the following part of $\langle E, E\rangle.\nu(\kappa^\omega - \kappa^{\omega^\sigma})$ survives (notice also $\langle E, E\rangle = 2\sum_{i=1}^{\ell} i^2 \dim \mathfrak{g}_j$)

$$\langle [X, \mathsf{P}.Y] - [Y, \mathsf{P}.X], E\rangle = j\mathsf{P}(X, Y) - i\mathsf{P}(Y, X), \quad X \in \mathfrak{g}_{-i},\ Y \in \mathfrak{g}_{-j}$$

which is a sort of *weighted antisymmetrization* of P, viewed as a 2-tensor. This gives us two conclusions:

- For all closed Weyl geometries σ, the $\mathfrak{z}_E$ component of the curvature κ of the defining Cartan connection ω equals to the weighted antisymmetric part of P^σ on the image $\sigma(\mathcal{G}_0)$
- In the presence of the affine bundle of scales $\mathcal{S}_{\nu_P}$, the curvatures of the induced connections $\gamma^{\mathfrak{g}}_{\mathcal{S}}$ on $\mathcal{S}$ are given by the differences of the component $\kappa_{\mathfrak{z}_E}$ of the canonical curvature κ and the weighted antisymmetrizations of the Rho tensors P^σ.

Analogies to conformal circles

The conformal circles and their various generalizations have been studied intensively, see e.g. [BaiE], [Eas], [CheM]. We present an extremely simple approach to such distinguished curves for all parabolic geometries. In fact, we simply extend the classical results on affine connections, which say that a curve is a (non-parameterized) geodesic line if and only if its development is a line.

2.32. Cartan's circles. Let M be a manifold with a parabolic geometry of the type G/P, i.e. we are given $p\colon \mathcal{G} \to M$, $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$. We can define a family of distinguished curves on M as follows: For each $X \in \mathfrak{g}_-$ there is the horizontal vector field $\omega^{-1}(X)$ on $\mathcal{G}$ and every frame $u \in \mathcal{G}$ determines the unique flow line (integral curve) $\mathrm{Fl}_t^{\omega^{-1}(X)}(u)$. We call the curves

$$\mathbb{R} \ni t \mapsto \alpha^{u,X}(t) = p(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)) \in M$$

the *horizontal flows* (also *Cartan's circles* or simply *generalized circles*, cf. [Sha, p. 210]).

The tangent bundle TM is identified with $\mathcal{G} \times_{\mathrm{Ad}} (\mathfrak{g}/\mathfrak{p})$, where $\mathfrak{g}/\mathfrak{p}$ is viewed as $\mathfrak{g}_-$. This identification may be realized by $\{u, X\} \mapsto Tp(\omega^{-1}(X)(u))$. Thus, for each fixed tangent vector $\xi = \{u, X\} \in T_{p(u)}M$, the horizontal flow $\alpha^{u,X}$ is tangent to ξ. Of course, the curve $\alpha^{u,X}$ depends heavily on the choice of both u and X. We shall see, that there may be many horizontal flows tangent to the same vector, but there could be also only one (up to reparametrization). In certain sense, this behavior is similar to that of a family of circles tangent to a given vector in the plane.

2.33. Developments of curves. First we embed the manifold M into the canonical bundle $FM = \mathcal{G} \times_P (G/P) \simeq \tilde{\mathcal{G}} \times_G (G/P) \simeq \tilde{\mathcal{G}}/P$

$$M \ni p(u) \mapsto \{u, o\} \in \mathcal{G} \times_P (G/P)$$

where $u \in \mathcal{G}$, $o = [e] \in G/P$. (Of course, the value is independent of the choice of u.) Since $\tilde{\mathcal{G}}$ carries the canonical principal connection $\tilde{\omega}$, there is the induced connection on FM. This is a general connection on the associated bundle FM. We shall write $\mathrm{Pt}^{\tilde{\omega}}(c, t, y)$ for its parallel transport of the point y along the curve $c(s)$ at the time $s = t$. Let us notice that the fibers of FM are diffeomorphic to the homogeneous space G/P, but there is no distinguished diffeomorfism available. On the other hand, the existence of the global section $M \to FM$ provides some analogy to the tangent bundle on M. More explicitly, it is not difficult to prove that the restrition of the vertical bundle VFM to $M \subset FM$ is canonically isomorphic to the tangent bundle $TM = \mathcal{G} \times_P \mathfrak{g}_-$

$$T_xM \ni \{u, X\} \mapsto \{0_u, X\} \in V_xFM \subset T\mathcal{G} \times_P T(G/P).$$

The *(local) development of a curve* $\alpha(t) \subset M \subset FM$ at its point $x = \alpha(0)$ is the curve $\beta\colon \mathbb{R} \to F_xM$ defined for all t close to zero by the condition

$$\mathrm{Pt}_t^{\tilde{\omega}}(\alpha, t, \beta(t)) = \alpha(t).$$

2.34. Lemma. *The development of each horizontal flow $\alpha^{u,X}$ in the point $x = p(u) \in M$ is the curve $\beta^{u,X}(t) = \{u, [\exp tX]\}$.*

Proof. We claim that the parallel transport of $u \in \mathcal{G} \subset \tilde{\mathcal{G}}$, along $\alpha^{u,X}$, is

$$\mathrm{Pt}^{\tilde{\omega}}(\alpha^{u,X}, t, u) = \mathrm{Fl}_t^{\omega^{-1}(X)}(u).\exp(-tX) \tag{1}$$

and the Lemma then follows easily: We have just to write down the parallel transport of $\{u, [\exp tX]\}$ along our curve $\alpha^{u,X}$

$$\{u, [\exp tX]\} \mapsto \{\mathrm{Fl}_t^{\omega^{-1}(X)}(u).\exp(-tX), [\exp tX]\} = \{\mathrm{Fl}_t^{\omega^{-1}(X)}(u), [e]\}$$

which is exactly the required formula.

So we are left with the proof of formula (1). The parallel transport is defined uniquely by the conditions

(a) $\tilde{p}(\mathrm{Pt}^{\tilde{\omega}}(\alpha^{u,X}, t, u)) = \alpha^{u,X}(t)$
(b) $\mathrm{Pt}^{\tilde{\omega}}(\alpha^{u,X}, 0, u) = u$
(c) $\tilde{\omega}(\frac{\partial}{\partial t}_{|t=0} \mathrm{Pt}^{\tilde{\omega}}(\alpha^{u,X}, t+s, u)) = 0$

which are required whenever all expressions exist. Clearly, the right hand side of (1) satisfies (a) and (b) and we have to compute the derivative in (c).

$$\begin{aligned}
&\tfrac{\partial}{\partial t}_{|t=0}\Big(\mathrm{Fl}_{s+t}^{\omega^{-1}(X)}(u).\exp -(s+t)X\Big) = \\
&\quad = \tfrac{\partial}{\partial t}_{|t=0}\Big(\mathrm{Fl}_{s}^{\omega^{-1}(X)}(u).\exp -(s+t)X\Big) + Tr^{\exp(-sX)}.\omega^{-1}(X)(\mathrm{Fl}_s^{\omega^{-1}(X)}(u)) \\
&\quad = -\zeta_X(\mathrm{Fl}_s^{\omega^{-1}(X)}(u).\exp(-sX)) + Tr^{\exp(-sX)}.\omega^{-1}(X)(\mathrm{Fl}_s^{\omega^{-1}(X)}(u)) \\
&\quad = Tr^{\exp(-sX)}.\Big(-\zeta_{\mathrm{Ad}\exp(-sX).X}(\mathrm{Fl}_s^{\omega^{-1}(X)}(u)) + \omega^{-1}(X)(\mathrm{Fl}_s^{\omega^{-1}(X)}(u))\Big)
\end{aligned}$$

By definition, the horizontal vector in $T_u\tilde{\mathcal{G}}$ over $\xi = \{u, X\} \in T_xM$, with respect to $\tilde{\omega}$, is $\omega^{-1}(X)(u) - \zeta_X(u)$. Thus, using the right-invariance of $\tilde{\omega}$, we obtain

$$\begin{aligned}
\tilde{\omega}(\mathrm{Fl}_s^{\omega^{-1}(X)}(u))\Big(Tr^{\exp(-sX)}.\Big(-\zeta_X(\mathrm{Fl}_s^{\omega^{-1}(X)}(u)) + \omega^{-1}(X)(\mathrm{Fl}_s^{\omega^{-1}(X)}(u))\Big)\Big) = \\
= \mathrm{Ad}\exp(sX) \circ \tilde{\omega}(\omega^{-1}(X) - \zeta_X)(\mathrm{Fl}_s^{\omega^{-1}(X)}(u)) = 0
\end{aligned}$$

and (c) holds as well. □

2.35. In the rest of this subsection we shall study the links between the horizontal flows and the Weyl geometries, but let us first touch the question 'how many different horizontal flows leaving a fixed point in a given direction are there?'.

By the previous Lemma, there is a bijective correspondence between local horizontal flows leaving $x \in M$ and distinct distinguished curves $\beta^{u,X}$. At the same time, it suffices to discuss the curves $t \mapsto [\exp tX] \in G/P$.

If the tangent space TM is irreducible (i.e. $\mathfrak{g}$ is $|1|$-graded), then for each tangent vector $\xi = \{u, X\}$, all curves $\beta^{u.\exp Z, X} = \beta^{u, \mathrm{Ad}_{(\exp Z)} X}$, $Z \in \mathfrak{g}_1$ are tangent to ξ. On the other hand, the Baker-Campbell-Hausdorff formula yields

$$\begin{aligned}
\exp \mathrm{Ad}_{(\exp Z)} tX &= \exp(tX + t[Z, X] + \tfrac{1}{2}t[Z, [Z, X]]) \\
&= \exp tX.\exp(-tX).\exp(tX + \dots) \\
&= \exp tX.\exp t^2(\tfrac{1}{2}[X, [X, Z]] + \text{terms in } \mathfrak{p} \text{ or higher order in } t) \\
&= \exp(tX + \tfrac{1}{2}t^2[X, [X, Z]]).\exp(\mathrm{o}(t^2)) \mod P
\end{aligned}$$

Thus, if $[X,[X,Z]]$ is not zero, then already the second derivative of the curve $\beta^{u,X}$ at the origin changes. For general gradings, the behavior is much more subtle.

2.36. Lemma. *For each $X \in \mathfrak{g}_-$, $u \in \mathcal{G}$, there locally exists a G_0-equivariant section $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$, such that $\mathrm{Fl}_t^{\omega^{-1}(X)}(u) \in \sigma(\mathcal{G}_0)$ for all t. For each such σ, the deformation tensor P^σ vanishes on the tangent vectors to the horizontal flow $\alpha^{u,X}$.*

Proof. Let us write $c(t) = p_0^\ell(\mathrm{Fl}_t^{\omega^{-1}(X)}(u))$. The requirement on σ then reads $\sigma(c(t)) = \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$. We can start with some equivariant section σ_0 and look for a one-form Υ which will give us the right σ. It suffices to define the values of $\Upsilon(c(t))$ by

$$\mathrm{Fl}_t^{\omega^{-1}(X)}(u) = \sigma_0(c(t)).\exp\Upsilon(c(t)).$$

Such a one form Υ locally always exists because the mapping $t \mapsto c(t)$ is locally injective.

Let us write $u_t := \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$, i.e. $Tp(\omega^{-1}(X)(u_t)) = \{u_t, X\} \in TM$ is tangent to $\alpha^{u,X}(t)$ at t. Now, given a σ as above, we know $\omega^{-1}(X)(u_t) = (\omega^\sigma)^{-1}(X)(u_t) + \zeta_{\mathsf{P}^\sigma.X}(u_t)$. But since $\omega^{-1}(X)(u_t) \in T\sigma(\mathcal{G}_0)$, we have $\omega^\sigma(\omega^{-1}(X)(u_t)) = X$ as well. Thus $\mathsf{P}^\sigma(u_t).X = 0$ for all t for which the horizontal flow is defined. □

In the 1-graded we can start with a sc and choose Υ clos so that the Lemm works with a scale as well. Look at th more general case!

2.37. Corollary. *A curve $\alpha\colon \mathbb{R} \to M$ is a (non-parameterized) horizontal flow on some neighborhood of the origin if and only if there is a generalized Weyl geometry $\sigma\colon \mathcal{G}_0 \to \mathcal{G}$, such that α is a geodesic for γ^σ and at the same time the deformation tensor P^σ vanishes along the curve α on this neighborhood.*

Proof. Follows directly from Lemma 2.36 and the fact that auto-parallel fields with respect to ω and γ^σ coincide if P vanishes. □

2.38. Normal coordinates. Let us define 'canonical coordinates' $\varphi_u\colon \mathfrak{g}_- \to M$ given by a choice of the frame u, at least locally around the origin in $\mathfrak{g}_-$. For each $u \in \mathcal{G}$, there is a neighborhood U of $0 \in \mathfrak{g}_-$ such that the flows $\mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ exist for all $X \in U$ and $t \le 1$. We define

$$\bar\varphi_u\colon \mathfrak{g}_- \supset U \to \mathcal{G}, \quad \bar\varphi_u(X) = \mathrm{Fl}_1^{\omega^{-1}(X)}(u)$$
$$\varphi_u\colon \mathfrak{g}_- \supset U \to M, \quad \varphi_u(X) = p(\bar\varphi_u(X)).$$

By the definition of the Cartan connections, the mapping $\omega \times \pi\colon T\mathcal{G} \to \mathfrak{g} \times G$ is a global diffeomorphism and so φ_u is a locally defined diffeomorphism. The projection $p_0^\ell\colon \mathcal{G} \to \mathcal{G}_0$ also determines the local G_0-equivariant section $\sigma_u\colon \mathcal{G}_0 \to \mathcal{G}$ satisfying

$$\sigma_u \circ p_0^\ell \circ \bar\varphi = \bar\varphi.$$

We shall see in a moment that these special coordinates φ_u and the generalized Weyl geometries σ_u have particularly nice properties. Let us also observe that the choice of the frame u defines also a local trivialization of $\mathcal{G} \to M$.

In fact, the situation is quite analogous to the Riemannian geometries and their normal coordinates, only the discussion on the transformations under the change of the frame u is quite subtle in general. More explicitly, for each $b \in P$, $X \in \mathfrak{g}_-$

$$(\mathrm{Fl}_1^{\omega^{-1}(\mathrm{Ad}\, b.X)}(u)).b = \mathrm{Fl}_1^{\omega^{-1}(X)}(u.b)$$

which implies

$$\varphi_{u.b}(X) = p(\mathrm{Fl}_1^{\omega^{-1}(\mathrm{Ad}\, b.X)}(u)).$$

If $b \in G_0$, then $\mathrm{Ad}\, b.X \in \mathfrak{g}_-$ again, and we can recognize the transformation rules for normal coordinates known from Riemannian geometry. Once $b = \exp Z \in P_+$ we have to distill the part in $X + [X, Z] + \dots$ contributing to the horizontal flows.

2.39. Theorem. *Let $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ be a Cartan connection on $p\colon \mathcal{G} \to M$. The normal coordinates φ_u determined by the frame $u \in \mathcal{G}$, the corresponding generalized Weyl geometry σ_u, and its Rho tensor P^{σ_u} have the following properties*

(1) *the symmetrization of the iterated invariant differential*

$$\mathcal{G} \xrightarrow{(\nabla^\omega)^k \mathsf{P}^{\sigma_u}} \otimes^{k+1}\mathfrak{g}_-^* \otimes \mathfrak{p}_+ \xrightarrow{\mathrm{Sym}} \odot^{k+1}\mathfrak{g}_-^* \otimes \mathfrak{p}_+$$

vanishes at u, for all orders $k \geq 0$. In particular, $\mathsf{P}^{\sigma_u}(u) \in \mathfrak{g}_-^* \otimes \mathfrak{p}_+$ vanishes.

(2) *all geodesics of γ^σ going through $p(u)$ are generalized circles on a neighborhood of $p(u)$*

(3) *the curvature R^{σ_u} of the principal connection γ^σ at $p_0^\ell(u) \in \mathcal{G}_0$ coincides with the $\mathfrak{g}_0$-component of the curvature κ of ω at u.*

Proof. Let us write briefly P instead of P^σ. Its value at u vanishes according to Lemma 2.36.

Next, we shall look at the first derivative of P. We have

$$\nabla^\omega_Z \mathsf{P}(u)(X) = \tfrac{\partial}{\partial t}_{|0} \mathsf{P}(\mathrm{Fl}_t^{\omega^{-1}(Z)}(u))(X) = \tfrac{\partial}{\partial t}_{|0} \mathsf{P}(\mathrm{Fl}_1^{\omega^{-1}(tZ)}(u))(X).$$

Thus, Lemma 2.36 implies $\nabla^\omega_Z \mathsf{P}(u)(Z) = 0 \in \mathfrak{p}_+$, for all $Z \in \mathfrak{g}_-$. Consequently the mapping $(X, Z) \mapsto \nabla^\omega_Z \mathsf{P}(u)(X)$ is antisymmetric.

Surprisingly enough, the same elementary argument works in general. Let us consider the $(k+1)$-linear mapping α

$$\otimes^k \mathfrak{g}_- \times \mathfrak{g}- \ni (Y_1, \dots, Y_k, X) \mapsto \nabla^\omega_{Y_k} \dots \nabla^\omega_{Y_1} \mathsf{P}(u)(X) \in \mathfrak{p}_+.$$

By definition,

$$\alpha(Y_1, \dots, Y_k, X) = \tfrac{\partial}{\partial t_k}_{|0} \left(\dots \tfrac{\partial}{\partial t_1}_{|0} \left(\mathsf{P} \circ \mathrm{Fl}_1^{\omega^{-1}(t_1 Y_1)} \circ \dots \circ \mathrm{Fl}_1^{\omega^{-1}(t_k Y_k)}(u)(X) \right) \dots \right).$$

Thus, $\alpha(X, X, \dots, X) = \frac{\partial}{\partial t_k}_{|0} \dots \frac{\partial}{\partial t_1}_{|0} (\mathsf{P} \circ \mathrm{Fl}^{\omega^{-1}(X)}_{t_1 + \dots + t_k})(u)(X)$ vanishes for all $X \in \mathfrak{g}_-$. The complete symmetrization of α equals to the polarization of the mapping $X \mapsto \alpha(X, \dots, X)$ and so it vanishes as well.

The other two claims are now obvious from the construction: The Rho tensor P^σ vanishes along all geodesics of γ^σ and so the conditions from Corollary 2.37 are satisfied. The difference of the curvature of γ^σ and the pullback of the $\mathfrak{g}_0$-component of κ via σ is given by the $\mathfrak{g}_0$-component of $\partial \mathsf{P}$, i.e. the mapping $(X, Y) \mapsto [X, \mathsf{P}.Y] - [Y, \mathsf{P}.X]$, see 2.27. Since P vanishes at u, the third claim is obvious. □

Correspondence spaces

Our general approach to parabolic geometries offers a straightforward generalization of the extremely useful twistor correspondences. We have just to mimic the treatment of the flat models in [BasE].

Let M be a manifold endowed with a parabolic structure of type G/P. Thus, we have the canonical bundle $\mathcal{G} \to M$, endowed with the Cartan connection ω. Now, let us choose a parabolic subgroup $Q \subset P \subset G$, so that we have the manifold $\mathcal{Q} = \mathcal{G}/Q$ and the principal Q-bundle $\mathcal{G} \to \mathcal{Q}$.

