NATURAL OPERATORS ON CONFORMAL MANIFOLDS

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Preface

This dissertation is based on a written version of my lecture series held during my visiting professorship at the University of Vienna in the Fall term 1991/1992. I acknowledge gratefully the support and kind hospitality of the University during my visit. The motivating interest of the listeners forced me to study deeply the subject which became one of the main interests of my scientific research nowadays. Since the last year, a joint seminar of J. Bureš, V. Souček and myself devoted especially to this topic works at the Charles University in Prag. The general setting for the study of the natural operators originates in the work of the seminar of I. Kolář during the last ten years in Brno and in the Middle-European Seminar organized jointly by I. Kolář and P. Michor in Brno and Vienna since 1985. The recent monograph [Kolář, Michor, Slovák, 93] collects the most of the results of this cooperation.

The submitted version of the lecture notes, first distributed at the University of Vienna in 1992, has been revised and essentially extended. The Sections 4 and 8 present my original results, the rest of the text collects the necessary background for the theory of natural operators on conformal manifolds which is really difficult to be found in one place. The exposition covers the topics assumed as well known (to specialists) in the survey paper [Baston, Eastwood, 90] and those regularly applied in the fairly many other recent papers concerning the naturality problems in conformal geometry. So a graduate student of differential geometry should be able to start an active work in this area after studying the lecture notes. The bibliography is far from being complete, however I have involved all papers which I have seen by myself and which thereby have influenced the text.

My approach combines the general methods developed for the study of the naturality problems in the above mentioned monograph [Kolář, Michor, Slovák, 93], which are more suited for solving concrete (even non-linear) problems, but which have not been worked out in the category of conformal manifolds there, and the methods from the representation theory employed by some of the cited authors (which apply then only to linear problems, of course). The latter methods are very powerful and they lead to very nice general classification results, but on the other hand, these results are rather implicit. I believe, that my approach should lead to new concrete results in the near future as well. The whole text might seem strange since we are seeking for natural operators, but neither we apply the results nor we state what they are good for. But the applications are rather non-trivial as a rule, the interested reader can find some of them in [Baston, Eastwood, 90], [Fefferman, 79], [Fefferman, Graham, 85] for the conformal invariants and [Atiyah, Bott, Patodi, 73], [Gilkey, 84] for the Riemannian invariants. Typically, a classification result on all natural operators helps to describe properties of rather concrete geometric objects. Moreover, the theory of the natural operators is itself rich enough to be treated separately.

The reader is assumed to be familiar with standard finite dimensional differential geometry. The study of some parts of the monograph [Kolář, Michor, Slovák, 93] will be probably necessary for a detailed understanding. Further, a detailed treatment of the representation theory cannot be involved in the text, but I offer at least brief overviews, mainly in the Appendix.

In the first preparatory section I try to motivate the naturality problems, to indicate the connections to representation theory and to fix some notation. The next section explains the basic setting for naturality problems and in the third one, I develop the necessary theory of natural tensors (the so called Weyl's theory). Section 4 is based on my recent papers [Slovák, 92a], [Slovák, 92b]. In fact, it presents the first step towards the classification in Section 8, since all conformal invariants must be first of all Riemannian invariants. Furthermore, the results present a nice application showing the power of the general approach to (non-linear) naturality problems mentioned above.

Next, I describe thoroughly the flat conformal structures and their morphisms which is applied immediately to the description of all first order linear natural conformal operators which do not vanish on conformally flat manifolds in Section 6. In fact, this section covers a result by Fegan from 1970 which is a special case of the later general classification. But I like to present some of the ideas of the later development in a more concrete setting. Among these operators, there are some living on bundles involving more structure, the spin bundles. These are treated in Section 7 by means of the Clifford algebras. In particular, this introduces the reader to the famous Dirac operators.

Section 8 presents a general classification of all natural operators on conformally flat manifolds based on the representation theory of parabolic subalgebras in the orthogonal groups and the classifications of Riemannian invariants from Section 4. This is the core of the dissertation. The results were partially known, but I have never found a concise proof in the literature. The presented classification also corrects some unprecise claims from the survey [Baston, Eastwood, 90].