2.40. Lemma. *For each parabolic subgroup $Q \subset P \subset G$, the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is also a Cartan connection for the bundle $\mathcal{G} \to \mathcal{Q}$. Moreover, if ω is a normal Cartan connection on M, then it is also normal on $\mathcal{Q}$.*

Proof. Since $Q \subset P$, ω is still a Cartan connection on the principal Q-bundle $\mathcal{G} \to \mathcal{Q}$, cf. the definition. The formula for the codifferential reads (cf. 6.9)

$$\partial^* \kappa(u)(X) = \sum_\alpha [\eta^\alpha, \kappa(X, \xi_\alpha)] - \tfrac{1}{2} \sum_\alpha \kappa([\eta^\alpha, X]_{\mathfrak{g}_-}, \xi_\alpha)$$

where ξ_α, η^α are dual basis of $\mathfrak{g}_-$ and $\mathfrak{p}_+$. We can choose the basis for $\mathfrak{p}_+$ in a way which admits its extension to basis of $\mathfrak{q}_+$ and the additional elements $\eta^{\alpha'}$ will all be in the reductive part of $\mathfrak{p}$. Thus, if we apply the formula for ∂^* in the case of the parabolic Q, all the additional terms in the sums will be killed by the horizontality of the curvature κ. Therefore the Cartan connection ω remains normal also on $\mathcal{G} \to \mathcal{Q}$. □

We say that ω defines the *induced parabolic structure* on $\mathcal{Q}$.

2.41. Observations.

1. *The induced parabolic geometry on $\mathcal{Q}$ is flat if and only if the original parabolic geometry on M is flat on $\mathcal{Q}$.*
2. *The parabolic geometry induced by a torsion-free one on M is not torsion-free in general.*
3. *Whenever there is some cohomology of the pair $(\mathfrak{g}, \mathfrak{q})$ with non-positive homogeneity, the parabolic structure induced on $\mathcal{Q}$ by a regular Cartan connection ω on $\mathcal{G} \to M$ need not be regular any more.*

Proof. The first observation is obvious since the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is the same one for both geometries and so the curvature is given by the same structure equation. The length of the grading of $\mathfrak{g}$ given by $\mathfrak{q}$ is strictly bigger than that of $(\mathfrak{g}, \mathfrak{p})$. In particular, the homogeneities of elements in $\mathfrak{g}_-$ decrease in general. On the other hand, the curvature κ is the same one on $\mathcal{Q}$ but with different splitting into homogeneous components. The available degrees of homogeneity can only decrease though. This explains the remaining claims. We shall meet explicit examples later. □

2.42. Remarks. In fact, due to the construction, the induced parabolic geometries on the *correspondence space* $\mathcal{Q}$ are somewhat special. Indeed, some parts of the curvature on $\mathcal{G} \to \mathcal{Q}$ have to vanish automatically (more or less these must be those enabling the 'integration of the fibers' in $\mathcal{G}/Q$ over individual points in M).

In terms of the Dynkin diagrams with some crossed nodes (describing $P \subset G$), we obtain the correspondence spaces $\mathcal{Q}$ for each choice of a set of further nodes to be crossed. Of course, in order to complete the analogy to the twistor correspondences, we should be able to reverse this construction as well. This means, we should like to start with a parabolic geometry on $\mathcal{Q}$, to remove some crosses (i.e. to choose $P' \supset Q$) and to seek for a manifold M' equipped with the compatible geometry of type G/P'. This would clearly require certain integrability conditions on the curvature of ω. Together with the automatic partial flatness of the correspondence space $\mathcal{Q}$, the latter conditions will probably force vanishing of the whole curvature κ of ω in many cases. A careful study of this problem has to be done yet.

How to deal with examples?

Let us formulate a sequence of steps which should lead quickly to a better understanding of the basics of any particular parabolic geometry. We shall work out several explicit examples in Sections 4-5.

2.43. Recipe.

(1) Find the filtration of the $\mathfrak{p}$-module $\mathfrak{g}$ with respect to the adjoint action of P and understand the individual $\mathfrak{g}_0$-submodules; find a suitable base of the center of $\mathfrak{g}_0$ including the grading element E and compute the actions of these base elements on $\mathfrak{g}_0$-modules; understand the geometries in terms of this filtration.

(2) Find all $\mathfrak{g}_0$-submodules in $H^2(\mathfrak{g}_-, \mathfrak{g})$ and relate them to submodules in $\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathfrak{g}$; understand the local invariants of the goemetries.

(3) Find a reasonable bundle of scales $\mathcal{S}$ (at least locally) and describe explicitly its relation to the G_0-structures on TM and (generalized) Weyl geometries induced by scales.

(4) Understand explicitly the generalized Weyl geometries, their torsions and curvatures, the twistor connections, horizontal flows, etc.

Of course, only the first two steps are completely algorithmic. The decomposition of the filtrations is a standard task in finite dimensional representation theory (starting with the highest root of $\mathfrak{g}$ we find all the submodules, one after the other; the actions of central elements are easily readable from the Dynkin diagram and the inverse to the Cartan matrix, see Section 6 for more details). The second step requires to use Konstant's BBW theorem and to find explicit information about the generators of the cohomologies.

The third step will always need some insight in the special geometry in question. The aim is in fact to identify the bundle $\mathcal{S}$ with some line bundle intrinsic to the particular geometric situation. The archetypical example is the conformal Riemannian geometry: we understand $\mathcal{S}$ as a square root of the line bundle of the distinguished conformal metrics.

The rest should be a separate interesting research project for many choices of G and P.

3. Invariant Operators

Each action λ of the Lie group P on a manifold S yields natural bundles $\mathcal{E}(\lambda) = \mathcal{G} \times_P S$ over all manifolds M with parabolic geometries of type G/P. In particular, if λ is a linear representation of P on a vector space $\mathbb{E}$ then $\mathcal{E}(\lambda)$ is a natural vector bundle. Differential operators between such bundles $\mathcal{E}(\lambda)$, $\mathcal{E}(\mu)$ are most suitably described by mappings on the jet prolongations $J^k\mathcal{E}(\lambda)$.

Roughly speaking, the invariant operators are those differential operators which are defined for all bundles $\mathcal{E}(\lambda)$, $\mathcal{E}(\mu)$ with certain fixed λ and μ, in a sort of universal way, independent of any further choices. On the subcategory of locally flat geometries of type G/P, we can employ the standard definition of natural operators, i.e. we consider the differential operators commuting with the action of morphisms (see [KMS] for a general theory for such questions). Since there are many representations of P but very few morphisms in general, this definition does not extend easily to the whole category. We will not discuss the possibilities for axiomatic definitions of invariant operators now. Rather we shall present some procedures producing families of operators, which clearly are invariant in any reasonable sense.

We shall begin with a detailed discussion on the algebraic structure of jets of sections of natural bundles on manifolds with parabolic geometries. In view of our general aims, we shall restrict ourselves to linear representations λ of P on $\mathbb{E}$. The symbol $\mathbb{E}_\lambda$ will always denote the resulting P-module (and also the $\mathfrak{p}$-module), we shall also use the same λ for the Lie algebra homomorphism $\mathfrak{p} \to \mathfrak{gl}(\mathbb{E}_\lambda)$. In fact the whole P is always the semi-direct product of its reductive Levi part G_0 and $\exp \mathfrak{p}_+$, so that may work with $(G_0, \mathfrak{p})$-modules.

Algebraic structure of jets

3.1. First order jets. In the special case of the homogeneous vector bundle $\mathcal{E}(\lambda) = G \times_P \mathbb{E}_\lambda$ over the homogeneous space $G \to G/P$, the jet prolongations $J^k\mathcal{E}(\lambda)$ inherit the action of G. If we view sections in $C^\infty\mathcal{E}(\lambda)$ as P-equivariant functions $s \in C^\infty(\mathcal{G}, \mathbb{E}_\lambda)^P$, then the 1-jets of sections at the distinguished point $o \in G/P$ are identified with 1-jets of these equivariant functions at the unit $e \in G$ and the action is given by $g.(j^1_e s) = j^1_e(s \circ \ell_{g^{-1}})$ for all $g \in G$. Indeed, for each section s and any local section u of $\mathcal{G}$ we have

$$\begin{aligned} g.j^1_o(x \mapsto \{u(x), s(u(x))\}) &= j^1_o(x \mapsto \{\ell_g \circ \ell_{g^{-1}} \circ u(x), s(\ell_{g^{-1}} \circ u(x))\}) \\ &= j^1_0(x \mapsto \{u(x), s \circ \ell_{g^{-1}}(u(x))\}). \end{aligned}$$

Thus the induced action of $Z \in \mathfrak{p}$ on the section s is given by differentiation in the direction of the right invariant vector field ζ^R_Z on G, $Z.j^1_e s = -j^1_e \zeta^R_Z.s$.

We shall write the jets as $j^1_e s = (v, \varphi) \in \mathbb{E}_\lambda \oplus (\mathfrak{g}^*_- \otimes \mathbb{E}_\lambda)$, where we identify $T_e G \simeq \mathfrak{g}_- \oplus \mathfrak{p}$ via ω, i.e. $v = s(e)$ and $\varphi(X) = \omega^{-1}(X).s(e)$. Now we can express the action of $Z \in \mathfrak{p}$ in these terms (we use the commuting of the left invariant fields and the right invariant fields and the fact that their values at e coincide)

$$\begin{aligned}(v,\varphi) &\mapsto -j^1_e\zeta^R_Z.s = \big(-\zeta^R_Z.s(e), (X\mapsto -\omega^{-1}(X).(\zeta^R_Z.s)(e))\big)\\ &= \big(-\omega^{-1}(Z).s(e), (X\mapsto -\omega^{-1}(Z).(\omega^{-1}(X).s)(e))\big)\\ &= \big(\lambda(Z)(v), (X\mapsto -\omega^{-1}(X).(\omega^{-1}(Z).s)(e) - [\omega^{-1}(Z),\omega^{-1}(X)].s(e))\big) \qquad (1)\\ &= \big(\lambda(Z)(v), (X\mapsto \lambda(Z)\circ\varphi(X) - \varphi\circ \mathrm{ad}_-(Z)(X) + \lambda(\mathrm{ad}_{\mathfrak{p}}(Z)(X)(v)))\big)\\ &= \big(\lambda(Z)(v), \lambda(Z)\circ\varphi - \varphi\circ\mathrm{ad}_-(Z) + \lambda(\mathrm{ad}_{\mathfrak{p}}(Z)(\_))(v)\big).\end{aligned}$$

So we define the $\mathfrak{p}$-module $J^1\mathbb{E}_\lambda$ as $\mathbb{E}_\lambda\oplus(\mathfrak{g}^*_-\otimes\mathbb{E}_\lambda)$ with the $\mathfrak{p}$-action given by (1). We call this module the *first jet prolongation* of $\mathbb{E}_\lambda$. Obviously, for each $\mathfrak{p}$-module homomorphism $\alpha\colon\mathbb{E}_\lambda\to\mathbb{E}_\mu$ the mapping $J^1\alpha\colon(v,\varphi)\mapsto(\alpha(v),\alpha\circ\varphi)$ is a well defined $\mathfrak{p}$-module homomorphism $J^1\mathbb{E}_\lambda\to J^1\mathbb{E}_\mu$. Thus J^1 is a functor on $\mathfrak{p}$-modules. We shall also write $J^1\lambda$ for the corresponding representation.

Next, consider an arbitrary principal P-bundle $\mathcal{G}$ with Cartan connection ω. The P-module $\mathbb{E}_\lambda$ gives rise to the associated bundle $\mathcal{E}(\lambda)$ and its first jet prolongation $J^1\mathcal{E}(\lambda)$.

3.2. Proposition. *The invariant differentiation ∇^ω defines the mapping*

$$\iota\colon C^\infty(\mathcal{G},\mathbb{E}_\lambda)^P\to C^\infty(\mathcal{G},J^1\mathbb{E}_\lambda)^P,\quad \iota(s)(u)=(s(u),(X\mapsto\nabla^\omega s(u)(X)))$$

which yields a diffeomorphism $J^1\mathcal{E}(\lambda)\simeq\mathcal{G}\times_P J^1\mathbb{E}_\lambda$. For each fiber bundle morphism $f\colon\mathcal{E}(\lambda)\to\mathcal{E}(\mu)$ given by a P-module homomorphism $\alpha\colon\mathbb{E}_\lambda\to\mathbb{E}_\mu$, the first jet prolongation J^1f corresponds to the P-module homomorphism $J^1\alpha$.

Proof[1]**.** Let us recall that $\nabla^\omega s(u)(X)=\omega^{-1}(X)(u).s$. Thus the mapping $\iota\colon s\mapsto(s,\nabla^\omega s)$ is well defined and depends on first jets only, but we have to check its equivariance. This means exactly the commuting with the derivatives in the directions of fundamental vector fields $\omega^{-1}(Z)$, $Z\in\mathfrak{p}$. So we aim at $-\zeta_Z\iota(s)(u)=J^1\lambda(Z)\circ\iota(s)(u)$. To see this, we just have to copy the computation 3.1.(1) and to remember that the curvature of any Cartan connection is horizontal:

$$\begin{aligned}-\zeta_Z.\iota(s)(u) &= \big(-\omega^{-1}(Z).s(u), (X\mapsto -\omega^{-1}(Z).(\omega^{-1}(X).s)(u))\big)\\ &= \big(\lambda(Z)(s(u)), (X\mapsto -\omega^{-1}(X).(\omega^{-1}(Z).s)(u) - \omega^{-1}([Z,X]).s(u))\big)\\ &= \big(\lambda(Z)(s(u)), (X\mapsto \lambda(Z)\circ\nabla^\omega s(u)(X) - \nabla^\omega s(u)\circ\mathrm{ad}_-(Z)(X)\\ &\qquad + \lambda(\mathrm{ad}_{\mathfrak{p}}(Z)(X))(s(u)))\big)\end{aligned}$$

Clearly, we have constructed a diffeomorphism $J^1\mathcal{E}(\lambda)\to E\times_P J^1\mathbb{E}_\lambda$.

Finally, consider a homomorphism $\alpha\colon\mathbb{E}_\lambda\to\mathbb{E}_\mu$. The corresponding homomorphism $f\colon\mathcal{E}(\lambda)\to\mathcal{E}(\mu)$ is defined by $\{u,v\}\mapsto\{u,\alpha(v)\}$, and so the induced action on sections is $(x\mapsto\{u(x),s(u(x))\})\mapsto(x\mapsto\{u(x),\alpha\circ s(u(x))\})$. Taking 1-jet of this expression we obtain just the homomorphism $J^1\alpha$. □

[1] Similar arguments appeared first implicitly in [CSS1] and in a very special case in the Master Thesis by Martin Panák

3.3. Semi-holonomic jets. Since we have posed no restrictions on the representation λ above, we can iterate the functors J^1 on the associated vector bundles as well as on the P-modules. Proposition 3.2 then implies that the kth iteration $J^1 \dots J^1 \mathcal{E}(\lambda)$ is an associated bundle to $\mathcal{G}$ with the corresponding P-module $J^1 \dots J^1 \mathbb{E}_\lambda$.

Let us look more carefully at $J^1 J^1 \mathbb{E}_\lambda$ and $J^1 J^1 \mathcal{E}(\lambda)$. There are two obvious $\mathfrak{p}$-module homomorphisms $J^1 J^1 \mathbb{E}_\lambda \to J^1 \mathbb{E}_\lambda$, the first one given by the projection $p_\lambda \colon (v, \varphi) \mapsto v$ defined on each first jet prolongation and the other obtained by the action of J^1 on p_λ. Thus there is a submodule $\bar{J}^2 \mathbb{E}_\lambda$ in $J^1 J^1 \mathbb{E}_\lambda$ on which these two projections coincide. As a vector space, this is

$$\bar{J}^2 \mathbb{E}_\lambda = \mathbb{E}_\lambda \oplus (\mathfrak{g}_-^* \otimes \mathbb{E}_\lambda) \oplus (\mathfrak{g}_-^* \otimes \mathfrak{g}_-^* \otimes \mathbb{E}_\lambda).$$

The two $\mathfrak{p}$-module homomorphisms $J^1 p_\lambda$, $p_{J^1\lambda}$ give rise to fiber bundle morphisms $J^1 J^1 \mathcal{E}(\lambda) \to J^1 \mathcal{E}(\lambda)$ which are just the two standard projections on second non-holonomic jet prolongations. So we conclude that the second semi-holonomic prolongation $\bar{J}^2 \mathcal{E}(\lambda)$ is naturally equivalent to $\mathcal{G} \times_P \bar{J}^2 \mathbb{E}_\lambda$.

Iterating this procedure, we obtain the kth semi-holonomic jet prolongations and $J^1(\bar{J}^k \mathbb{E}_\lambda)$ equipped with two natural projections onto $\bar{J}^k \mathbb{E}_\lambda$, which correspond to the usual projections on the first jet prolongation of semi-holonomic jets. Their equalizer is then the submodule $\bar{J}^{k+1} \mathbb{E}_\lambda$. As a vector space (and G_0-module),

$$\bar{J}^k \mathbb{E}_\lambda = \bigoplus_{i=0}^{k} (\otimes^i \mathfrak{g}_-^* \otimes \mathbb{E}_\lambda).$$

3.4. Proposition. *For each integer k, the kth semi-holonomic jet prolongation $\bar{J}^k \mathcal{E}(\lambda)$ carries the natural structure of associated fiber bundle $\mathcal{G} \times_P \bar{J}^k \mathbb{E}_\lambda$. Moreover, the invariant differential defines the natural embedding*

$$J^k \mathcal{E}(\lambda) \ni j^k_u s \mapsto \{u, (s(u), \nabla^\omega s(u), \dots, (\nabla^\omega)^k s(u))\} \in \bar{J}^k E_\lambda \simeq \mathcal{G} \times_P \bar{J}^k \mathbb{E}_\lambda.$$

Proof. The first part of the statement has been already shown. What remains is to discuss the equivariance properties of the invariant differentials. However also this follows from the first order case easily by induction, using only the definition of the semi-holonomic prolongations. □

3.5. Remarks. A few observations and comments are in place.

It is just the existence of the natural associated bundle structure on $\bar{J}^k \mathcal{E}(\lambda)$ (i.e. depending on ω only) which gives rise to the differential operator $D_\Phi : C^\infty \mathcal{E}(\lambda) \to C^\infty \mathcal{E}(\mu)$ for each P-module homomorphism $\Phi \colon \bar{J}^k \mathbb{E}_\lambda \to \mathbb{E}_\mu$. In view of the existence of the canonical Cartan connections on the parabolic geometries, this means that each such P-module homomorphism defines an invariantly defined operator on manifolds with the appropriate geometric structures. On the other hand, not all invariant operators arise in this way, as well known e.g. from the conformal Riemannian geometry, see e.g. [CSS1, Eas, EasS, Slo1,Slo2].

The existence of the canonical embedding provided by the iterated differential suggests a straightforward method for explicit constructions of such operators. Given a P-module homomorphism $\Phi\colon \bar{J}^k\mathbb{E}_\lambda \to \mathbb{E}_\mu$ we compose this with the iterated differentials to obtain quite explicit analytic expressions for the operators. On the other hand, we can also start with an arbitrary G_0-module homomorphism Φ, compose it with the differentials and discuss the equivariance of the resulting expression. Its expansion in terms of the underlying generalized Weyl geometries yields an algorithmic method for finding operators, see e.g. [CSS1, Slo2].

While the semi-holonomic prolongations $\bar{J}^k\mathcal{E}(\lambda)$ are constructed by a purely algebraic construction, the embedding of $J^k\mathcal{E}(\lambda)$ depends of course heavily on the curvature of the Cartan connection. This makes the discussion on the algebraic conditions for the existence of invariant operators which are not coming from P-module homomorphisms much more difficult.

3.6. Weighted orders. The general Ricci identity for invariant differentials (see 2.13) shows that the iterated invariant differentials of a section are in certain extent determined by their evaluation on the elements $X \in \mathfrak{g}_{-1}$. What we need is the additional condition $\kappa_i(X,Y) = 0$ for all $X, Y \in \mathfrak{g}_{-1}$ and $i < -1$. This is always true if $\kappa^i = 0$ for all $i \le 0$, which is a consequence of the structure equation on $\mathcal{G}$, see 2.2. Thus, for regular parabolic geometries, the evaluation of the ℓth iterated differential $(\nabla^\omega)^\ell s$ on $\mathfrak{g}_{-1}$ in order to know the first differential on the whole $\mathfrak{g}_-$ (here ℓ is the length of the grading of $\mathfrak{g}$ as usual).

The latter observation suggests to refine the notion of the order of an operator:

For each $X \in \mathfrak{g}_{-k}$, $1 \le k \le \ell$, we say that the operator ∇^ω_X, has the *weighted order* k. The total order of an operator D_Φ for a P-module homomorphism Φ is then defined as usual.

Restricted jets

3.7. Our next goal is to define an algebraic object corresponding to partially defined jets, i.e. describing derivatives in some directions only. First we rewrite slightly the $\mathfrak{p}$-action on $\mathbb{E}_\lambda \oplus (\mathfrak{g}_-^* \otimes \mathbb{E}_\lambda)$. Since the Killing form provides the dual pairing $\mathfrak{g}_-^* \simeq \mathfrak{p}_+$ we have for all $Y \otimes v \in \mathfrak{g}_-^* \otimes \mathbb{E}_\lambda$, $X \in \mathfrak{g}_-$, $Z \in \mathfrak{p}$

$$\begin{aligned}(Y \otimes v) \circ \operatorname{ad}_-(Z)(X) &= \langle \operatorname{ad}_-(Z)(X), Y\rangle v \\ &= \langle [Z,X], Y\rangle v \quad \text{(since } \mathfrak{p}_+ \text{ is orthogonal to } \mathfrak{p}) \\ &= -\langle X, [Z,Y]\rangle v \quad \text{(the invariance of the Killing form)} \\ &= -([Z,Y] \otimes v)(X).\end{aligned}$$

For a fixed dual linear basis $\xi_\alpha \in \mathfrak{g}_-$, $\eta^\alpha \in \mathfrak{p}_+$ we can also rewrite the term

$$\lambda(\operatorname{ad}_{\mathfrak{p}}(Z)(X))(v) = \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}}.v$$

Thus the 1-jet action on $J^1\mathbb{E}_\lambda = \mathbb{E}_\lambda \oplus (\mathfrak{p}_+ \otimes \mathbb{E}_\lambda)$ is (the dots indicate the $\mathfrak{p}$-action given by λ)

$$(1)\quad J^1\lambda(Z)(v_0, Y_1 \otimes v_1) = (Z.v_0, Y_1 \otimes Z.v_1 + [Z, Y_1] \otimes v_1 + \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}}.v_0).$$

There is the $\mathfrak{p}$-invariant vector subspace $\{0\} \oplus (\mathfrak{p}_+^2 \otimes \mathbb{E}_\lambda) \subset J^1\mathbb{E}_\lambda$ and we define the $\mathfrak{p}$-module

$$J^1_{\mathcal{R}}\mathbb{E}_\lambda = J^1\mathbb{E}_\lambda/(\{0\} \oplus (\mathfrak{p}_+^2 \otimes \mathbb{E}_\lambda)) \simeq \mathbb{E}_\lambda \oplus ((\mathfrak{p}_+/\mathfrak{p}_+^2) \otimes \mathbb{E}_\lambda) \simeq \mathbb{E}_\lambda \oplus (\mathfrak{g}_{-1}^* \otimes \mathbb{E}_\lambda).$$

The formula for the $\mathfrak{p}$-action $J^1_{\mathcal{R}}\lambda$ reads

$$(2)\quad J^1_{\mathcal{R}}\lambda(Z)(v_0, Y_1 \otimes v_1) = (Z.v_0, Y_1 \otimes Z.v_1 + [Z, Y_1]_{\mathfrak{g}_1} \otimes v_1 + \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}]_{\mathfrak{p}}.v_0)$$

where $\eta^{\alpha'}$ and $\xi_{\alpha'}$ are dual bases of $\mathfrak{g}_{\pm 1}$. The latter formula gets much simpler if λ is a G_0-representation extended trivially to the whole $\mathfrak{p}$. Then for each $W \in \mathfrak{g}_0$, $Z \in \mathfrak{g}_1$

$$(3)\qquad J^1_{\mathcal{R}}\lambda(W)(v_0, Y_1 \otimes v_1) = (W.v_0, Y_1 \otimes W.v_1 + [W, Y_1] \otimes v_1)$$

$$(4)\qquad J^1_{\mathcal{R}}\lambda(Z)(v_0, Y_1 \otimes v_1) = (0, \sum_{\alpha'} \eta^{\alpha'} \otimes [Z, \xi_{\alpha'}].v_0)$$

while the action of $\mathfrak{p}_+^2$ is trivial. Exactly as with the functor J^1, the action of $J^1_{\mathcal{R}}$ on $(G_0, \mathfrak{p})$-module homomorphisms is given by the composition.

The associated fiber bundle $J^1_{\mathcal{R}}\mathcal{E}(\lambda) := \mathcal{G} \times_P J^1_{\mathcal{R}}\mathbb{E}_\lambda$ will be called the *restricted first jet prolongation*. The invariant differential provides a natural mapping $J^1\mathcal{E}(\lambda) \to J^1_{\mathcal{R}}\mathcal{E}(\lambda)$.

The inductive construction of the semi-holonomic jet prolongations of $(G_0, \mathfrak{p})$-modules can be now repeated with the functor $J^1_{\mathcal{R}}$. The resulting $\mathfrak{p}$-modules are the equalizers of the two natural projections $J^1_{\mathcal{R}}(\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda) \to \bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda$ and, as $\mathfrak{g}_0$-modules, they are equal to

$$\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda = \bigoplus_{i=0}^k (\otimes^i \mathfrak{g}_1 \otimes \mathbb{E}_\lambda).$$

We shall write $\bar{J}^k_{\mathcal{R}}\mathcal{E}(\lambda)$ for the associated fiber bundles corresponding to the modules $\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda$. The resulting modules and bundles are called the *restricted semi-holonomic prolongations* of $\mathcal{E}(\lambda)$ and $\mathbb{E}_\lambda$, respectively.

As before, the iterated invariant differential yields a natural mapping $J^k\mathcal{E}(\lambda) \to \bar{J}^k_{\mathcal{R}}\mathcal{E}(\lambda)$, so that each P-module homomorphism $\Phi\colon \bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda \to \mathbb{E}_\mu$ defines the differential operator $D_\Phi : C^\infty(\mathcal{E}(\lambda)) \to C^\infty(\mathcal{E}(\mu))$.

Jet-module homomorphisms

3.8. The algebraic structure of jet-modules. We shall deal with the restricted jets together with the usual ones. The idea is very simple, we have just to iterate the action on one-jets.

Let us start with the second order case. Each element $(v_0, Y_1 \otimes v_1, W_1 \otimes W_2 \otimes v_2) \in \bar{J}^2\mathbb{E}_\lambda$ is understood as

$$((v_0, Y_1 \otimes v_1), Y_1 \otimes (v_1, 0)) + ((0,0), W_1 \otimes (0, W_2 \otimes v_2)) \in J^1 J^1 \mathbb{E}_\lambda$$

and for all $Z \in \mathfrak{p}$ we have

$$\begin{aligned}
(1)\quad & \bar{J}^2\lambda(Z)(v_0, Y_1 \otimes v_1, W_1 \otimes W_2 \otimes v_2) = \\
&= \bar{J}^2\lambda(Z)\big(((v_0, Y_1 \otimes v_1), Y_1 \otimes (v_1, 0)) + ((0,0), W_1 \otimes (0, W_2 \otimes v_2))\big) = \\
&= \Big(Z.v_0, Z.(Y_1 \otimes v_1) + \sum_\alpha \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}}.v_0, \\
& Z.(W_1 \otimes W_2 \otimes v_2) + \sum_\alpha \big(\eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}}.(Y_1 \otimes v_1) + Y_1 \otimes \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}}.v_1\big) + \\
& \sum_{\alpha,\beta} \eta^\beta \otimes \eta^\alpha \otimes [[Z, \xi_\beta]_{\mathfrak{p}}, \xi_\alpha]_{\mathfrak{p}}.v_0\Big)
\end{aligned}$$

The action on the restricted jets is read off these lines easily. Simply Y_1, W_1, W_2 have to be in $\mathfrak{g}_1 \simeq \mathfrak{p}_+/\mathfrak{p}_+^2$. For $Z \in \mathfrak{g}_0$ we obtain the tensorial product of the obvious actions,

$$\bar{J}^2_{\mathcal{R}}\lambda(Z)(v_0, Y_1 \otimes v_1, W_1 \otimes W_2 \otimes v_2) = (Z.v_0, Z.(Y_1 \otimes v_1), Z.(W_1 \otimes W_2 \otimes v_2))$$

while for $Z \in \mathfrak{p}_+$ essentially all summands survive. The formula, however, simplifies heavily if the action of $\mathfrak{p}_+$ on $\mathbb{E}_\lambda$ is trivial. Then $Z_1 \in \mathfrak{g}_1$ and $Z_2 \in \mathfrak{g}_2$ yield

$$\begin{aligned}
& \bar{J}^2_{\mathcal{R}}\lambda(Z_1)(v_0, Y_1 \otimes v_1, W_1 \otimes W_2 \otimes v_2) = (0, \sum_{\alpha'} \eta^{\alpha'} \otimes [Z_1, \xi_{\alpha'}].v_0), \\
(2)\quad & \sum_{\alpha'} (\eta^{\alpha'} \otimes [Z_1, \xi_{\alpha'}].(Y_1 \otimes v_1) + Y_1 \otimes \eta^{\alpha'} \otimes [Z_1, \xi_{\alpha'}].v_1)) \\
& \bar{J}^2_{\mathcal{R}}\lambda(Z_2)(v_0, Y_1 \otimes v_1, W_1 \otimes W_2 \otimes v_2) = \\
(3)\quad & (0, 0, \sum_{\alpha', \beta'} \eta^{\beta'} \otimes \eta^{\alpha'} \otimes [[Z_2, \xi_{\beta'}], \xi_{\alpha'}].v_0)
\end{aligned}$$

while all $Z \in \mathfrak{p}_+^3$ act trivially.

The above computations can be easily generalized:

3.9. Proposition. *Let* $Y_1 \otimes \dots \otimes Y_r \otimes v \in \otimes^r \mathfrak{p}_+ \otimes \mathbb{E}_\lambda$, $(0, \dots, Y_1 \otimes \dots \otimes Y_r \otimes v, 0, \dots, 0) \in \bar{J}^{k+r}\mathbb{E}_\lambda$ *and* $Z \in \mathfrak{p}$. *Then*

$$\bar{J}^{k+r}\lambda(Z)(0, \dots, Y_1 \otimes \dots \otimes Y_r \otimes v, 0, \dots, 0) = (0, \dots, 0, \varphi_r, \varphi_{r+1}, \dots, \varphi_{r+k})$$

where

$$\varphi_{r+s} = \sum_{\substack{\alpha_1, \dots, \alpha_s \\ 0 \le i_1 \le \dots \le i_s \le r}} Y_1 \otimes \dots \otimes Y_{i_1} \otimes \eta^{\alpha_1} \otimes Y_{i_1+1} \otimes \dots \otimes Y_{i_2} \otimes \eta^{\alpha_2} \otimes \dots$$
$$\otimes Y_{i_s} \otimes \eta^{\alpha_s} \otimes [\dots [Z, \xi_{\alpha_1}]_\mathfrak{p}, \dots, \xi_{\alpha_s}]_\mathfrak{p}.(Y_{i_s+1} \otimes \dots \otimes Y_r \otimes v).$$

In particular, if $s = 0$ *we obtain the standard tensor product of the representations, i.e.* $\varphi_r = Z.(Y_1 \otimes \dots \otimes Y_r \otimes v)$.

The action on the restricted jets $\bar{J}^{k+r}_\mathcal{R}\mathbb{E}_\lambda$ *is obtained by restriction of this formula to the dual basis* $\xi_{\alpha'}$, $\eta^{\alpha'}$ *of* $\mathfrak{g}_{\pm 1}$ *and the actions of the iterated brackets on* $\mathfrak{g}_1 \simeq \mathfrak{p}_+/\mathfrak{p}_+^2$.

The obvious projection $\pi\colon \bar{J}^{k+r}\mathbb{E}_\lambda \to \bar{J}^{k+r}_\mathcal{R}\mathbb{E}_\lambda$ *is a* P*-module homomorphism.*

Proof. We have already seen that the formula holds for $0 \le r + k \le 2$. So let us assume, it holds also for all $k' + r' < k + r$. The semi-holonomic $(k + r)$-jet is a 1-jet of an $(k + r - 1)$-jet and what we have to do is to apply the standard first jet prolongation of the representation $\bar{J}^{k+r-1}\lambda$. Thus we consider

$$\big((0, \dots, Y_1 \otimes \dots \otimes Y_r \otimes v, 0, \dots), Y_1 \otimes (0, \dots, Y_2 \otimes \dots \otimes Y_r \otimes v, 0, \dots)\big) \in J^1\bar{J}^{k+r-1}\mathbb{E}_\lambda$$

and compute how $J^1(\bar{J}^{k+r-1}\lambda)(Z)$ acts. By the induction hypothesis, we obtain

$$\Big((0, \dots, Z.(Y_1 \otimes \dots \otimes Y_r \otimes v), \dots, \sum_{\substack{0 \le i_1 \le \dots \le i_s \le r \\ \alpha_1, \dots, \alpha_s}} (Y_1 \otimes \dots \otimes Y_{i_1} \otimes \eta^{\alpha_1} \otimes \dots$$
$$\otimes Y_{i_s} \otimes \eta^{\alpha_s} \otimes [\dots [Z, \xi_{\alpha_1}]_\mathfrak{p}, \dots, \xi_{\alpha_s}]_\mathfrak{p}.(Y_{i_s+1} \otimes \dots \otimes Y_r \otimes v)), \dots\big),$$
$$Z.Y_1 \otimes \big(0, \dots\big) + Y_1 \otimes \big(0, \dots, Z.(Y_2 \otimes \dots Y_r \otimes v), \dots, \sum_{\substack{\alpha_1, \dots, \alpha_s \\ 1 \le i_1 \le \dots \le i_s \le r}} Y_2 \otimes \dots Y_{i_1} \otimes$$
$$\eta^{\alpha_1} \otimes \dots \otimes Y_{i_s} \otimes \eta^{\alpha_s} \otimes [\dots [Z, \xi_{\alpha_1}]_\mathfrak{p}, \dots, \xi_{\alpha_s}]_\mathfrak{p}.(Y_{i_s+1} \otimes \dots \otimes Y_r \otimes v), \dots\big) +$$
$$\eta^{\alpha_1} \otimes \big(0, \dots, [Z, \xi_{\alpha_1}]_\mathfrak{p}.(Y_1 \otimes \dots Y_r \otimes v), \dots, \sum_{\substack{\alpha_2, \dots, \alpha_s \\ 0 \le i_2 \le \dots \le i_s \le r}} Y_1 \otimes \dots \otimes Y_{i_2} \otimes$$
$$\eta^{\alpha_2} \otimes \dots \otimes Y_{i_s} \otimes \eta^{\alpha_s} \otimes [\dots [Z, \xi_{\alpha_1}]_\mathfrak{p}, \dots, \xi_{\alpha_s}]_\mathfrak{p}.(Y_{i_s+1} \otimes \dots \otimes Y_r \otimes v), \dots\big)\Big)$$

All components up to the order $k + r - 1$ clearly yield the required formula (in fact this was the induction hypothesis) which appears once in the 'value part' of the 1-jet, and once more in the 'derivative part'. The $(k + r)$th component appears already only in the derivative part and consists of two summands, where the first one exhausts all possibilities with Y_1 in the beginning of the term, while the other summand produces all the remaining possibilities, i.e. those beginning with η^{α_1}. Thus their sum yields exactly the required formula. □

As a corollary of the latter Proposition, we obtain a useful criterion for homomorphisms of the restricted jets:

3.10. Proposition. *Let $\mathbb{E}_\lambda$, $\mathbb{E}_\mu$ be irreducible P-modules and let Φ be a G_0-module homomorphism $\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda \supset \otimes^k\mathfrak{g}_1 \otimes \mathbb{E}_\lambda \to \mathbb{E}_\mu$. Then Φ extends trivially to a P-module homomorphism if and only if*

$$\Phi\bigg(\sum_{0\le i\le k-1} Y_1\otimes\cdots\otimes Y_i\otimes\eta^{\alpha'}\otimes[Z,\xi_{\alpha'}].(Y_{i+1}\otimes\cdots\otimes Y_{k-1}\otimes v)\bigg)=0$$

for all $Z \in \mathfrak{g}_1$, $Y_1,\dots,Y_{k-1} \in \mathfrak{g}_1$, $v \in \mathbb{E}_\lambda$. Moreover, each P-module homomorphism $\Phi\colon \bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda \to \mathbb{E}_\mu$ is obtained in this way.