In the last section, I discuss the problem whether the latter operators extend to operators on the whole category of conformal manifolds. This is a very subtle question and even the definition of the conformally invariant operators varies from author to author. This happens since the conformal manifolds are not locally homogeneous and, moreover, the most of interesting vector bundles do not live on all manifolds (the existence of the conformal weights makes the difference with respect to the Riemannian case). One approach is to take the implicit description of all Riemannian invariants, to modify slightly the definition of the naturality and to try to find out those operators which are invariant with respect to all scalar deformations of the metrics. This is the way undertaken by Branson, Ørsted, Wünsch and others. We shall discuss another approach, the point of which is to classify first the linear operations on the conformally flat manifolds and then to use certain geometrical methods to extend the latter operators to all conformal manifolds. The geometry involved is based on the canonical Cartan connection on conformal manifolds which is treated first and then I indicate how the general methods work.

Some short parts of the exposition follow [Kolář, Michor, Slovák, 93], in particular 3.1 – 3.8 and 3.21 of this text are based on Section 24 (prepared by I. Kolárř), 3.15 – 3.20 and Section 4 extend my exposition from Section 33 of the monograph. The style of the whole text is rather brief, an active cooperation of the reader is assumed.

Brno, 1993 Jan Slovák

1. Introduction

1.1. Geometric operators. In general, operators are rules transforming sections of one bundle into sections of another one. In differential geometry, we often meet manifolds with some more structure, like Riemannian or symplectic manifolds and the isomorphisms respecting these distinguished structures. Then the most important bundles are those with a distinguished action of the isomorphisms on their sections, the bundles of geometric objects. The geometric (or invariant or natural) operators are those operators which intertwine the distinguished actions of the isomorphims. The latter expresses that the definition of such operators does not involve any special choice and the operators are then defined invariantly on all objects from the category in question. These rough ideas are behind the formal definitions in Section 2, see in particular 2.12. Let us demonstrate the concept of the natural operators on the simplest case, the operations on functions.

Let us start with the operators $D\colon C^\infty(M,\mathbb{R})\to C^\infty(E)$ of order 1, i.e. $Df(x)$ depends only on the first derivatives of f at x, and the symbol E denotes the unknown target vector bundle with an action of the isomorphisms. We first require the invariance with respect to the action of all diffeomorphisms given by $\varphi_*(f) = f\circ\varphi^{-1}$ and we ask the (rather trivial) question: What are the linear operators D defined on $C^\infty(\mathbb{R}^m,\mathbb{R})$ intertwining the actions of all local diffeomorphisms $\varphi\colon\mathbb{R}^m\to\mathbb{R}^m$? Since the action is transitive on $\mathbb{R}^m$, it is enough to restrict ourselves to a single point $x\in\mathbb{R}^m$, say $x=0$, and since we assume the order is one, D is in fact determined by a mapping $\tilde D\colon \mathbb{R}\oplus\mathbb{R}^{m*}\to E_x$ (now E_x is the standard fiber of the unknown bundle). This mapping $\tilde D$ is linear and its dual mapping goes $\tilde D^*\colon E_x^*\to\mathbb{R}^*\oplus\mathbb{R}^m$. First of all the mappings commute with the linear isomorphisms and so $\tilde D^*$ intertwines the induced actions of $GL(m,\mathbb{R})$ on the standard fibers. But the right hand side is precisely the decomposition into $GL(m,\mathbb{R})$-irreducible components and so the unknown standard fiber must be either $\mathbb{R}$ or $\mathbb{R}^{m*}$. By the Schur's lemma, the first possibility corresponds to scalar multiples of the identity operator, the second one yields a scalar multiple of $D^*((dx^i)^*)(f) = (dx^i)^*(Df) = \frac{\partial f}{\partial x^i}$. Thus $Df = \frac{\partial f}{\partial x^i}dx^i$ or $Df = f$ up to constant multiples and the only possible target is the cotangent bundle or $C^\infty(M,\mathbb{R})$. In this way we have classified all invariant local linear operators of order one on functions.

There is a general classification result proved independently by [Terng, 78] and [Kirillov, 77]: *All natural linear operators on arbitrary tensor bundles (invariant with respect to the tensorial action of all local diffeomorphisms) are compositions of exterior differentials and invariant algebraic tensor operations (i.e. operations of order zero).* Hence there are no operations of higher order on functions natural with respect to all diffeomorphisms.