Proof. We know that $\mathfrak{p}_+$ acts trivially on $\mathbb{E}_\lambda$. For all $Z \in \mathfrak{g}_k$ and $(Y_1\otimes\cdots\otimes Y_r\otimes v) \in \otimes^r\mathfrak{g}_1 \otimes \mathbb{E}_\lambda \subset \bar{J}^{r+s}_{\mathcal{R}}\mathbb{E}_\lambda$, $s \ge k$ we obtain

$$\bar{J}^{r+s}_{\mathcal{R}}\lambda(Z)(Y_1\otimes\cdots\otimes Y_r\otimes v) = \sum_{\substack{0\le i_1\le\cdots\le i_k\le r\\ \alpha'_1,\dots,\alpha'_k}} Y_1\otimes\cdots\otimes Y_{i_1}\otimes\eta^{\alpha'_1}\otimes Y_{i_1+1}\otimes\dots$$
$$\otimes Y_{i_2}\otimes\eta^{\alpha'_2}\otimes\cdots\otimes Y_{i_k}\otimes\eta^{\alpha'_k}\otimes[\dots[Z,\xi_{\alpha'_1}]_{\mathfrak{p}},\dots,\xi_{\alpha'_k}]_{\mathfrak{p}}.(Y_{i_k+1}\otimes\cdots\otimes Y_r\otimes v)$$

In particular, if $Z \in \mathfrak{g}_1$, then the formula is:

$$\bar{J}^{r+s}_{\mathcal{R}}(Z)(Y_1\otimes\cdots\otimes Y_r\otimes v) = \sum_{0\le i\le r} Y_1\otimes\cdots\otimes Y_i\otimes\eta^{\alpha'}\otimes[Z,\xi_{\alpha'}].(Y_{i+1}\otimes\cdots\otimes Y_r\otimes v)$$

By the hypothesis, $\mathbb{E}_\lambda$, $\mathbb{E}_\mu$ are irreducible, so in particular $\mathfrak{p}_+$ acts trivially. Then the grading element in $\mathfrak{g}_0$ acts differently on each G_0-module component $\otimes^r\mathfrak{g}_1\otimes\mathbb{E}_\lambda$ in $\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda$ and so each P-module homomorphism $\Phi\colon \bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda \to \mathbb{E}_\mu$ must be a G_0-module homomorphism $\otimes^r\mathfrak{g}_1\otimes\mathbb{E}_\lambda \to \mathbb{E}_\mu$ trivially extended to the whole $\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda$, for some $0 \le r \le k$. Without any loss of generality, we may always assume $r = k$ and then the necessary and sufficient condition on a given $\mathfrak{g}_0$-module homomorphism Φ to be a $\mathfrak{p}$-module homomorphism is its vanishing on the image of the action of $\mathfrak{p}_+$. Moreover, since the whole $\mathfrak{p}_+$ is generated by $\mathfrak{g}_1$, the image of the action intersected with the top component $\otimes^k\mathfrak{g}_1\otimes\mathbb{E}_\lambda$ coincides with the image of the last but one component under the action of $\mathfrak{g}_1$. □

3.11. Remarks. The latter criterion provides a powerful tool for discussion on natural differential operators. Each such Φ, composed with the projection π : $\bar{J}^k\mathbb{E}_\lambda \to \bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda$ yields a P-module homomorphism on the semiholonomic jets, and thus a differential operator $D_{\Phi\circ\pi}$.

The explicit realization of this operator through the embedding $J^k\mathcal{E}(\lambda) \to \bar{J}^k\mathcal{E}(\lambda)$ via the iterated differential with respect to the canonical Cartan connection ω suggests that we could consider the standard jets in $J^k\mathcal{E}(\lambda)$ as associated bundles corresponding to a sort of quotient of the semi-holonomic jet modules, where the defining relations are given by the values of the curvature and its invariant derivatives (cf. the Ricci identity in 2.13). Of course, the resulting modules happen to be non-isomorphic at different points, in general. In particular, on the homogeneous spaces we recover the tautological embedding of the standard jet modules into the semi-holonomic ones provided by ω. This point of view definitely deserves further investigation and might possibly help in the study of operators which do not come from our algebraic jet module homomorphisms. (The symmetries of the curvature and its derivatives imply certain restrictions on the structure of the resulting quotients.)

There is also the interesting question, how far are the general P-module homomorphisms $\Phi\colon \bar{J}^k\mathbb{E}_\lambda \to \mathbb{E}_\mu$ determined by their restrictions $\Phi_0 = \Phi_{|\bar{J}^k_{\mathcal{R}}\mathbb{E}_\lambda}$ to the image of the embedding of the G_0-module of the restricted jets. Obviously, Φ_0 is a G_0-module homomorphism and there is a good evidence that Φ is fully determined by Φ_0. However, Φ does not vanish on the G_0-module complement of the restricted jets, in general. Thus the problem: What are the conditions on a G_0-module homomorphism Φ_0 to extend to a P-homomorphism Φ, definitely deserves further investigation. Proposition 3.9 gives a sufficient condition only.

3.12. First order operators. Let us discuss some simple applications of Propositions 3.10, 3.11. First we shall deal with first order operators. Each irreducible representation λ of $\mathfrak{g}_0$ is determined by the scalar action w of the grading element E and the restriction λ' of λ to the orthogonal complement $E^\perp = \mathfrak{g}_0'$. The scalar w defines a generalization of the conformal weight of objects in conformal Riemannian geometries. It seems to be reasonable to normalize the *conformal weight* of the representation λ in such a way, that the line bundle modeling the scale bundle $\mathcal{S}$ will be of weight one.

Let us fix the representation λ' and consider the scalar w as a free parameter. Our aim is to find all homomorphisms $\Phi\colon J^1\mathbb{E}_{w,\lambda'} \to \mathbb{E}_\mu$ with irreducible representations $\mu = (\tilde{w}, \mu')$ of $\mathfrak{g}_0$. Of course, Φ must be in particular a homomorphism of $\mathfrak{g}_0$-modules. Therefore, Φ is a projection to one irreducible component $\mathbb{E}_\mu$ in $J^1\mathbb{E}_\lambda$. Either $\lambda = \mu$ or $\mathbb{E}_\mu \subset \mathfrak{g}_k \otimes \mathbb{E}_\lambda$ for suitable k (notice that E acts differently for each k). For all $Z \in \mathfrak{g}_i$, $i > 0$, and $(v_0, Y \otimes v_1) \in J^1\mathbb{E}_\lambda$, the formula in 3.9 then yields the condition

$$0 = \Phi\left([Z,Y] \otimes v_1 + \sum_\alpha \eta^\alpha \otimes [Z,\xi_\alpha]_{\mathfrak{g}_0}.v_0\right).$$

Inserting $v_0 = 0$ we conclude that Φ factors through the restricted jets $J^1_{\mathcal{R}}\mathbb{E}_\lambda$. The latter formula with $Z \in \mathfrak{g}_1$ and $v_1 = 0$ then reads (with dual basis $\eta^{\alpha'}$, $\xi_{\alpha'}$ of $\mathfrak{g}_{\pm 1}$)

$$0 = \Phi\left(\sum_{\alpha'} \eta^{\alpha'} \otimes [Z,\xi_{\alpha'}]_{\mathfrak{g}_0}.v_0\right).$$

Now, Φ is the projection to an irreducible component in $\mathfrak{g}_1 \otimes \mathbb{E}_\lambda$ and its argument can be viewed as the mapping

$$\mathfrak{g}_1 \otimes \mathbb{E}_\lambda \to \mathfrak{g}_{-1}^* \otimes \mathbb{E}_\lambda,\ (Z,v) \mapsto (X \mapsto [Z,X].v).$$

Assume now that $\mathbb{E}_{\mu'}$ appears in $\mathfrak{g}_1 \otimes \mathbb{E}_{\lambda'}$ with multiplicity one. Then the above mapping is a constant multiple of the identity on this component and the above condition is a linear equation on w. Thus, for each multiplicity one component $\mathbb{E}'_\mu \in \mathfrak{g}_1 \otimes \mathbb{E}_{\lambda'}$ there is a uniquely defined scalar action w of the grading element E such that Φ turns out to be a homomorphism $J^1_{\mathcal{R}}\mathbb{E}_{w,\lambda'} \to \mathbb{E}_{(w+1),\mu'}$. In fact, we can say even more: $\mathfrak{g}_1$ usually splits into further irreducible components and the action of the center of $\mathfrak{g}_0$ distinguishes them. Thus we can apply the above discussion to the individual components of $\mathfrak{g}_1$.

Similar considerations appeared first in [Feg] in the very special case of first order invariant operators on conformal Riemannian manifolds.

3.13. Some more operators. The discussion on higher order jets is much more difficult in general, but we can treat similarly the morphisms on symmetrizations of higher order restricted jets. The resulting scalars w can be also expressed explicitly with the help of finite dimensional representation theory of semisimple Lie algebras. The case of $|1|$-graded algebras $\mathfrak{g}$ has been treated in great detail in [CSS3].

Another interesting source of examples is provided by $\mathfrak{g}$-modules. Consider such a module with its filtration by P-submodules $W = W_0 + \cdots + W_k$. In particular, the top component W_k decomposes as direct sum of irreducible P-modules $\mathbb{E}_\lambda$. If we pick up any of these components, we can take its semiholonomic jets $\bar{J}^k\mathbb{E}_\lambda$ and look for components in the individual levels $\otimes^i\mathfrak{p}_+ \otimes \mathbb{E}_\lambda$ which also appear in the $\mathfrak{g}_0$-submodules W_{k-i}. This will often lead to operators similar to the well known D-operators in conformal Riemannian and similar geometries, see e.g. [Eas]. In particular, we always obtain interesting operators for the adjoint representation on $W = \mathfrak{g}$.

The dual picture

Instead of seeking for P-module homomorphisms $\Phi\colon \bar{J}^k\mathbb{E}_\lambda \to \mathbb{E}_\mu$, we can pass to their dual morphisms $\Phi^*\colon \mathbb{E}^*_\mu \to (\bar{J}^k\mathbb{E}_\lambda)^*$. We shall see in a moment why this is very reasonable.

3.14. Verma modules. It is well known for holonomic jets that the dual modules $J^k\mathbb{E}_\lambda$ enjoy the nice algebraic structure of a finite dimensional part of the induced module

$$V_\mathfrak{p}(\lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E}^*_\lambda.$$

In particular, the dual to the inverse limit $J^\infty\mathbb{E}_\lambda$ is the whole *generalized Verma module* $V_\mathfrak{p}(\lambda)$. Thus, instead of looking for homomorphisms defined on the highly reducible P-modules $J^k\mathbb{E}_\lambda$ we have only to discuss morphisms defined on the highest weight modules $V_\mathfrak{p}(\lambda)$. These are quite well known in representation theory, see e.g. [BasE] for further links. Fortunately, our 'less symmetric' modules $\bar{J}^k\mathbb{E}_\lambda$ have quite similar duals which were first studied in [EasS].

We start with a modification of the definition of $\mathfrak{U}(\mathfrak{g})$. Our algebra $\bar{\mathfrak{U}}(\mathfrak{g})$ is defined as the quotient of the free tensor algebra $T(\mathfrak{g})$ by the ideal I which is generated by $\{X\otimes Y - Y\otimes X - [X,Y];\ \text{for all}\ X\in\mathfrak{p}, Y\in\mathfrak{g}\}$. Thus, we force the compatibility of the commutator with the bracket only for those brackets with at least one element in $\mathfrak{p}$.

3.15. Definition. The *semi-holonomic Verma module* induced from the P-module $\mathbb{E}_\lambda$ is the $(\mathfrak{g}, P)$-module

$$\bar{V}_{\mathfrak{p}}(\lambda) = \bar{\mathfrak{U}}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E}^*_\lambda.$$

The obvious filtration of $\bar{U}(\mathfrak{g})$ gives rise to the filtration of the semi-holonomic Verma module

$$\mathbb{E}_\lambda = \mathcal{F}_0\bar{V}_{\mathfrak{p}}(\lambda) \subset \mathcal{F}_1\bar{V}_{\mathfrak{p}}(\lambda) \subset \dots$$

by $\mathfrak{U}(\mathfrak{p})$-modules, with

$$\mathcal{F}_{k+1}\bar{V}_{\mathfrak{p}}(\lambda)/\mathcal{F}_k\bar{V}_{\mathfrak{p}}(\lambda) \simeq \otimes^{k+1}\mathfrak{g}_- \otimes \mathbb{E}_\lambda$$

as G_0–modules.

3.16. Lemma. *The $\mathfrak{U}(\mathfrak{p})$-modules $(\bar{J}^k\mathbb{E}_\lambda)^*$ dual to the semi-holonomic jet modules are naturally identified with the $\mathfrak{U}(\mathfrak{p})$-submodules $\mathcal{F}_k\bar{V}_{\mathfrak{p}}(\lambda)$ in the semi-holonomic Verma module $\bar{V}_{\mathfrak{p}}(\lambda)$.*

Proof. The claim is obvious on the level of G_0-modules. Consider $Z \in \mathfrak{p}_+$ and let us show the explicit computation for $k = 1$. The general case follows analogously from the full formula in Proposition 3.9.

If $X \otimes v^* \in \mathfrak{g}_- \otimes \mathbb{E}^*_\lambda \subset (J^1\mathbb{E}_\lambda)^*$, $Y \otimes v \in \mathfrak{p}_+ \otimes \mathbb{E}_\lambda \subset J^1\mathbb{E}_\lambda$, then

$$\begin{aligned}\langle X \otimes v^*, Z.(Y \otimes v)\rangle &= \langle X \otimes v^*, [Z, Y] \otimes v + Y \otimes Z.v\rangle \\ &= \langle X, [Z, Y]\rangle\langle v^*, v\rangle - \langle X, Y\rangle\langle Z.v^*, v\rangle \\ &= -\langle [Z, X], Y\rangle\langle v^*, v\rangle - \langle X, Y\rangle\langle Z.v^*, v\rangle\end{aligned}$$

If $[Z, X] \in \mathfrak{g}_-$, then the resulting expression equals to

$$\langle -[Z, X] \otimes v^* + X \otimes Z.v^*, Y \otimes Z\rangle.$$

Otherwise, the first summand disappears and we are left with $\langle X \otimes Z.v^*, Y \otimes v\rangle$.

The latter computation does not give the full information on $Z.(X \otimes v^*)$ though. We also have to consider its action on the image of the lower components. Given $v \in \mathbb{E}_\lambda \subset J^1\mathbb{E}_\lambda$ we have

$$\begin{aligned}\langle X \otimes v^*, Z.v\rangle &= \langle X \otimes v^*, Z.v + \eta^\alpha \otimes [Z, \xi_\alpha]_{\mathfrak{p}}.v\rangle \\ &= \langle X, \eta^\alpha\rangle\langle v^*, [Z, \xi_\alpha]_{\mathfrak{p}}.v\rangle \\ &= -\langle [Z, X]_{\mathfrak{p}}.v^*, v\rangle.\end{aligned}$$

As the result of both computations we obtain the formula saying that $Z \in \mathfrak{p}$ 'bubbles' through the elements X leaving always a new term behind, the bracket $[Z, X]$. The same happens then with the new term, until the resulting bracket element is not in $\mathfrak{p}$. Then it remains were it appears. This is exactly the action of Z in the semi-holonomic induced modules. □

3.17. Remarks. The most powerful tool in the theory of the generalized Verma modules is the central (or infinitesimal) character, see 6.12. We do not have a straightforward analogy for semi-holonomic Verma modules, but we can extend the translation functors due to Zuckermann and Jantzen. In fact we can show that once the essential homomorphisms building these functors would exist in the semi-holonomic setting, then also the whole translation principal worked as well. Special situations were studied in great detail in [EasS]. Roughly speaking, there are just two main points: (1) usually there are many homomorphisms of semi-holonomic Verma modules with a given symbol, (2) we have to find 'initial data' to start the translations with. It is known already from conformal Riemannian geometry, that there are homomorphisms of (holonomic) Verma modules which do not have semi-holonomic analogues, see again [EasS].

Thus, we are able to get very general structural results on the existence of the homomorphisms of semi-holonomic Verma modules. On the other hand, even if we would find the singular vectors in $\bar{V}_{\mathfrak{p}}(\lambda)$ defining those homomorphisms, it is not evident how to find analytic formulae for the operators in a direct algorithmic way. A combination of the direct discussion on the jet level with the dual picture seems to be most promising.

4. $|1|$-graded examples

In the next two sections, we shall indicate how to follow the Recipe 2.43 in concrete examples. We shall discuss the geometries in general dimensions, however we usually draw the diagrams for some of them only. In fact, the main algebraic tools from representation theory work with the complex modules, but we may use all these results for our real objects with the help of the complexification procedure. In all our examples, this causes no essential problems.

This section is devoted to the parabolic geometries with the irreducible tangent bundles, i.e. the length of the gradings must be $\ell = 1$. In the notation of the Dynkin diagrams with crossed nodes, this means that the sum of the coefficients of the highest root of $\mathfrak{g}$ at the simple roots corresponding to the crossed nodes must be one, see 6.3. Thus, there might be only one cross in the diagram and its position is further restricted, the full list of the complex forms appears in Table 14.

Since the general theory of Section 2 simplifies heavily for $|1|$-graded cases, we start our exposition with brief review.

Review of general properties

4.1. Corollaries. Let $\mathfrak{g}$ be a $|1|$-graded algebra, $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Consider a Lie group G with Lie algebra $\mathfrak{g}$ and let P be the subgroup of elements whose adjoint representation respects the filtration of $\mathfrak{g}$ as $\mathfrak{p}$-module.

Each parabolic geometry of the type G/P, i.e. a Cartan connection on a principal P-bundle $\mathcal{G} \to M$, is regular. All normal parabolic geometries are uniquely constructed from reductions of the linear frame bundles P^1M to the structure groups G_0, except $\mathfrak{g}$ is a real form of $\mathfrak{sl}(m+1,\mathbb{C})$, cf. [CSS2], [CSch]. The exceptions were already discussed in 2.5.

The whole torsion part of the curvature κ of the Cartan connection ω is constant on the fibers of $\mathcal{G}$.

All the Weyl geometries on M share the same torsion and each choice of a scale yields also the reduction of TM to the semisimple part of G_0. More generally, if the parabolic geometry is normal, then the torsion is zero whenever the homogeneity one component of the cohomology $H^2(\mathfrak{g}_-, \mathfrak{g})$ vanishes. In this case, all the Weyl geometries share a comon component of the curvature corresponding to non-vanishing cohomology of homogeneity two. If only homogeneity three is available, then the local invariant of the geometry in question is built of the first derivatives of the curvatures.

All identities and formulae simplify heavily too. The main reason is the commutativity of $\mathfrak{g}_{\pm 1}$.

4.2. Corollaries. In particular, the Ricci and Bianchi identities from 2.13 get

$$\boxed{\begin{gathered}(\nabla^\omega_X \circ \nabla^\omega_Y - \nabla^\omega_Y \circ \nabla^\omega_X)s = \lambda(\kappa_{\mathfrak{p}}(X,Y)) \circ s - \nabla^\omega_{\kappa_-(X,Y)} s \\ \sum_{\text{cycl}} \big([\kappa(X,Y), Z] - \kappa(\kappa_-(X,Y), Z) - \nabla^\omega_Z \kappa(X,Y)\big) = 0\end{gathered}}$$

The formulae in 2.16 and 2.17 for modules with trivial actions of $\mathfrak{g}_1$ specify always to

$$\boxed{\begin{gathered}(\nabla^\omega_X - \nabla^{\gamma^\sigma}_X)s(u) = \lambda([X, \tau(u)]) \circ s(u) \\ \nabla^{\hat\gamma} s(X) = \nabla^\gamma s(X) + \lambda([X, \Upsilon]) \circ s\end{gathered}}$$

4.3. Corollaries. There is only one choice for the definition of closed (generalized) Weyl geometries and affine bundles of scales, but the global existence of the bundle of scales has to be discussed separately for the individual geometries.

The transformations of the Weyl geometries σ and those of the induced connections $\gamma^\sigma_{\mathcal{S}}$ on the scale bundles $\mathcal{S}$ are given by the same one forms Υ, up to a multiple (cf. 2.23) and the curvature of $\gamma^\sigma_{\mathcal{S}}$ is given by the difference of the $\mathfrak{z}_E$-component of the curvature κ of the defining Cartan connection and the antisymmetrization of the deformation tensor P^σ (cf. 2.31).

The transformation rule for P under the change Υ of the (generalized) Weyl geometry σ follows from 2.18

$$\boxed{\hat{\mathsf{P}}.X = \mathsf{P}.X - \nabla^{\gamma_1}_X \Upsilon - \tfrac{1}{2}[\Upsilon, [\Upsilon, X]]}$$

4.4. Proposition. *For all torsion-free normal parabolic geometries of type G/P on M with irreducible tangent bundles, the closed (generalized) Weyl geometries are exactly those with symmetric Rho tensors.*

Proof. Let ω be the normal and torsion-free Cartan connection, κ^ω its curvature. The Bianchi identity implies (here e_i and e^i form dual basis of $\mathfrak{g}_{\pm 1}$)

$$[\kappa^\omega_{\mathfrak{g}_0}(X,Y), e_i] = [\kappa^\omega_{\mathfrak{g}_0}(X, e_i), Y] + [\kappa^\omega_{\mathfrak{g}_0}(e_i, Y), X]$$

and so the $\partial*$-closedness implies

$$\begin{aligned}0 &= \partial^*\kappa^\omega_{\mathfrak{g}_0}(X)(Y) - \partial^*\kappa^\omega_{\mathfrak{g}_0}(Y)(X)\\ &= \sum_i([[\kappa^\omega_{\mathfrak{g}_0}(X,e_i),Y],e^i] + [[\kappa^\omega_{\mathfrak{g}_o}(e_i,Y),X],e^i]) = \sum_i[[\kappa^\omega_{\mathfrak{g}_0}(X,Y),e_i],e^i].\end{aligned}$$

Now, the trace part in $\mathfrak{g}_0 \subset \mathfrak{gl}(\mathfrak{g}_{-1})$ is given by the evaluation of the Killing form

$$\sum_i \langle [\kappa^\omega(X,Y),e_i],e^i\rangle = -\sum_i \langle [[\kappa^\omega_{\mathfrak{g}_0}(X,Y),e_i],e^i],E\rangle = 0$$

Since the trace part is generated by E, its vanishing is equivalent to the vanishing of the $\mathfrak{z}_E$-component of κ^ω and the claim follows from the general discussion in 2.31. □

4.5. Let us choose a closed Weyl geometry σ_0, any Weyl geometry σ, and let Υ be the one-form transforming σ into σ_0. If the torsion of our structure is zero, then the change of the curvature on $\mathcal{S}$ caused by Υ is (the free arguments from $\mathfrak{g}_{-1}$ are denoted by $(-)$)

$$\operatorname{Alt}(\langle -\nabla^{\gamma^\sigma}_{(-)}\Upsilon,(-)\rangle - \tfrac{1}{2}\langle[\Upsilon,[\Upsilon,(-)]],(-)\rangle) = -d\Upsilon - \tfrac{1}{2}\operatorname{Alt}\langle[\Upsilon,(-)],[\Upsilon,(-)]\rangle = -d\Upsilon$$

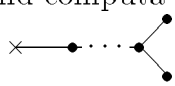
In particular, a choice of Υ does not change the curvature $\kappa^\sigma_{\mathcal{S}}$ if and only if $d\Upsilon = 0$. This is nicely compatible with the fact that the scales are parametrized by functions and the difference between the two connection forms on $\mathcal{S}$ is given by Υ, up to a multiple.

Now, since $\mathfrak{p}_+$ is abelian, the consecutive change from σ_1 to σ_2 and σ_3, achieved by means of Υ and Υ', equals to the change determined by $\Upsilon + \Upsilon'$. Thus, for each fixed σ, there is a class of one-forms $[\Upsilon]_\sigma$ transforming σ into a closed Weyl geometry. They all differ by closed forms. In the presence of Hodge theory on differential forms, the latter observation picks up locally a distinguished scale for each (generalized) Weyl geometry. This is exactly the case in the conformal Riemannian geometries, see e.g. [Gau].

In the presence of the torsion, the antisymmetrization of the covariant derivative does not equal to the exterior derivative and the formulae get more messy.

In the rest of this section, we shall illustrate the theory on several quite well known geometries.

Conformal Riemannian geometries

The basic features of conformal Riemannian geometries were reviewed in the introductory Section 1. Now we shall follow the steps in Recipe 2.43 in order to recover them again and we shall comment the necessary arguments and computations. In the Dynkin diagram notation, the pair $(\mathfrak{g},\mathfrak{p})$ is encoded by  in even dimensions and  in odd dimensions.

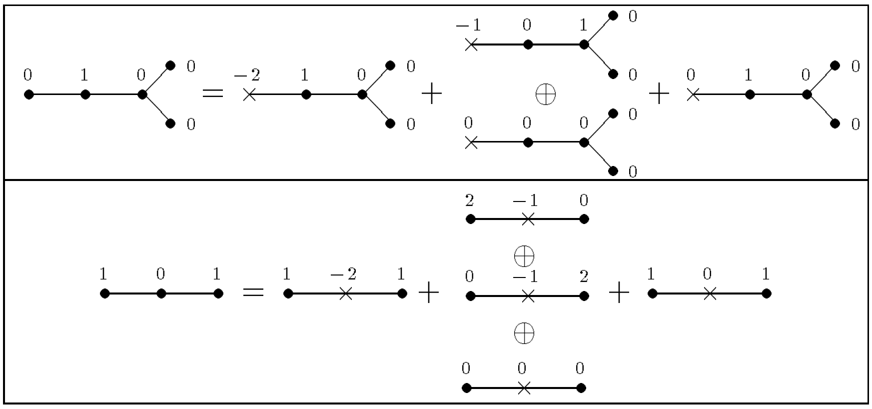


Table 1

Step 1. In the even dimensional case, the filtration of $\mathfrak{so}(2n+1,1,\mathbb{R})$ is easily found by looking at the orbit of the highest root $\overset{0}{\times}\!-\!\overset{1}{\times}\cdots\overset{0}{\times}\!<\!{}^{\times\, 0}_{\times\, 0}$. The zero weight has to be added to $\mathfrak{g}_0$, this is the one-dimensional center $\mathfrak{z}_E$. The result is displayed in Table 1.

As discussed in 6.7, the action of the grading element E on a highest weight vector is expressed by the scalar product of the coefficients over the nodes in the Dynkin diagram with the first column in the inverse Cartan matrix. Thus we obtain the vector $(1\ 1\ \dots\ 1\ {}^{1/2}_{1/2})$ in the even dimensional cases and $(1\ \dots\ 1\ \frac{1}{2})$ in odd dimensions.

The computations for odd dimensional geometries are slightly different, since there always are the zero weights in the defining representations which do not belong to the orbit of the highest weight. Fortunately, the highest weights of $\mathfrak{g}_{-1}$, the semisimple part of $\mathfrak{g}_0$, and $\mathfrak{g}_1$ are still in the orbit of $\overset{0}{\times}\!-\!\overset{1}{\times}\cdots\overset{0}{\times}\!\Rightarrow\!\overset{0}{\times}$, except the case $\mathfrak{g}=\mathfrak{so}(5,\mathbb{C})$. In the latter case, the orbit does not touch $\mathfrak{g}_0$, but we can locate it easily in the tensor product of $\mathfrak{g}_1 = \langle \overset{0}{\times}\!\Rightarrow\!\overset{2}{\times}, \overset{2}{\times}\!\Rightarrow\!\overset{-2}{\times}, \overset{1}{\times}\!\Rightarrow\!\overset{0}{\times} \rangle$ and $\mathfrak{g}_{-1} = \langle \overset{-2}{\times}\!\Rightarrow\!\overset{2}{\times}, \overset{0}{\times}\!\Rightarrow\!\overset{-2}{\times}, \overset{-1}{\times}\!\Rightarrow\!\overset{2}{\times} \rangle$. The results are displayed in Table 2.

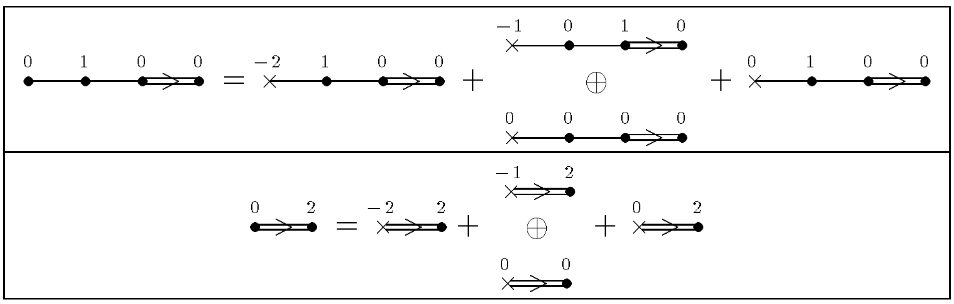


Table 2

As in all |1|-graded cases, a conformal Riemannian structure on a manifold M is defined by the reduction of TM to the structure group G_0, i.e. to the group $CO(m,\mathbb{R})$. This means exactly the choice of a scalar product on each T_xM, up to multiples.

Step 2. Konstant's version of the BBW theorem yields easily the G_0-modules in the cohomology $H^2(\mathfrak{p}_+, \mathfrak{g})$, which are dual to the cohomologies we need. The homogeneity is then found by the action of E and because of the extremely simple structure of $\mathfrak{g}$, this yields the complete information about the curvatures. See Table 3 for the results.

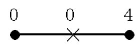
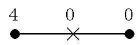
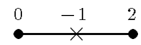
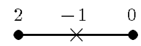
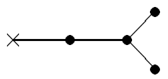
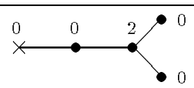
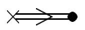
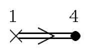
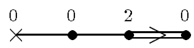
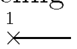
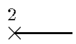
pair $(\mathfrak{g}, \mathfrak{p})$	cohomologies	homogeneity	curvature components
	 	2 2	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to$  $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to$ 
		2	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_0$
		3	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_1$
		2	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_0$

Table 3

These cohomological results can be interpreted easily in the classical terms. The common ∂^*-closed torsion of all Weyl geometries is zero (because there is no cohomology of homogeneity one). If the dimension is greater then four, then there is the unique local invariant of the structures which is given by the shared component of the curvature of the Weyl geometries, valued in the bundle described in the second column. On four dimensional conformal manifolds, this local invariant further splits into the self-dual and anti self-dual parts. Finally, there is no shared part of the curvatures of Weyl geometries in dimension three, but the Cotton-York tensor yields the local invariant of the structure.

Step 3. By the general theory and the formula for the action of E, the modeling line bundle of the scale bundle $\mathcal{S}$ should correspond to the highest weight  $\cdots$. Obviously such a representation of P exists. On the other hand, the metrics in the distinguished conformal class live in an affine line bundle modeled over densities with highest weight  $\cdots$. We should like to see, how our scales (as sections of $\mathcal{S}$) correspond to the metrics.

Recall again that metrics on M are reductions of the linear frame bundle P^1M to the structure group $O(m, \mathbb{R})$. Thus they are in bijective correspondence with the sections of the quotient $P^1M/O(m, \mathbb{R})$. Therefore, the metrics from the distinguished conformal class are in bijection with sections of $\mathcal{G}_0/O(m, \mathbb{R})$ and the orthogonal group plays the role of $\ker \nu_P$ from 2.22. This correspondence is easily understood locally: A section of $P^1\mathbb{R}^m$ is a $GL(m, \mathbb{R})$-valued function, say $A(x)$. Each value $A(x)$ defines the positive definite matrix $g(x) = A(x)A(x)^T$, which has the property that $A(x)^T g(x)^{-1} A(x)$ is the identity matrix $\mathbb{I}_m$ for all x. In particular, if $A(x) = e^{f(x)}B(x)$ with $B(x) \in O(m, \mathbb{R})$, then $g(x) = e^{2f(x)}\mathbb{I}_m$. This yields the above identification and we see, that our $\mathcal{S}$ plays the role of a square root of the bundle of conformal metrics.

Step 4. Since $\mathfrak{g}$ is $|1|$-graded, the interpretation of the Weyl geometries is simple. By the general theory, the connections γ^σ are all linear connections belonging to the bundle $\mathcal{G}_0$, which share the common minimal torsion. In our case, this torsion must be zero since there is no cohomology in the homogeneity one. Thus we obtain exactly all torsion free connections belonging to the G_0 structure on M.

Now, the application of our general formulae lead to all the well known objects like the twistor connections, conformal circles, canonical coordinates, etc. See e.g. [Eas] for much information on interesting objects in conformal Riemannian geometries.

Projective and almost quaternionic geometries

The next two series of examples are special cases of the so called almost Grassmannian structures. They correspond to the choices ×—•···• (projective structures) and •—×—•···• (almost quaternionic structures).

Step 1. First of all we have to discuss which group G with the Lie algebra $\mathfrak{g}$ we choose. From the formal point of view, the most natural choice is with G_0 being the adjoint group of the $\mathfrak{g}_0$ module $\mathfrak{g}_{-1}$, which does not impose any further properties on our structures. This does not coincide with the most obvious choices, e.g. $G = SL(p+q, \mathbb{C})$, $p = 1, 2$, in the complex case, since then the action of $G_0 = S(GL(p,\mathbb{C}) \times GL(q,\mathbb{C}))$ on $\mathfrak{g}_{-1}$ has a non-trivial (discrete) kernel in general. Thus, either we have to work modulo this kernel, or our manifolds M will be oriented (as an additional ingredient of our structure). For the sake of simplicity, we take the second option in our discussion below.

The projective geometries are one of the exceptional examples of second order structures which were discussed in 2.5. The almost quaternionic geometries correspond to the choice of the real form $\mathfrak{sl}(1+q,\mathbb{H})$ of $\mathfrak{sl}(2+2q,\mathbb{C})$. The dimension of the manifolds is $4q$ in this case. Table 4 shows the filtrations of the complex groups. The low dimensional almost quaternionic case (with $q = 1$) coincides with the 4-dimensional conformal geometries (the last line in our table is related to 8-dimensional geometries). The filtrations are computed easily starting with the highest root of $\mathfrak{g}$.

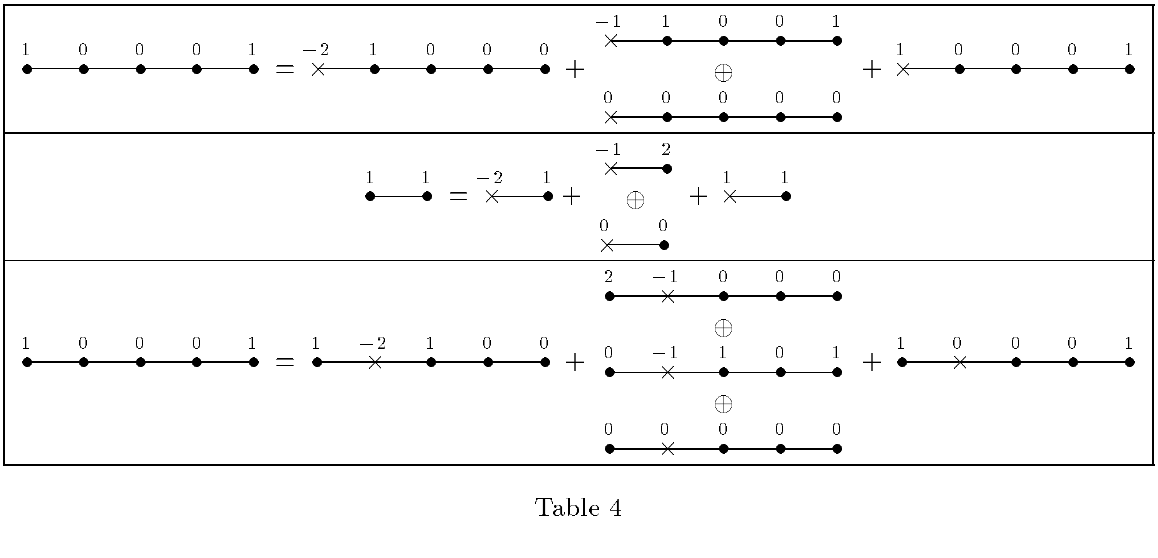


Table 4

By the general theory, the action of the grading element E is given by the vector $\frac{1}{m+1}(m\ (m-1)\ \dots\ 1)$ in the m-dimensional projective case, or by the vector $\frac{1}{2(q+1)}(2q\ 4q\ 2(2q-1)\ \dots\ 2)$ in the $4q$-dimensional almost quaternionic case.

Step 2. Again, the above data together with Kostant's version of BBW yield the cohomologies as displayed in Table 5 (cf. 6.10). The computations for the projective structures are very similar to the conformal Riemannian cases with an analogy to Cotton-York tensor in the lowest dimension and the Weyl curvature in all remaining ones. The computations for the other structures deserve more comments. In particular, due to the location of the cross, we have exactly two possibilities for actions with $w \in W$ of length two. This is the source for the two different components. In the lowest dimensions, both of them are of degree two (see the 4-dimensional conformal case), but they split into a torsion part and a Weyl part in all higher dimensions. A direct inspection of the weights in the tensor product $\mathfrak{g}_1 \otimes \mathfrak{g}_1 \otimes \mathfrak{g}_0$ yields also the target of the curvature component (the other two components could never produce the coefficient 3).

pair $(\mathfrak{g}, \mathfrak{p})$	cohomologies	hom.	curvature components
$\times$—$\bullet$	$\overset{4}{\times}$—$\overset{1}{\bullet}$	3	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_1$
$\times$—$\bullet$—$\bullet$—$\bullet$—$\bullet$	$\overset{1}{\times}$—$\overset{1}{\bullet}$—$\overset{0}{\bullet}$—$\overset{1}{\bullet}$—$\overset{1}{\bullet}$	2	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_0$
$\bullet$—$\times$—$\bullet$—$\bullet$—$\bullet$	$\overset{0}{\bullet}$—$\overset{0}{\times}$—$\overset{1}{\bullet}$—$\overset{0}{\bullet}$—$\overset{3}{\bullet}$ $\overset{3}{\bullet}$—$\overset{-2}{\times}$—$\overset{1}{\bullet}$—$\overset{1}{\bullet}$—$\overset{0}{\bullet}$	2 1	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to$ $\overset{0}{\bullet}$—$\overset{-1}{\times}$—$\overset{1}{\bullet}$—$\overset{0}{\bullet}$—$\overset{1}{\bullet}$ $\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$

Table 5

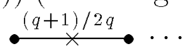
Projective structures – Steps 3 & 4. Let us cosider the affine bundle of volume forms on a oriented manifold M. The modeling vector bundle is the line bundle of densities corresponding to the highest weight $\overset{m}{\times}$— $\cdots$. Since M is oriented, we can choose its mth root, a bundle of densities with the appropriate highest weight $\overset{m}{\times}$— $\cdots$. The choice of a scale leads to the reduction of the linear frame bundle $P^1M = \mathcal{G}_0$ to the group $SL(m, \mathbb{R})$. Such reductions are in correspondence with sections of $P^1M/SL(m, \mathbb{R})$ and these sections can be again understood locally as matrix valued functions. The correspondence to volume forms is then given by their determinants. Thus, our scale bundle $\mathcal{S}$ can be considered as the mth root of the bundle of volume forms. In particular, each global scale determines a volume form globally.