There are two very well known examples of second order operators on $C^\infty(\mathbb{R}^m)$

$$\Delta f = \frac{\partial^2 f}{\partial x^1\partial x^1}+\dots+\frac{\partial^2 f}{\partial x^m\partial x^m} \qquad \text{Laplace operator}$$

$$\Box f = -\frac{\partial^2 f}{\partial x^1\partial x^1}+\dots+\frac{\partial^2 f}{\partial x^m\partial x^m} \qquad \text{Klein-Gordon (wave) operator.}$$

As we have mentioned, they cannot be modified to become invariant with respect to all diffeomorphisms which is equivalent to the statement: these local expressions cannot be extended to invariantly defined operators on functions on arbitrary manifolds. However, we can still reach this if we restrict ourselves to manifolds with suitable structure. We shall consider Riemannian manifolds or pseudo-Riemannian manifolds but first we have to fix some notation.

1.2. Abstract index formalism. The typical subjects of natural operations are tensor fields with several covariant and contravariant components. The latter means, we take the vector space $V = \mathbb{R}^m$ or $V = \mathbb{C}^m$, and consider the tensor product $\otimes^q V \otimes \otimes^p V^*$ with the standard representation of $GL(m, \mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For each m-dimensional manifold M we define the tensor bundle $T^{(p,q)}M$ as the associated vector bundle to the first order frame bundle on M corresponding to the above tensor product with $V = \mathbb{R}^m$. The tensor fields are sections of these tensor bundles or their subbundles. In local coordinates, a tensor in a point $x \in M$ is an N-tuple of scalars for suitable N, the tensor fields are then N-tuples of scalar-valued functions $f^{i_1 \dots i_q}_{j_1 \dots j_p}$. On complex m-dimensional manifolds we get the complex tensor bundles on replacing $\mathbb{R}$ by $\mathbb{C}$. If we use the complex scalars on real manifolds, we get the complexifications of the real bundles in question.

There are several basic operations like permutations of the copies of V or V^* in the tensor products, linear combinations of such permutations and evaluations with respect to one chosen copy of V and one copy of V^*, the so called *contraction* or *trace*. In order to be able to indicate such operations without explicit use of local coordinates, we shall use a kind of 'abstract markers' or 'labels' for the copies of V and V^*. So V^i and V^j means two distinct copies of V and the expressions t^a, t^{b_j}, $f^{i_1 \dots i_p}$, etc. will always denote tensors in V^a, V^{b_j}, $V^{i_1} \otimes \dots \otimes V^{i_p}$, or the corresponding tensor fields, respectively. The same labels used as subscripts indicate isomorphic but distinct copies of the dual V^* and the concatenation of such symbols expresses the tensor product. Hence, in general we should distinguish carefully the order of the subscripts and superscripts, i.e. we should write $t^a{}_b \in V \otimes V^*$ but $t_b{}^a \in V^* \otimes V$. It is generally adopted in a large part of geometry to forget about the order of subscripts and superscripts, but we shall be forced to follow this convention exactly when dealing with Riemannian manifolds and spinors later on.

Now, it is easy to write down the above mentioned operations. The permutations of the copies of V or V^* result in precisely the same permutations of the subscripts or superscripts. The linear combinations of tensors are denoted simply as linear combinations of the formal expressions. In particular, the alternation and symmetrization are important enough to have a special notation: $(a \dots b)$ means symmetrization over the indicated indices, $[a \dots b]$ is the alternation, $\{a \dots b\}$ is the sum over cyclic permutations. We adopt the so called summation convention which means that any occurrence of the same label once among the superscripts and once in the subscripts denotes a contraction with respect to the indicated entries.

If we distinguish a linear isomorphism $g_{ab} \colon V \to V^*$, i.e. $g_{ab} \in V^* \otimes V^*$, then there is its inverse $g^{ab} \in V \otimes V$. We can apply these isomorphisms to each copy of V or V^* in the tensor products which can be indicated as a contraction with the proper tensor g_{ab} or g^{ab}. Then it is suitable to add the convention that $g_{ab} t^{\dots b \dots}_{\dots \ \dots} = t^{\dots \ \dots}_{\dots a \dots}$ and to consider the contractions over all repeated indices in the latter sense. In particular, $g^a{}_b = g^{ac} g_{cb} = \delta^a{}_b$, the 'Kronecker delta'. The latter will apply in our discussion on operations on pseudo-Riemannian manifolds. Of course, then we have to take care of the order of the indices, this is very important if g_{ab} is not symmetric.