Since the projective structures are exceptional (i.e. the structure group of $\mathcal{G}_0$ is too big to encode the whole structure), the Weyl geometries cannot be arbitrary connections on $\mathcal{G}_0$ with the minimal available torsion (which is zero again). Their further restrictions come from the choice of the whole bundle $\mathcal{G}$ as a reduction of the second order frame bundle on M. In fact it is easy to verify their classical description using our tools: We claim that they all share the (non-parameterized) geodesics. Indeed, the geodesics are flow lines of auto-parallel vector fields and our formula for the transformation of the covariant deirvatives says that $\nabla^{\gamma^\sigma}_X \xi(u)$ with

$\xi(u) = X$ varies by $[[X, \Upsilon], X]$. Now, in index notation, the iterated bracket is given by

$$[[X^a, \Upsilon_b], Y^c] = (X^a \Upsilon_b + X^c \Upsilon_c \delta^a_b) Y^b$$

which is a scalar multiple of X^a if $Y^a = X^a$. Thus we can always rescale the original vector field in order to obtain an auto-parallel vector field again.

Almost-quaternionic structures – Steps 3 & 4. The almost quaternionic structures are much more interesting. First of all we notice the presence of a possible non-zero torsion, which will be shared by all the Weyl geometries. By definition, the *quaternionic structures* are then just those where this torsion vanishes. There still remains the Weyl part of the curvature which is shared by all Weyl geometries for the quaternionic structures, but gets mixed with the torsion contributions in general. The meaning of the structure is also clear from the description of $\mathfrak{g}$: There are the 'defining bundles' which correspond to the standard representations $\mathbb{C}^{2*}$ and $\mathbb{C}^{2q}$ of $GL(2, \mathbb{C})$ and $GL(2q, \mathbb{C})$ (viewed as G_0 representations in the obvious way). The complexification of the tangent bundle on the oriented manifold M is identified with the tensor product of these defining bundles. At the same time, the top degree forms of these bundles have to be identified since the structure group is further reduced from $GL(2, \mathbb{C}) \times GL(2q, \mathbb{C})$ to $S(GL(2, \mathbb{C}) \times GL(2q, \mathbb{C}))$ (notice again the role played by the chosen orientation of M). The line bundle  exists and so does the bundle of affine scales. Again, its relation to the volume forms on M is easily understood: the sections of $\mathcal{S}$ correspond to reductions of the complexified tangent spaces from $S(GL(2, \mathbb{C}) \times GL(2q, \mathbb{C}))$ to its semisimple part $SL(2, \mathbb{C}) \times SL(2q, \mathbb{C})$, thus they determine the choice of a (complex) volume form on $TM \otimes \mathbb{C}$ which induces a real volume form on TM.

In the lowest possible dimension, i.e. for $q = 1$, we recover the spinor approach to the (complexified) conformal Riemannian geometries. Since we require vanishing of half of the curvature in the definition of the higher dimensional quaternionic structures, they behave similarly to the self-dual conformal structures.

Again, we can easily discuss analogies to twistor connections, D-operators, analogues to conformal circles, etc. Many concepts and results discussed in [BaiE], [BEG], [Sal] can be nicely achieved by our tools.

5. |2|-graded examples

A real version of CR-structures

Step 1. The algebras in question are $\mathfrak{g} = \mathfrak{sl}(m+2, \mathbb{R})$. These are the split forms of the complex algebras $\mathfrak{sl}(m+2, \mathbb{C})$ and the grading is given by the Dynkin diagram with the most left and most right nodes crossed. Again, we have to discuss the chosen Lie group G. The most obvious choice $G = SL(m+2, \mathbb{R})$ leads to $G_0 = S(GL(1, \mathbb{R}) \times GL(m, \mathbb{R}) \times GL(1, \mathbb{R}))$, $P = G_0 \rtimes \exp \mathfrak{p}_+$, and the action of this G_0 on $\mathfrak{g}_-$ is not effective. Thus we either have to work modulo the kernel of this action, or we have to involve a lift of the adjoint group to P into the definition of $\mathcal{G}$ and the Cartan connection.

Table 6 displays the filtrations of $\mathfrak{g}$ for $m = 1$ (which is an exceptional low dimensional case) and $m = 3$ (which represents well the general behavior). Since there are two crossed nodes in the diagram, $\mathfrak{g}_{\pm 1}$ consist of two G_0-submodules each, which are pair-wise dual. In fact all our considerations apply to the other real forms $\mathfrak{su}(p+1, q+1)$ as well. So the expected behavior of our structures should be similar to the standard CR-structures. The main advantage of our approach is that we can involve torsion, i.e. we cover also the so called almost CR-structures (defined e.g. on hypersurfaces in almost complex manifolds).

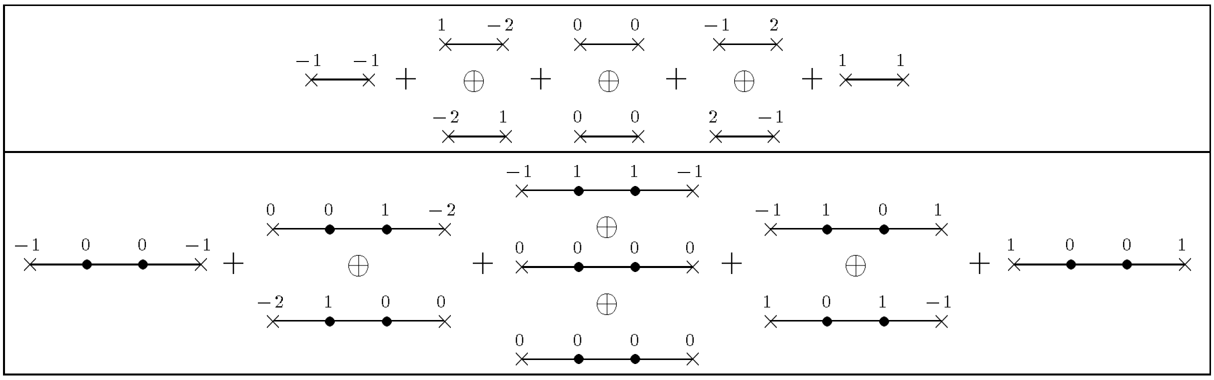


Table 6

The definition of the corresponding geometries is by far not so simple as in the $|1|$-graded cases. Of course, our most general approach would say just 'the structure is defined by a choice of the principal P-bundle $\mathcal{G}$ equipped with the Cartan connection ω'. However, we prefer to show explicitly what is the structure on the manifold M which can be used to construct $\mathcal{G}$ and the normal regular Cartan connection ω on $\mathcal{G}$ uniquely, in the sence of Remark 2.6 and Theorem 2.4.

We first have to understand the $\mathfrak{g}_0$-module $\mathfrak{g}_-$ equipped with the G_0-equivariant Lie bracket. Obviously, the dimensions of the manifolds must be $2m+1$. In the block matrix form, we can describe the grading of $\mathfrak{g}$ as follows

$$\begin{pmatrix} * & \mathfrak{g}_1^L & \mathfrak{g}_2 \\ \mathfrak{g}_{-1}^L & * & \mathfrak{g}_1^R \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1}^R & * \end{pmatrix}$$

where the stars indicate the $\mathfrak{g}_0$ entries. Let us call $\mathfrak{g}_{-1}^L$ and $\mathfrak{g}_{-1}^R$ the 'left' and 'right' submodules, respectively. Analogously we refer to the $\mathfrak{g}_0$-submodules in $\mathfrak{g}_1$.

The grading element E acts on the coefficients over the nodes in Dynkin diagrams by the vector $(1\ 1\ \dots\ 1)$, the sum of the first and last columns in the inverse Cartan matrix, cf. 6.7. Another element E' in the center acts by the difference of the same columns, i.e. by the vector $\frac{1}{m+2}(m\ (m-2)\ \dots\ -(m-2)\ -m)$. They are given by block matrices (E' is found easily by its action on the simple root vectors)

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E' = \frac{m}{m+2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{2}{m}\mathbb{I}_m & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Obviously, E' is orthogonal to E. The homogeneity degrees (d_1, d_2) of the individual modules in the filtration (with respect to actions of E and E' respectively) are indicated in the following scheme

$$\begin{array}{ccccc} & & (0,0) & & \\ & (-1,1) & & (1,-1) & \\ (-2,0) & & (0,0) & & (2,0) \\ & (-1,-1) & & (1,1) & \\ & & (0,0) & & \end{array}$$

In particular, E' acts trivially on $\mathfrak{g}_{\pm 2}$ and $\mathfrak{g}_0$ and it distinguishes the left and right submodules in $\mathfrak{g}_{\pm 1}$. We also see that the submodules $\mathfrak{g}^R_{\pm 1}$ are the upper ones in the filtration above. Clearly $(\mathfrak{g}^L_{-1})^* = \mathfrak{g}^L_1$ as $\mathfrak{g}_0$-module and similarly for $\mathfrak{g}^R_{\pm 1}$. The bracket restricted to $\mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-1}$ is non-degenerate, while it vanishes on $\mathfrak{g}^L_{-1} \wedge \mathfrak{g}^L_{-1}$ and $\mathfrak{g}^R_{-1} \wedge \mathfrak{g}^R_{-1}$. The adjoint group G_0 acting effectively on $\mathfrak{g}_-$ is obtained by choosing G as the quotient of the group of all $A \in GL(m+2, \mathbb{R})$ with $|\det A| = 1$ by its center. For odd m this is exactly the special linear group, but we obtain a non trivial center for even m and G_0 will have two components in that case.

Now we can define a regular normal parabolic geometry of the type in question: *A real almost CR-structure on a manifold M is given by a filtration $TM = T^{-2}M \supset T^{-1}M$ with $\dim T^{-1}_x M = 2m$ for all $x \in M$ together with a reduction of the associated graded vector bundle to the structure group G_0, subject to the structure equation.* The structure equation is specified as follows. The grading and the Lie bracket defines the *Levi form*

$$\mathcal{L}\colon T^LM \times T^RM \to TM/T^{-1}M.$$

The existence of the G_0-structure on $\operatorname{Gr} TM$ defines another tensorial form $T^LM \times T^RM \to TM/T^{-1}M$ which is non-degenerate, and two G_0-invariant complementary subbundles T^LM, $T^RM \subset T^{-1}M$. The structure equation then requires that the latter tensorial form coincides with the Levi form $\mathcal{L}$. This has further consequences. Since $[\mathfrak{g}^L_{-1}, \mathfrak{g}^L_{-1}] = 0$, the restriction of the Levi form to $T^LM \wedge T^LM$ vanishes and analogously $\mathcal{L}$ restricted to $T^RM \wedge T^RM$ is zero. Thus there are two additional tensorial forms on $T^{-1}M$ defined by means of the Lie bracket

$$\mathcal{L}^L\colon T^LM \wedge T^LM \to T^RM \quad \textit{(the left Levi form)}$$
$$\mathcal{L}^R\colon T^RM \wedge T^RM \to T^LM \quad \textit{(the right Levi form)}$$

In particular, the Lie bracket of vector fields maps both $C^\infty(T^LM) \times C^\infty(T^LM)$ and $C^\infty(T^RM) \times C^\infty(T^RM)$ into $T^{-1}M = T^LM \oplus T^RM$ and the Levi form defines a conformal symplectic structure on $T^{-1}M$ with two distinguished complementary Lagrangian subbundles. Thus we also call these structures *Lagrangian contact structures*. Special cases of such structures were also studied in [Tak].

Equivalently these structures are described just by the (local) contact structure together with two distinguished complementary Lagrangian subbundles. The structure is called *integrable* if the distinguished Lagrangian subbundles are integrable, i.e. if the left and right Levi forms vanish identically.

pair $(\mathfrak{g}, \mathfrak{p})$	cohomologies	actions of E, E'	curvature components
$\times\!\!-\!\!\!-\!\!\times$	$\overset{5}{\times}\!\!-\!\!\!-\!\!\overset{-1}{\times}$ $\overset{-1}{\times}\!\!-\!\!\!-\!\!\overset{5}{\times}$	4, 2 4, -2	$\mathfrak{g}^L_{-1} \times \mathfrak{g}_{-2} \to \mathfrak{g}^L_1$ $\mathfrak{g}^R_{-1} \times \mathfrak{g}_{-2} \to \mathfrak{g}^R_1$
$\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\times$	$\overset{2}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-3}{\times}$ $\overset{-3}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{2}{\times}$ $\overset{-1}{\times}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\overset{-1}{\times}$	1, 3 1, -3 2, 0	$\mathfrak{g}^L_{-1} \times \mathfrak{g}^L_{-1} \to \mathfrak{g}^R_{-1}$ $\mathfrak{g}^R_{-1} \times \mathfrak{g}^R_{-1} \to \mathfrak{g}^L_{-1}$ $\mathfrak{g}^L_{-1} \times \mathfrak{g}^R_{-1} \to \overset{-1}{\times}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\overset{-1}{\times}$

Table 7

Step 2. The Kostant's BBW yields the necessary cohomologies. The knowledge of the homogeneities of the components in $\mathfrak{g}$ and some inspection of the weights in their tensor products yield further explicit information, see Table 7.

As a consequence of these data we obtain

Proposition. *The whole torsion component κ_- of the curvature of the canonical Cartan connection vanishes if and only if the corresponding defining subbundles $T^L M$ and $T^R M$ are integrable. The 3-dimensional Lagrangian contact structures are always integrable.*

Proof. The curvature component corresponding to the lowest homogeneity in the cohomology is constant along the fibers. Since this component of $\kappa(u)(X,Y)$ is in the torsion part, we can evaluate it on the underlying tangent bundle $T_x M$ by means of the isomorphism $u\colon \operatorname{Gr} T_x M \to \mathfrak{g}_-$. We obtain

$$[X,Y] - u\big(Tp([\omega^{-1}(X), \omega^{-1}(Y)](u))\big)$$

but this is exactly the sum of the values of the right and left Levi forms expressed via u. Therefore the vanishing of these forms is equivalent to the vanishing of the cohomology with the lowest homogeneity. By the general theorem, if this vanishes then the next possibly non-zero component appears according to the next non-zero cohomology. This does not belong to the torsion part in our case. $\square$

Let us also notice, that the existence of the complementary Lagrangian subspaces is dictated by the regularity requirement for the canonical normal connection. Of course, there can be more general Cartan connections on M without this property. On the other hand, the complete information on the second cohomology shows that all normal Cartan connections are automatically regular.

Steps 3 & 4. Since the center of $\mathfrak{g}_0$ is 2-dimensional, the closed Weyl geometries are defined by a choice of the complement of E in the center. We shall make the most natural choice, the orthogonal complement spaned by E'. By the general theory, the scalar actions of E and E' on the line bundle modeling the affine line bundle of scales are then one and zero. Thus we should obtain the line bundle $\overset{1/2}{\times}\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\overset{1/2}{\times}$.

The $\mathfrak{p}$-submodule $\mathfrak{g}_2 \subset \mathfrak{p}$ gives rise to the invariant subbundle $T^*_2 M \subset T^* M$, the anihilator of $T^{-1}M$. Let us assume now that we have chosen a lift $\mathcal{G}'_0$ of $\mathcal{G}_0$ to a principal bundle with structure group $S(GL(1,\mathbb{R})\times GL(m)\times GL(1,\mathbb{R}))$. This makes no problems locally and the line bundle $\overset{1/2}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1/2}{\times}$ exists as an assoicated bundle to $\mathcal{G}'$. Now, the action of $(a, B, c) \in S(GL(1,\mathbb{R}) \times GL(m,\mathbb{R}) \times GL(1,\mathbb{R}))$ on $\mathfrak{g}_2$ is given by the scalar $ac \in \mathbb{R}$ and the choice of a scale leads to the reduction of G_0 to elements of the form (a, B, a). Thus we can understand L locally as the square root of the affine bundle of non-zero sections of T_2^*M. Consequently, the choice of a scale σ yields a contact form θ^σ annihilating $T^{-1}M$. In particular, it determines a vector field transversal to $T^{-1}M$ and hence a splitting $TM = T^LM \oplus T^RM \oplus \mathbb{R}$. We also obtain the symplectic structure on $T^{-1}M$ defined by the Levi form. The linear connection γ^σ is then just the connection which preserves θ^σ (thus also the two Lagrangian subspaces), belongs to the induced G_0-structure on TM, and has the minimal torsion. Let us notice, that even if the bundles T^LM and T^RM are integrable, there still can appear some torsion, which comes from the contribution of the mapping $(X, Y) \mapsto [\mathsf{P}.X, Y]$, $X \in \mathfrak{g}_{-1}$, $Y \in \mathfrak{g}_{-2}$.

The defining representation on $\mathbb{R}^{m+2}$ gives rise to analogues of twistor bundles equipped with a canonical connection, etc. We shall not go into details here, but let us mention at least the horizontal flows $\alpha^{u,X}$ in directions transversal to $T^{-1}M$. The transversality means that the tangent vector $\xi(x) = \{u, X\} \in \mathcal{G} \times_P \mathfrak{g}_-$ at the origin of the flow has a non-zero component of X in $\mathfrak{g}_{-2}$. A horizontal flow $\alpha^{u.g,\tilde{X}}$ has the same tangent direction at x if and only if $\mathrm{Ad}_{\mathfrak{g}_-}\, g.\tilde{X} = X$. In view of the assumption on X this means $g = \exp Z$, $Z \in \mathfrak{g}_2$, and a computation similar to that in 2.35 shows that the flows will coincide up to parametrizations. These transversal horizontal flows are the analogues to the well known chains of Chern and Moser in CR-geometry.

Let us also illustrate the idea of the general correspondence spaces on this example. Following the general theory, there is a canonical fibration over each projective manifold M, which is equipped with the Lagrangian contact structure. The latter fibration is defined as the quotient $M' = \mathcal{G}/P$ of the canonical bundle $\mathcal{G}$ over M by our subgroup P (which is a subgroup in the structure group P' of $\mathcal{G} \to M$). Over 2-dimensional projective structures, we obtain integrable 3-dimensional Lagrangian contact structures and the Cotton-York part of the curvature of the projective structure gives rise to the non-trivial component $\mathfrak{g}_{-1}^L \times \mathfrak{g}_{-2} \to \mathfrak{g}_1^L$, while the other one vanishes. In all higher dimensions, one of the torsions of the structure on M' van- Check!
ishes while the other one vanishes if and only if the original projective structure is flat (and in that case also the third non-trivial component vanishes automatically).

The geometric construction of M' goes as follows: Let M be a projective manifold of dimension $m+1$ and define $M' = \mathbb{P}(T^*M)$. This is a manifold of dimension $2m+1$ with fibration $p\colon M' \to M$ and there is the exact sequence

$$0 \to VM' \to TM' \to p^*(TM) \to 0$$

Next, notice that each element $x' \in M'$ defines a nowhere vanishing linear form on M, up to a multiple. Hence there are distinguished subspaces of rank m in $p^*(TM)$ corresponding to the kernels. Now, each linear connection on M defines a splitting $p^*(TM) \to TM'$ since M' is the associated bundle $P^1M \times_{G_0} \mathbb{P}(\mathbb{R}^{(m+1)*})$. Thus we obtain the subspaces T^LM' in TM' complementary to $T^RM = VM'$ which play the role of the two defining subbundles. A direct check shows that a change of the connection within the given projective class does not effect the subspaces T^LM'. See notes from 18/10/97 for detai
Of course, the vertical bundle is involutive.

These correspondence spaces for projective manifolds have been already discussed in [Tak]. However, only the evident result that the structure on M' is locally flat if and only if the projective structure on M is locally flat was deduced there.