If not disabled explicitly, all italic indices in the further text will be used in the above context. If we shall need the concrete values in some coordinates, we shall use the same symbols but underlined.

1.3. Riemannian invariants. There are two important tools available: the rising and lowering of indices by means of the (pseudo-) metric and the canonical Levi-Cività connection. The latter can substitute the usual derivatives, the former allows to take traces (contractions). The covariant derivative with respect to the Levi-Cività connection is defined on each vector bundle associated with the (pseudo-) Riemannian linear frame bundle. Hence we consider the composition (the first covariant derivative coincides with the exterior derivative d)

$$C^\infty(M,\mathbb{R}) \xrightarrow{\nabla} C^\infty(T^*M) \xrightarrow{\nabla} C^\infty(T^*M \otimes T^*M).$$

The target of this composed operator decomposes into subbundles invariant with respect to (pseudo-) Riemannian local isomorphisms. We have

$$T^*M \otimes T^*M \simeq \Lambda^2 M \oplus S^2 T^*M \simeq \Lambda^2 M \oplus (S^2T^*)_0 M \oplus (M \times \mathbb{R})$$

where the mid term means the trace-free part of the symmetric forms while the last one corresponds to the traces $(t_{ij} \mapsto t_{[ij]} + (t_{(ij)} - \frac{1}{m} t_{aa} g_{ij}) + \frac{1}{m} t_{aa} g_{ij}$ in the Riemannian case, m being the dimension). The composition of the above operator with the projection onto the third term is the operator $\nabla^a\nabla_a \colon C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$, $f \mapsto \nabla^a\nabla_a f = g^{ab}\nabla_a\nabla_b f$ which coincides with the Laplace operator in the flat Riemannian case and the wave operator in the flat pseudo-Riemannian case.

The projection onto the antisymmetric part is zero (the Ricci identity) while the projection onto the symmetric trace-free part yields another invariantly defined operator.

1.4. The conformal invariance. We have seen that there are very few linear operators living on all manifolds and there is a plenty of them on Riemannian manifolds. But the restriction to manifolds with more structure brings also another interesting phenomenon – there exist more geometric objects, i.e. more bundles with distinguished actions of the isomorphisms in question. In the (pseudo-) Riemannian case, all the new objects live in some tensor bundles, they form only finer decompositions into irreducible parts. However, in general there might appear quite different new objects, i.e. the distinguished actions are not restrictions of some action of all diffeomorphisms. The conformal manifolds are manifolds equipped with a class of pseudo-metrics which are all equal up to a multiple by a scalar function. Hence the distances in the individual metrics from the class differ but the angles are the same ones. In particular the 'light cone' in the pseudo-Riemannian case is defined invariantly. There are more local isomorphisms of conformal manifolds than in the Riemannian case, but much less than the set of all local diffeomorphisms. We shall see that each of them is globally determined by its derivatives up to the second order in an arbitrary point. Nevertheless, there are not many invariantly defined operators and it is a rather hard problem to describe them. Only very few of them live in tensor bundles, but on each tensor bundle, the restricted action of the group $GL(m,\mathbb{R})$ to $O(m,\mathbb{R})$ on the standard fiber can be extended to the center $\mathbb{R}\subset GL(m,\mathbb{R})$ which also belongs to the conformal isomorphisms on the flat conformal manifold $\mathbb{R}^m$. This extensions are given by multiplication with an arbitrary fixed power of the elements of the center and the negative of the power is called the *conformal weight* of the resulting bundles. Such tensors with weights are also called (tensor valued) densities with conformal weight. They can be interpreted as follows: With respect to a fixed metric from the conformal class, the densities of weight α are represented by usual tensors, but if we deform the metric into $\hat{g}=f^2g$ (this is achieved by the action of $-f.\mathrm{id}$ at a point), then the corresponding tensors are multiplied by f^α.

Choosing the proper weights on the bundles, we can sometimes eliminate the effect of the deformation of the metric by a scalar function and some of the pseudo-Riemannian invariant operators become then conformally invariant. These rough definitions and ideas will be discussed in detail later on. Now we illustrate only the complexity of the problems on some concrete explicit calculations.

1.5. The conformal curvature. The Riemannian covariant derivative is invariantly defined. We shall see in Section 4 that all natural operators on (pseudo-) Riemannian manifolds are built from this covariant derivative and the Riemannian curvature. So we have to inspect how the covariant derivative transforms if we deform the metric.

If we deform $g\mapsto\hat{g}=f^2.g$ with a positive function f, then we get the deformed Christoffel symbols

$$\begin{aligned}\hat{\Gamma}^{\underline{i}}_{\underline{lk}}&=\frac{1}{2}\hat{g}^{\underline{ij}}(\hat{g}_{\underline{lj},\underline{k}}+\hat{g}_{\underline{jk},\underline{l}}-\hat{g}_{\underline{lk},\underline{j}})\\&=\Gamma^{\underline{i}}_{\underline{lk}}+\frac{1}{2}f^{-2}g^{\underline{ij}}(2f_{\underline{k}}fg_{\underline{lj}}+2f_{\underline{l}}fg_{\underline{jk}}-2f_{\underline{j}}fg_{\underline{lk}})\\&=\Gamma^{\underline{i}}_{\underline{lk}}+f^{-1}g^{\underline{ij}}(f_{\underline{k}}g_{\underline{lj}}+f_{\underline{l}}g_{\underline{jk}}-f_{\underline{j}}g_{\underline{lk}})\\&=\Gamma^{\underline{i}}_{\underline{lk}}+\Upsilon_{\underline{k}}\delta^{\underline{i}}_{\underline{l}}+\Upsilon_{\underline{l}}\delta^{\underline{i}}_{\underline{k}}-\Upsilon^{\underline{i}}g_{\underline{lk}}\end{aligned}$$

where the ('concrete') indices after comma denote the values of partial derivatives, the comma is omitted for functions and $\Upsilon_{\underline{a}}:=\nabla_{\underline{a}}(\log f)$. The latter coincides with the Lie derivative in the direction of the a's coordinate by definition. According to our general conventions, Υ_a denotes the corresponding 1-form while Υ^a is the corresponding vector field $g^{ab}\Upsilon_b$. The coordinate expression for the covariant derivative is

$$\nabla_{\underline{a}}X^{\underline{b}}=\frac{\partial X^{\underline{b}}}{\partial x^{\underline{a}}}+\Gamma^{\underline{b}}_{\underline{ka}}X^{\underline{k}},\quad\nabla_{\underline{a}}X_{\underline{b}}=\frac{\partial X_{\underline{b}}}{\partial x^{\underline{a}}}-\Gamma^{\underline{k}}_{\underline{ab}}X_{\underline{k}}.$$

If we insert our expression for the deformed Christoffel symbols and use the general abstract index notation, we can write

$$\hat{\nabla}_aX^b=\nabla_aX^b+\Upsilon_k\delta^b_aX^k+\Upsilon_aX^b-\Upsilon^bg_{ka}X^k \tag{1}$$

$$\hat{\nabla}_aX_b=\nabla_aX_b-\Upsilon_aX_b-\Upsilon_bX_a+\Upsilon^kg_{ab}X_k. \tag{2}$$

We recall that $\nabla_a X_b$ must be understood as one symbol, a 2-form.

The curvature can be defined by $R_{abc}{}^d X^c = (\nabla_a \nabla_b - \nabla_b \nabla_a) X^d$, where the iterated covariant derivative is taken with respect to two different connections, both induced from the same Levi-Cività connection on the Riemannian linear frame bundle. A direct computation yields

$$\hat{R}_{abcd} = f^2 (R_{abcd} + \Xi_{ac} g_{bd} - \Xi_{bc} g_{ad} + \Xi_{bd} g_{ac} - \Xi_{ad} g_{bc}) \tag{3}$$
$$\Xi_{ab} = \nabla_a \Upsilon_b - \Upsilon_a \Upsilon_b + \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab}$$

where the tensor field Ξ is symmetric (notice $\nabla_a \Upsilon_b = \nabla_a \nabla_b (\log f)$ and the second covariant derivative is symmetric on functions). The curvature on a (pseudo-) Riemannian manifold M is a section of the tensor bundle $\otimes^4 T^* M$ which is a sum of several subbundles invariant with respect to isometries. Hence also the curvature splits into several parts. Since the curvature satisfies several identities:

$$R_{abcd} = R_{cdab}, \quad R_{abcd} = -R_{abdc}, \quad R_{\{abc\}d} = 0$$

(the last one is the Bianchi identity), the most of these summands are zero. Let us find the non-zero ones.