The case $\times\!-\!\!-\!\times\!-\!\!-\!\bullet\cdots\bullet$

In the last two examples, we shall only briefly indicate what kinds of difficulties we have to face. More detailed exposition of basic features of these very interesting geometries will appear elsewhere. The filtration of $\mathfrak{g} = \mathfrak{sl}(k+1,\mathbb{R})$ is described in Table 8

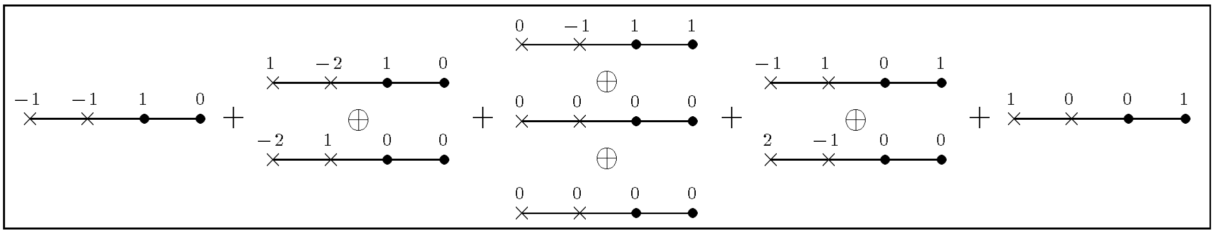


Table 8

The grading element E is the block diagonal matrix of sizes $(1,1,k-1)$ with blocks $\frac{1}{k+1}(2k-1, k-2, -3\mathbb{I}_{k-1})$ and it acts on the coefficients over the nodes by the vector $\frac{1}{k+1}((2k-1)\ (3k-3)\ (3k-6)\dots 6\ 3)$. Let us take the complementary element E' which is the block diagonal matrix $\frac{1}{k+1}(1,-k,\mathbb{I}_{k-1})$. This elements acts, similarly to the previous example, by the difference of the elements in the appropriate columns in the inverse Cartan matrix. So it acts by the vector $\frac{1}{k+1}(1\ (1-k)\ (2-k)\cdots -2\ -1)$. Let us write $\mathfrak{g}^R_{-1}$ and $\mathfrak{g}^L_{-1}$ for the upper and lower module in $\mathfrak{g}_{-1}$, respectively. Then E acts by -1 on both, while E' acts by 1 on $\mathfrak{g}^R_{-1}$ and by -1 on $\mathfrak{g}^L_{-1}$. The lowest dimensional example $\times\!-\!\!-\!\times$ has been already discussed, the only difference for $k=3$ is the module $\overset{0}{\times}\!-\!\overset{-1}{\times}\!-\!\overset{2}{\bullet} \subset \mathfrak{g}_0$.

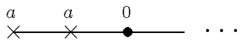
Now, the cohomologies are computed in Table 9.

pair $(\mathfrak{g},\mathfrak{p})$	cohomologies	action of E, E'	curvature components
$\overset{1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}$	$\overset{-4}{\times}\!-\!\overset{4}{\times}\!-\!\overset{0}{\bullet}$ $\overset{4}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet}$ $\overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{4}{\bullet}$	$1,-3$ $2,2$ $3,-1$	$\mathfrak{g}^R_{-1}\wedge\mathfrak{g}^R_{-1}\to\mathfrak{g}^L_{-1}$ $\mathfrak{g}_{-2}\wedge\mathfrak{g}^L_{-1}\to\mathfrak{g}^R_{-1}$ $\mathfrak{g}_{-2}\wedge\mathfrak{g}^R_{-1}\to\mathfrak{g}_0$
$\overset{1}{\times}\!-\!\overset{0}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{1}{\bullet}$	$\overset{-3}{\times}\!-\!\overset{1}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}$ $\overset{4}{\times}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{1}{\bullet}$ $\overset{0}{\times}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{3}{\bullet}$	$0,-2$ $2,2$ $3,-1$	$\mathfrak{g}^R_{-1}\wedge\mathfrak{g}^R_{-1}\to\mathfrak{g}_{-2}$ $\mathfrak{g}_{-2}\wedge\mathfrak{g}^L_{-1}\to\mathfrak{g}^R_{-1}$ $\mathfrak{g}_{-2}\wedge\mathfrak{g}^R_{-1}\to\mathfrak{g}_0$

Table 9

The most interesting part is the existence of the cohomology of degree zero. This implies that there might be a normal parabolic geometry which is not regular.

Notice that then this part of the curvature will be constant along the fibers of $\mathcal{G} \to \mathcal{G}_0$ and the component will obstruct the equality of the two algebraic brackets $T^R \wedge T^R \to TM/T_{-1}M$ defined by the Lie bracket and the G_0-structure on $\operatorname{Gr} TM$. Also the next component of homogeneity two is extremely interesting. Our intuition tells that this part of the curvature should be related to the possibly non-zero algebraic bracket $T^R M \wedge T^R M \to T^L M$ (an analogue to the rigth Levi form on the real almost-CR structures), however this one would correspond to homogeneity three.

Als the choice of the representation $\nu_P \colon P \to Gl^+(\mathbb{R})$ needed for the definition of the affine bundle of scales is much more subtle. In fact, the most natural choice of the orthogonal complement to E in the center of $\mathfrak{g}_0$ leads to the weight  , $a = \frac{m+2}{5m+1}$ which does not seem to make much sense. On the other hand, we can fix the complement in such a way that the line bundle $\mathcal{G}_0 \times_P \mathfrak{g}_1^L$ turns out to be the modeling line bundle. The block matrix form of the corresponding E' is $(m, m, -2\mathbb{I}_m)$ and E' acts on weights by the vector $(m\ 2m\ 2m-2\ \dots\ 2)$. Thus the affine scale bundle $\mathcal{S}_\nu$ exists for this choice and a global scale yields also a global non-zero section ξ of $T^L M$, and so the explicit splitting of TM.

The case 

The last example deals with a case where we have crossed only one node in the Dynkin diagram, but the coefficient at that particular simple root in the expression for the highest root of $\mathfrak{g}$ is two. Thus we obtain a |2|-graded algebra with irreducible $\mathfrak{g}_{-1}$.

As already discussed, these are the other examples of the exceptional geometries from 2.5.

It is more tricky now to compute the filtration, since there are more non-trivial orbits of the Weyl group in this case. The result is in the Table 10.

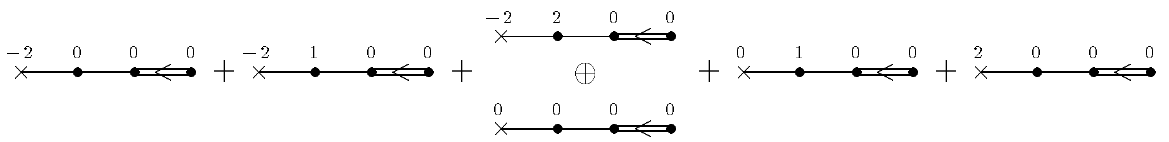
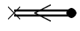


Table 10

In particular, we deal again with contact structures. The dimension of the manifolds is $(2k-2)+1$, where k is the number of the nodes. The center of G_0 is this time one-dimensional and the grading element E acts by the vector $(1\ 1 \dots 1\ 1)$. The low dimensional case  enjoys the same filtration. The cohomologies are listed in Table 11

The results are quite similar to projective geometries, there is a sort of Weyl curvature in higher dimensions while a Cotton-York tensor appears in dimension three. Since the center of $\mathfrak{g}_0$ is one-dimensional, there is only one choice for the bundle of scales which will be modeled over the square root of $\mathcal{G} \times_P \mathfrak{g}_2$.

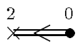
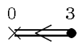
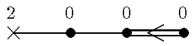
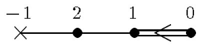
pair $(\mathfrak{g}, \mathfrak{p})$	cohomologies	action of E	curvature components
		3	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_1$
		2	$\mathfrak{g}_{-1} \wedge \mathfrak{g}_{-1} \to \mathfrak{g}_0$

Table 11

6. Appendix: Some facts from Representation Theory

Here we add brief explanation of some concepts and results well known in representation theory of semisimple Lie algebras and Lie groups. They are all available, but rather scattered in the literature. Moreover, the notation and terminology do not always coincide. The aim of this appendix is just to help a bit a differential geometer, who knows some basic facts about representations of semisimple Lie algebras and groups but who is not familiar with all this stuff. Much more information is available from [BasE] or some standard monograph, see e.g. [Hum], [Nay], [Sam].

The symbol $\mathbb{K}$ will always mean $\mathbb{R}$ or $\mathbb{C}$.

$|k|$-graded Lie algebras

6.1. Complex simple algebras. The complete classification list consists of four series of classical algebras A_ℓ, B_ℓ, C_ℓ, D_ℓ (labeled by their ranks) and four exceptional algebras E_6, E_7, E_8, F_4, G_2. We shall be mainly interested in the classical algebras. There are several simple objects encoding nicely all the information about the algebras. We should like to know the Dynkin diagram (with the labeling of the simple roots by its nodes), the highest root (expressed via simple roots α_i), the highest weight of the adjoint representation (expressed as linear combination of fundamental weights, i.e. through coefficients over the corresponding nodes), and the inverse of the Cartan matrix (since the fundamental weights are related to the simple roots by this matrix). Recall that the Cartan matrix (a_{ij}) is defined by $a_{ij} = \frac{2\langle\alpha_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}$ and this is read easily of the Dynking diagram ($a_{ij} = 0$ whenever the ith and jth nod are not adjacent, $a_{ii} = 2$, $a_{ij} = -1$ for a simple link, -2 for double link oriented from α_i to α_j, etc.). These four items are listed for all classical algebras in Table 12 and Table 13.

6.2. Simple Lie algebras with gradings. A finite grading on a Lie algebra is its decomposition (as a vector space)

$$\mathfrak{g} = \mathfrak{g}_{-p} \oplus \mathfrak{g}_{-p+1} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_q$$

such that the Lie bracket satisfies $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. If $p = q = k$, we say that $\mathfrak{g}$ is $|k|$-*graded*. Let us collect some useful information (see e.g. [CSch] for the proofs).

Proposition. *Let $\mathfrak{g}$ be a semisimple Lie algebra with finite grading. Then*

(1) *$\mathfrak{g}$ is $|k|$-graded for some $k \geq 0$.*
(2) *There is the unique grading element $E \in \mathfrak{g}$ such that ad_E is the multiplication by j on each $\mathfrak{g}_j$. Moreover $E \in \mathfrak{g}_0$.*

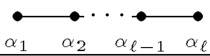
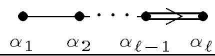
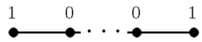
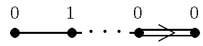
$\mathfrak{sl}(\ell+1,\mathbb{C})$ (type A_ℓ)	$\mathfrak{so}(2\ell+1,\mathbb{C})$ (type B_ℓ)
	
$\alpha_1+\dots+\alpha_\ell$	$\alpha_1+2\alpha_2+\dots+2\alpha_\ell$
	
$\frac{1}{\ell+1}\begin{pmatrix} \ell & \ell-1 & \dots & 2 & 1 \\ \ell-1 & 2(\ell-1) & \dots & 4 & 2 \\ \vdots & \vdots & i(\ell-i+1) & \vdots & \vdots \\ 1 & 2 & \dots & \ell-1 & \ell \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 2 & 2 & \dots & 2 & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2 & 3 & \dots & \ell-1 & \ell-1 \\ \frac{1}{2} & 1 & \frac{3}{2} & \dots & \frac{\ell-1}{2} & \frac{\ell}{2} \end{pmatrix}$

Table 12

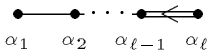
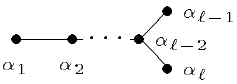
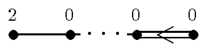
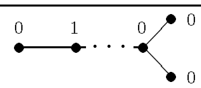
$\mathfrak{sp}(2\ell,\mathbb{C})$ (type C_ℓ)	$\mathfrak{so}(2\ell,\mathbb{C})$ (type D_ℓ)
	
$2\alpha_1+2\alpha_2\dots+2\alpha_{\ell-1}+\alpha_\ell$	$\alpha_1+2\alpha_2+\dots+2\alpha_{\ell-2}+\alpha_{\ell-1}+\alpha_\ell$
	
$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & \frac{1}{2} \\ 1 & 2 & 2 & \dots & 2 & \frac{2}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 1 & 2 & 3 & \dots & \ell-1 & \frac{\ell-1}{2} \\ 1 & 2 & 3 & \dots & \ell-1 & \frac{\ell}{2} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & \frac{1}{2} & \frac{1}{2} \\ 1 & 2 & 2 & \dots & 2 & \frac{2}{2} & \frac{2}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 1 & 2 & 3 & \dots & \ell-2 & \frac{\ell-2}{2} & \frac{\ell-2}{2} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \dots & \frac{\ell-2}{2} & \frac{\ell}{4} & \frac{\ell-2}{4} \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \dots & \frac{\ell-2}{2} & \frac{\ell-2}{4} & \frac{\ell}{4} \end{pmatrix}$

Table 13

(3) *The Killing form* $\langle\ ,\ \rangle$ *induces the isomorphisms* $\mathfrak{g}_i \simeq \mathfrak{g}^*_{-i}$, *while* $\langle \mathfrak{g}_i, \mathfrak{g}_j\rangle = 0$ *whenever* $j+i\neq 0$.

(4) $[\mathfrak{g}_{i+1},\mathfrak{g}_{-1}] = \mathfrak{g}_i$ *for all* $i<0$; *if no simple factor of* $\mathfrak{g}$ *is contained in* $\mathfrak{g}_0$, *then the same holds with* $i=0$.

(5) *If* $Z\in\mathfrak{g}_i$, $i>0$ *is an element with* $[Z,X]=0$ *for all* $X\in\mathfrak{g}_{-1}$, *then* $Z=0$. *The same holds with* $i=0$ *if no simple factor of* $\mathfrak{g}$ *is contained in* $\mathfrak{g}_0$.

6.3. Standard parabolic subalgebras. Consider a simple complex Lie algebra $\mathfrak{g}$ with fixed Cartan subalgebra $\mathfrak{h}$ and simple positive roots Δ_0^+. If we choose a subset $\Sigma\subset\Delta_0^+$, then there is the subalgebra $\mathfrak{p}_\Sigma$ generated by the Cartan algebra $\mathfrak{h}$ and all root spaces corresponding to those roots whose expressions as linear combinations of the simple roots have positive coefficients at elements in Σ, i.e.

$$\mathfrak{p}_\Sigma = \left(\mathfrak{h}\oplus\sum_{\alpha\in\langle-(\Delta_0^+\setminus\Sigma)\rangle}\mathfrak{g}_\alpha\right)\oplus\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha = \mathfrak{l}\oplus\mathfrak{u}_+$$

Obviously, $\mathfrak{p}_\Sigma$ contains the whole Borel subalgebra and so is parabolic. The subalgebras $\mathfrak{p}_\Sigma$ are called the *standard parabolic subalgebras* and the latter decomposition provides also their Levi decompositions. All parabolic subalgebras are conjugate to one of these.

The Σ-*height* of a root α is defined as the sum of its coefficients at the simle roots from Σ. Clearly, the decomposition according to the Σ-height yields a $|k|$-grading, where k is the Σ-height of the highest root. In particular, $\mathfrak{g}_0 = \mathfrak{l}$ and $\mathfrak{g}_1$ is generated by the rootspaces corresponding to the simple roots in $\Delta_0^+ \setminus \Sigma$, as $\mathfrak{g}_0$-module.

Conversely, given a $|k|$-graded simple complex algebra $\mathfrak{g}$, there is a Cartan algebra $\mathfrak{h}$ in $\mathfrak{g}$ and a choice of positive roots such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \dots \mathfrak{g}_k$ corresponds to a choice of simple roots Σ as above. We shall always write $\mathfrak{p}_+ = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k$ and $\mathfrak{g}_- = \mathfrak{g}_{-k} \oplus \dots \dots \mathfrak{g}_{-1}$. Thus, $\mathfrak{g}_-^* \simeq \mathfrak{p}_+$ and $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$.

6.4. We adopt the convention to indicate the simple roots in Σ by crossing out the corresponding nodes in the Dynkin diagram. The list of all $|1|$-graded classical complex simple algebras in Table 14 (up to isomorphisms) is obtained just by looking at the highest weights in Tables 12, 13. The are only two more among the exceptional algebras.

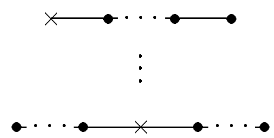
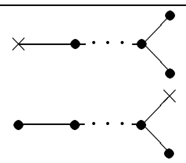
A_ℓ ($\frac{\ell}{2}$ or $\frac{\ell+1}{2}$ possibilities)	
B_ℓ (one possibility)	
C_ℓ (one possibility)	
D_ℓ (two possibilities)	

Table 14

6.5. Notation. Recall that the fundamental weights of a simple complex Lie algebra correspond to the nodes of the Dynkin diagram for $\mathfrak{g}$. Since all weights of $\mathfrak{g}$-modules can be written as linear combinations of the fundamental weights, we denote them by labeling the correpsonding nodes by the coefficients. In particular, the fundamental weights have the coefficient 1 over one node while all other nodes are labled by zero. The weight is dominant for $\mathfrak{g}$ if and only if all the coefficients are non-negative integers.

If there is a fixed parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$, we use the same notation as above but the Dynkin diagram has some nodes crossed. Notice that each weight denoted in such a way can be understood as a weight for the reductive Levi part $\mathfrak{g}_0$.

6.6. The Weyl group. As well known, all elements of the Weyl group $W \subset GL(\mathfrak{h}_0^*)$ of a simple complex Lie algebra are compositions of *simple reflections*, i.e. reflections with respect to hyperplanes orthogonal to simple roots.

For each root $\alpha \in \Delta$, the reflection S_α acts on the weight $\lambda \in \mathfrak{h}_0^*$ by $\mathcal{S}_\alpha(\lambda) = \lambda - \langle \lambda, H_\alpha \rangle \alpha$ where H_α is the coroot corresponding to α. Hence the coefficients over the nodes are given by $\langle S_\alpha(\lambda), H_i \rangle = \langle \lambda, H_i \rangle - \langle \lambda, H_\alpha \rangle \langle \alpha, H_i \rangle$ where H_i are the simple coroots. If α is a simple root, then $\langle \alpha, H_i \rangle$ is the Cartan integer which is encoded directly in the Dynkin diagram. This yields the formula for the new coefficients over the nodes after the action of a simple reflection:

Let a be the coefficient over the i-th node in the expression of λ. In order to get the coefficients over the nodes corresponding to $S_{\alpha_i}(\lambda)$, add a to the adjacent coefficients, with the multiplicity if there is a multiple edge directed towards the adjacent node, and replace a by $-a$.

For example, if λ is $\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}$ and we act by the middle simple reflection, we get the weight $\overset{a+b}{\bullet}\!-\!\overset{-b}{\bullet}\!-\!\overset{b+c}{\bullet}$. Similarly $\overset{a}{\bullet}\!\Rightarrow\!\overset{b}{\bullet}$ transforms under the action of the first simple reflection into $\overset{-a}{\bullet}\!\Rightarrow\!\overset{2a+b}{\bullet}$, while the second simple reflection yields $\overset{a+b}{\bullet}\!\Rightarrow\!\overset{-b}{\bullet}$.