The *Ricci curvature* R_{ac} is defined as the trace $R_{ac} = R_{abcb}$ and the trace $R := R_{aa}$ is called the *scalar curvature*. Let us write $C_{abcd} = R_{abcd} + S_{abcd}$ for the trace-free part of the curvature, i.e. both C_{abcd} and S_{abcd} are well defined. Let us try to find a symmetric tensor P_{ab} satisfying

$$S_{abcd} = P_{ac} g_{bd} - P_{bc} g_{ad} + P_{bd} g_{ac} - P_{ad} g_{bc}.$$

Since the tensor S_{abcd} is completely determined by its traces (see the definition), it suffices to consider the traces of this formal equation to find the tensor P_{ab}. We obtain

$$-R_{ac} = S_{abc}{}^b = m P_{ac} - P_{ac} + P_b{}^b g_{ac} - P_{ac} \tag{4}$$
$$= (m-2) P_{ac} + P_b{}^b g_{ac}$$
$$-R = S_{ab}{}^{ab} = 2m P_a{}^a - P_a{}^a - P_a{}^a = (2m-2) P_a{}^a \tag{5}$$

and so P_{ab} exists and is uniquely determined in dimensions greater then two. We shall write briefly $P := P_a{}^a$. In dimension two, the full curvature tensor is determined by its component $R_{\underline{1212}}$ and is therefore irreducible. In general the conformal geometry is essentially different in dimension two and we shall always assume $m \geq 3$ in the sequel.

Now, if we compare the deformation of R_{abcd} in (3) with the expression for the trace part $S_{abcd} = C_{abcd} - R_{abcd}$, we see that the whole deformation of $R_{abc}{}^d$ belongs to the trace part (the expression f^{-2} disappears during the rising of the index). Hence the trace-free part $C_{abc}{}^d$ is conformally invariant. We call it the *Weyl curvature* or *conformal curvature*. At the same time, we have found the deformation of P_{ab}:

$$\hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b - \frac{1}{2} \Upsilon_c \Upsilon^c g_{ab} \tag{6}$$

and the trace of this expression yields

$$\hat{P} = f^{-2} (P - \nabla^a \Upsilon_a + \frac{2-m}{2} \Upsilon^a \Upsilon_a). \tag{7}$$

1.6. The conformal Laplace operator. Let us compute the deformation of the usual Laplace operator $\nabla^a\nabla_a$, cf. 1.3. Using the above formulas, we get for functions h: $\hat{\nabla}_a h = \nabla_a h$ and

$$\hat{\nabla}_b\hat{\nabla}_a h = \nabla_b\nabla_a h - \Upsilon_b\nabla_a h + \Upsilon^k g_{ba}\nabla_k h$$
$$\hat{\nabla}^a\hat{\nabla}_a h = f^{-2}g^{ab}(\nabla_b\nabla_a h - \Upsilon_b\nabla_a h + \Upsilon^k g_{ba}\nabla_k h).$$

This formula does not seem to promise anything, but let us try to consider the functions h with some conformal weight. This means, the latter geometric objects can be represented by a function which changes together with a deformation of the metric and we have to apply the deformed Laplace operator to this 'deformed function'. If the conformal weight is α, we have $\hat{h} = f^\alpha h$. Hence using several times the formula $\nabla_a(f^\alpha h) = \alpha f^{\alpha-1}\nabla_a f h + f^\alpha\nabla_a h = f^\alpha(\alpha\Upsilon_a h + \nabla_a h)$ we obtain