The *affine action of the Weyl group* is defined by

$$w.\lambda = w(\lambda + \rho) - \rho$$

i.e. we have to apply the standard action to the weight shifted by the lowest form $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ and then shift the result back by $-\rho$. In terms of Dynkin diagrams this means to add one over each node, then act with w and finally subtract one over each node.

For each $w \in W$, the number of positive roots $\alpha \in \Delta^+$ which are transformed to $w.\alpha \in \Delta^-$ is called the *length of* w, we write $|w|$. Equivalently, the length of w is the minimal number of simple reflections in any expression for w in terms of simple reflections. We define the *sign of* w as $\mathrm{sgn} w = (-1)^{|w|}$.

Let $\mathfrak{p} \subset \mathfrak{g}$ be the parabolic subalgebra corresponding to $\Sigma \subset \Delta_0^+$. Then we define $W^{\mathfrak{p}} \subset W$ as the subset of all elements which map the weights dominant for $\mathfrak{g}$ into weights dominant for $\mathfrak{p}$.

6.7. $\mathfrak{p}$-modules. By the general theory, each irreducible $\mathfrak{p}$-module is a irreducible $\mathfrak{g}_0$-module equipped with the trivial action of $\mathfrak{p}_+$. Each such module is defined by the highest weight of the restriction of the representation to the semisimple part of $\mathfrak{g}_0$ and by the action of the center $\mathfrak{z}$ of $\mathfrak{g}_0$. It is very handy to encode such a representation by a weight of the whole $\mathfrak{g}$ which is allowed to have non-positive and non-integral weights over the crossed nodes. For each element in $\mathfrak{z}$ we are able to compute its action from these coefficients. In particular, E acts by the scalar product of the vector of the coefficients with the vector computed as the sum of those columns in the inverse Cartan algebra which correspond to the crossed nodes (the reason is that E acts by zero on $\alpha_i \in \Delta_0^+ \setminus \Sigma$ and it acts by one on $\alpha_j \in \Sigma$; the fundamental weights are obtained for α_i by multiplication by the inverse Cartan matrix).

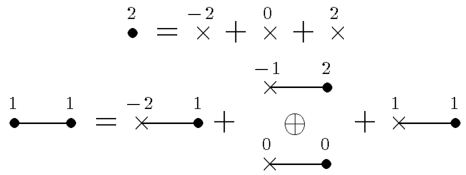
Most of the $\mathfrak{p}$ modules V are not irreducible, but they are indecomposable. Still, they enjoy a filtration by $\mathfrak{p}$-submodules

$$V = V_1 + V_2 + \cdots + V_r$$

such that the 'right hand ends' $V_i + \cdots + V_r$ are submodules for $1 \le i \le r$ and all quotients V_i/V_{i+1}, $1 \le i \le r$ (here $V_{i+1} = \{0\}$) are direct sums of irreducible $\mathfrak{p}$-modules. Of course, the 'left ends' are then quotients of V.

We can encode each such filtration by columns of the labled Dynkin diagrams encoding the highest weights of the irreducible components in V_i (as $\mathfrak{g}_0$-modules).

For example, a simple computation of the action of Weyl group on the highest weight of the adjoint representations we obtain easily the filtrations of $\mathfrak{p}$-modules for $\mathfrak{sl}(2,\mathbb{C})$ with the standard Borel subalgebra $\mathfrak{p}$, and $\mathfrak{sl}(3,\mathbb{C})$ with the first node crossed:



Cohomologies of Lie algebras

6.8. Consider for a moment an arbitrary Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module A. The cochains of degree q with coefficients in A are defined as the space $C^q(\mathfrak{g}; A)$ of all (continuous) skew-symmetric q-linear A-valued forms on $\mathfrak{g}$. By the definition, $C^q(\mathfrak{g}, A) = \operatorname{Hom}(\Lambda^q\mathfrak{g}; A)$ carries a natural $\mathfrak{g}$-module structure. We define the differential $\partial\colon C^q(\mathfrak{g}; A) \to C^{q+1}(\mathfrak{g}; A)$ by the formula

$$\begin{aligned}\partial c(X_0,\dots,X_q) = &\sum_{0\le i\le q} (-1)^i X_i.c(X_0,\dots\overset{\hat{i}}{}\dots X_q)\\ &+\sum_{0\le i<j\le q} (-1)^{i+j} c([X_i,X_j],X_0,\dots\overset{\hat{i}}{}\dots\overset{\hat{j}}{}\dots,X_q)\end{aligned}$$

One verifies easily $\partial^2 = 0$ and we obtain a complex by setting $C^q(\mathfrak{g}; A) = 0$ and $\partial(C^q(\mathfrak{g}; A)) = 0$ if $q < 0$. This complex is denoted by $C^*(\mathfrak{g}; A)$ and the corresponding cohomologies are denoted by $H^q(\mathfrak{g}; A)$ and called the cohomologies of $\mathfrak{g}$ with coefficients in A.

We need a special case only. Our $|k|$-graded algebras are $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$ and $\mathfrak{g}$ is a $\mathfrak{g}_-$-module via the restriction of the adjoint action. What we need is the *Lie algebra cohomology* $H^*(\mathfrak{g}_-; \mathfrak{g})$. Now, the grading of $\mathfrak{g}$ induces a natural grading on the cochains, $C^*(\mathfrak{g}_-; \mathfrak{g}) = \sum_{p,q} C^p_q(\mathfrak{g}_{-1}; \mathfrak{g})$ where $C^p_q(\mathfrak{g}_-; \mathfrak{g}) \subset C^p(\mathfrak{g}_-; \mathfrak{g})$ is the $\mathfrak{g}_0$-submodule of homogeneous homomorphisms of degree q, i.e. those with $c(\mathfrak{g}_{i_1} \wedge\cdots\wedge \mathfrak{g}_{i_p}) \subset \mathfrak{g}_{i_1+\cdots+i_p+q}$. Obviously, ∂ respects the homogeneity, i.e. $\partial\colon C^p_q \to C^{p+1}_q$. In the case of $|1|$-graded Lie algebras, $\mathfrak{g}_{-1}$ is abelian, only the first term in ∂ remains, and we get the so called *Spencer bigraded cohomology* $H^{p,q}(\mathfrak{g}_{-1}; \mathfrak{g})$.

The action of $\mathfrak{g}_0$ on the homogeneous components induces an action on the cochains which intertwines the differential and so there is a distinguished $\mathfrak{g}_0$-module structure on $H^*_*(\mathfrak{g}_-; \mathfrak{g})$.

6.9. The Hodge structure. Consider any $\mathfrak{g}$-module V, for example $V = \mathfrak{g}$. Due to the duality $\mathfrak{g}_- \simeq \mathfrak{p}_+^*$, the spaces $C^q(\mathfrak{g}_-, V) = \Lambda^q\mathfrak{g}_-^* \otimes V$ are identified with $(\Lambda^q\mathfrak{p}_+^* \otimes V^*)^*$. Thus the dual mapping to $\partial\colon C^{q-1}(\mathfrak{p}_+, V^*) \to C^q(\mathfrak{p}_+, V^*)$ is understood as $\partial^*\colon C^q(\mathfrak{g}_-, V) \to C^{q-1}(\mathfrak{g}_-, V)$. Oviously, $\partial^* \circ \partial^* = 0$.

Our aim is to understand the structure of the cohomology $H^*_*(\mathfrak{g}_-, \mathfrak{g})$. The main technical point is the existence of inner products on all $C^p_q(\mathfrak{g}_-, \mathfrak{g})$ such that ∂ and ∂^* are adjoint with respect to these products. Thus, we obtain the usual Hodge structure on $C^*_*(\mathfrak{g}_-, \mathfrak{g})$ and each $C^p_q(\mathfrak{g}_-, \mathfrak{g})$ decomposes as the direct sum of the image of ∂, image of ∂^* and kernel of $\Box = \partial \circ \partial^* + \partial^* \circ \partial$. In particular, each cohomology class contains a unique harmonic representative, i.e. a closed and co-closed cochain. There also is the (real) linear isomorphism

$$H^p_q(\mathfrak{g}_-, \mathfrak{g}) \simeq H^p_{-q}(\mathfrak{p}_+, \mathfrak{g}).$$

See [CSch] for a detailed exposition of all these properties.

A bit of effort leads to explicit formulae, e.g. for all $c \in C^2(\mathfrak{g}_-, \mathfrak{g})$, $X \in \mathfrak{g}_-$, and dual basis ξ_α, η^α of $\mathfrak{g}_-$ and $\mathfrak{p}_+$

$$\partial^* c(X) = \sum_\alpha [\eta^\alpha, c(X, \xi_\alpha)] - \frac{1}{2} \sum_\alpha c([\eta^\alpha, X]_{\mathfrak{g}_-}, \xi_\alpha)$$

see [CSch] for more details.

6.10. Kostant's BBW-theorem. *If A is a finite dimensional irreducible $\mathfrak{g}$-module of highest weight λ, then the irreducible finite dimensional representations of $\mathfrak{g}_0$ with highest weight μ occur in $H^*(\mathfrak{p}_+; A)$ if and only if there is an element $w \in W^{\mathfrak{p}} \subset W$ such that $\mu = w.\lambda = w(\lambda + \delta) - \delta$ and in that case it occurs in degree $|w|$ with multiplicity one.*

See e.g. [Vog, p. 123] for the proof. The original Kostant's formulation involves also an explicite description of the unique harmonic representative in terms of w, see [Kos] or [CSch].

In our situation, λ is the maximal root of $\mathfrak{g}$ and the affine action of $W^{\mathfrak{p}}$ is described in 6.6, and $H^p_q(\mathfrak{g}_-, \mathfrak{g}) \simeq H^p_{-q}(\mathfrak{p}, \mathfrak{g})^*$ as (real) $\mathfrak{g}_0$-modules.

In particular, if we want to compute $H^*_1(\mathfrak{g}_-; \mathfrak{g})$, we have to evaluate the affine action of those elements of length one which transform $\mathfrak{g}$-dominant weights into $\mathfrak{p}$-dominant weights. Obviously, only the simple reflections given by the crossed nodes can do that. For example, we obtain (the duals are easily computed by evaluating the action of E)

$$H^1_*(\overset{2}{\times}) = (\overset{-4}{\times})^* = \overset{4}{\times} \in H^1_2(\mathfrak{g}_-, \mathfrak{g})$$

$$H^1_*(\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}) = (\overset{-3}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet})^* = \overset{0}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet} \in H^1_1(\mathfrak{g}_-, \mathfrak{g})$$

Similarly, we can compute the second cohomologies. Now we have two simple reflections at disposal. Thus, we can either use two crossed nodes (if there so many), or we might start at a adjacent node to a cross. For example,

$$H^1_*(\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}) = (\overset{-5}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet})^* = \overset{4}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \in H^1_3(\mathfrak{g}_-, \mathfrak{g})$$

$$H^1_*(\overset{1}{\times}\!\!-\!\!\!-\!\!\overset{0}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet}) = (\overset{-4}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet})^* = \overset{1}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{1}{\bullet} \in H^1_2(\mathfrak{g}_-, \mathfrak{g})$$

The cohomologies of the complexified real algebras are the complexifications of the real cohomologies.

Verma modules

6.11. The enveloping algebra.

The*universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ is defined as the quotient $T(\mathfrak{g})/I$ of the (real or complex) tensor algebra generated by the elements of $\mathfrak{g}$, with respect to the two-sided ideal I in $T(\mathfrak{g})$, $I = \langle x \otimes y - y \otimes x - [x, y];\ x, y \in \mathfrak{g}\rangle$. There is the induced increasing filtration $\mathfrak{U}^k(\mathfrak{g})$ from that on $T(\mathfrak{g})$ and the inclusion $i\colon \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$. We have $i([x, y]) = i(x)i(y) - i(y)i(x)$ for all x, $y \in \mathfrak{g}$ and $\mathfrak{U}(\mathfrak{g})$ has the following universal property: For each associative algebra A over $\mathbb{K}$ with identity and each linear mapping $\varphi\colon \mathfrak{g} \to A$ satisfying $\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for all x, $y \in \mathfrak{g}$, there is a unique algebra homomorphism $\bar\varphi\colon \mathfrak{U}(\mathfrak{g}) \to A$ such that $\bar\varphi \circ i = \varphi$ and $\bar\varphi(1) = 1$.

According to the Birkhoff-Witt theorem, the canonical inclusion i extends to vector space isomorphisms $\sum_0^k S^k(\mathfrak{g}) = \mathfrak{U}^k(\mathfrak{g})$. These isomorphisms build an algebra isomorphism $S(\mathfrak{g}) = \sum_k S^k(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})$ if and only if $\mathfrak{g}$ is abelian.

As a consequence of the Birkhoff-Witt theorem we get some canonical identifications. Given a vector space basis x_i of $\mathfrak{g}$, the vector space $\mathfrak{U}^k(\mathfrak{g})$ is generated by the expressions $x_{i_1} \dots x_{i_l}$, $i_1 \le i_2 \le \dots \le i_l$, $l \le k$. If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of vector spaces, then $\mathfrak{U}(\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{b}) = U(\mathfrak{a}) \otimes U(\mathfrak{b})$ where $U(\mathfrak{a})$ means the linear span of the elements $x_1 \dots x_l$ with $x_i \in \mathfrak{a}$ and similarly for $U(\mathfrak{b})$.

The real universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a Lie algebra of a connected Lie group G is isomorphic to the algebra of left invariant vector fields (or right invariant vector fields) on G, i.e. to the enveloping algebra of left-invariant (or right-invariant) differential operators on the smooth functions on G.

The adjoint representation $\mathrm{ad}_x\colon \mathfrak{g} \to \mathfrak{g}$, $x \in \mathfrak{g}$ extends into a derivation on $\mathfrak{U}(\mathfrak{g})$. If $\mathfrak{g}$ is semisimple, then this representation is completely reducible. The subset $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$ of elements y with $\mathrm{ad}_x(y) = 0$ for all $x \in \mathfrak{g}$ is called the center of $\mathfrak{U}(\mathfrak{g})$. This is equivalent to the usual requirement that y commutes with all elements in $\mathfrak{U}(\mathfrak{g})$.

6.12. $\mathfrak{U}(\mathfrak{g})$-modules. Given a representation of a complex Lie algebra $\mathfrak{g}$, i.e. an algebra homomorphism $\varphi\colon \mathfrak{g} \to \mathrm{End}V$ for some complex vector space V, there is the unique algebra homomorphism $\bar\varphi\colon \mathfrak{U}(\mathfrak{g}) \to \mathrm{End}V$. If the representation is irreducible, then the actions of the elements from the center $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$ of the complex algebra must be multiplications by scalars. This can be viewed as an algebra homomorphism $\xi\colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$, the so called *central character* of the representation φ (also called *infinitesimal character*).

Suppose now, we have two irreducible representation V_λ, V_ρ corresponding to two dominant weights λ and ρ for a semisimple complex Lie algebra $\mathfrak{g}$ and an intertwining linear mapping $D\colon V_\lambda \to V_\rho$, i.e. a $\mathfrak{U}(\mathfrak{g})$-module homomorphism. Let us write ξ_λ and ξ_ρ for the infinitesimal characters of V_λ and V_ρ. For every $v \in V_\lambda$, $z \in \mathfrak{Z}(\mathfrak{g})$ we have $zD(v) = D(zv) = D(\xi_\lambda(z)v) = \xi_\lambda(z)D(v)$ and so either $\xi_\lambda = \xi_\rho$ or $D = 0$. The same conclusion is true if both representations are generated by a single highest weight vector.

6.13. Verma modules. Let us consider first an arbitrary complex Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{p}$. Given a representation of $\mathfrak{p}$ in a finite dimensional vector space V, we define the *induced representation*

$$\mathrm{Ind}(\mathfrak{g}, V) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} V.$$

The representation space V is canonically embedded into the induced representation $\mathrm{Ind}(\mathfrak{g}, V)$ via $V \mapsto 1 \otimes_{\mathbb{C}} V \simeq \mathfrak{U}(\mathfrak{p}) \otimes_{\mathfrak{U}(\mathfrak{p})} V$.

In particular, if $\mathfrak{g}$ is semisimple, $\mathfrak{p}$ is a Borel subalgebra and if we consider the one-dimensional characters λ of the Borel subalgebra $\mathfrak{p}$, then the induced representations are called the *Verma modules* and denoted by V_λ (sometimes a shift in the weight (by the lowest form) is used in the notation for symmetry reasons. Starting with a highest weight λ, the whole induced module is also a highest weight module.

In general, it is difficult to work with the induced representations since the structure of $\mathfrak{U}(\mathfrak{g})$ is complicated. However, if $\mathfrak{g}$ is semisimple and $\mathfrak{p}$ parabolic, the whole situation is much more similar to the theory of Verma modules. Let us recall $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{p}$ as a vector space direct sum of Lie subalgebras. Thus, given a finite dimensional representation of $\mathfrak{p}$ in $\mathbb{E}$, we have $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} \mathbb{E} \simeq \mathfrak{U}(\mathfrak{g}_-) \otimes_{\mathbb{C}} \mathbb{E}$ (as vector spaces) by virtue of the Birkhoff-Witt theorem. We shall denote this call such modules *generalized Verma modules* and we use the notation $V_\mathfrak{p}(\mathbb{E})$. If the representation is irreducible and corresponds to a dominant weight λ for $\mathfrak{p}$, then the $\mathfrak{U}(\mathfrak{g})$-module $V_\mathfrak{p}(\mathbb{E}_\lambda)$ is generated by the highest weight vector $1 \otimes v$ where v is the highest weight vector in $\mathbb{E}_\lambda$.

In particular, if the subalgebra $\mathfrak{g}_-$ is abelian, then $\mathfrak{U}(\mathfrak{g}_-) = S(\mathfrak{g}_-)$, the symmetric algebra and the latter is equal to the algebra $S((\mathfrak{g}_-)^*)$ of polynomials on $\mathfrak{g}_-$.

6.14. Homomorphisms of Verma modules. Consider dominant weights λ and ρ for complex parabolic $\mathfrak{p} \subset \mathfrak{g}$ and a homomorphism $D\colon V_\mathfrak{p}(\mathbb{E}_\lambda) \to V_\mathfrak{p}(\mathbb{E}_\rho)$ of $\mathfrak{U}(\mathfrak{g})$-modules. The whole modules are generated by the highest weight vectors $1 \otimes v_\lambda$ and $1 \otimes v_\rho$. Each element $z \in \mathfrak{Z}(\mathfrak{g})$ from the center must preserve the highest weight vectors and acts by scalar multiplication by $\xi_\lambda(z)$ and $\xi_\rho(z)$, the central characters of the representations. Hence a non-zero morphism can exist only if the infinitesimal characters coincide. A classical theorem by Harish-Chandra states that $\xi_\lambda = \xi_\rho$ if and only if $\lambda + \delta$ and $\rho + \delta$ are conjugate under the action of the Weyl group W of $\mathfrak{g}$, here δ is the lowest form (half the sum of all positive roots). This means that both weights have to be in the same orbit of the affine action.

In particular, if λ is dominant for $\mathfrak{g}$, then all weights ρ dominant for $\mathfrak{p}$ with the same infinitesimal character $\xi_\lambda = \xi_\rho$ are given by $\{w.\lambda\ ;\ w \in W^\mathfrak{p}\}$.

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