$$\begin{aligned}
\hat{\nabla}^a\hat{\nabla}_a\hat{h} &= g^{ab}f^{-2}\Big(\nabla_b(f^\alpha(\alpha\Upsilon_a h + \nabla_a h)) - \Upsilon_b f^\alpha(\alpha\Upsilon_a h + \nabla_a h)\\
&\qquad - \Upsilon_a f^\alpha(\alpha\Upsilon_b h + \nabla_b h) + \Upsilon^k g_{ba}f^\alpha(\alpha\Upsilon_k h + \nabla_k h)\Big)\\
&= g^{ab}f^{\alpha-2}\Big(\alpha(\nabla_b\Upsilon_a)h + \alpha^2\Upsilon_b\Upsilon_a h + \alpha\Upsilon_a(\nabla_b h) + \alpha\Upsilon_b(\nabla_a h) + \nabla_b\nabla_a h - \alpha\Upsilon_b\Upsilon_a h\\
&\qquad - \Upsilon_b(\nabla_a h) - \alpha\Upsilon_a\Upsilon_b h - \Upsilon_a(\nabla_b h) + \alpha\Upsilon^k g_{ba}\Upsilon_k h + \Upsilon^k g_{ba}(\nabla_k h)\Big)\\
&= f^{\alpha-2}\big(\nabla^a\nabla_a h + (\alpha^2 - 2\alpha + \alpha m)\Upsilon_a\Upsilon^a h + \alpha(\nabla^a\Upsilon_a)h + (2\alpha - 2 + m)\Upsilon^a(\nabla_a h)\big)
\end{aligned}$$

If we compare this formula with the deformation of P derived in 1.5.(7), we find two similar terms, $-\nabla^a\Upsilon_a$ and $\frac{2-m}{2}\Upsilon^a\Upsilon_a$. The first term in our formula corresponds to the usual Laplace operator and so it seems that we could eliminate the deformation by adding a suitable multiple of P and considering suitable conformal weights. The effect of the weight should cancel the last term in the formula, i.e. $2\alpha - 2 + m = 0$. This yields $\alpha = \frac{2-m}{2}$ and with this weight we have

$$\hat{P}\hat{h} = f^{\frac{-2-m}{2}}\left(Pf - \nabla^a\Upsilon_a f + \frac{2-m}{2}\Upsilon^a\Upsilon_a f\right).$$

Further, $(\alpha^2 - 2\alpha + \alpha m) = -\left(\frac{2-m}{2}\right)^2$ and so

$$\begin{aligned}
\hat{\nabla}^a\hat{\nabla}_a\hat{h} + \frac{2-m}{2}\hat{P}\hat{h} &= f^{\frac{-2-m}{2}}\left(\nabla^a\nabla_a h + \frac{2-m}{2}(P - f^2\hat{P})h\right) + \frac{2-m}{2}\hat{P}\hat{h}\\
&= f^{\frac{-2-m}{2}}(\nabla^a\nabla_a f + \frac{2-m}{2}Pf).
\end{aligned}$$

Now, we can consider the values of the operator $\nabla^a\nabla_a + \frac{2-m}{2}P$ on the conformal densities with weight $\frac{2-m}{2}$ as conformal densities with the weight $\frac{-2-m}{2}$ and we get a conformally invariant operator, the so called *conformal Laplace operator*.

In the dimension four we get the operator

$$D = \nabla^a\nabla_a - P = \nabla^a\nabla_a + \frac{1}{6}R$$

which transforms the (scalar) densities with weight -1 into (scalar) densities with weight -3.

2. Invariant operators

In the sequel, we shall write $C^\infty Y$ for the space of all local smooth sections of a fibered manifold Y.[1]

2.1. Local linear operators. Let Y, Y' be fibered manifolds with a common base M. A *local operator* is a mapping $D\colon C^\infty Y \to C^\infty Y'$ such that for all $s \in C^\infty Y$ defined at $x \in M$, $Ds(x)$ depends only on the germ of s at x. If Y is a vector bundle, then $C^\infty Y$ carries a natural vector space structure (defined pointwise). An operator $D\colon C^\infty Y \to C^\infty Y'$ is called *smooth* if smoothly parameterized curves of sections are transformed into smoothly parameterized ones.

Theorem. [Peetre, 59] *Let E, E' be two (finite dimensional) vector bundles with common base M. Each local linear operator $D\colon C^\infty E \to C^\infty E'$ has locally finite order, i.e. for each relatively compact coordinate neighborhood U on M there is an order k such that the values of the operator depend only on the partial derivatives of the sections up to the order k over the points from U.*

We shall not prove this theorem, it follows from a much more general non-linear result proved in [Slovák, 88], see also 4.5 or [Kolář, Michor, Slovák, 93].

Let us point out, that the formulation of this theorem is not satisfactory, first of all because of the lack of the invariant definition of the order. The solution is to use the language of jets which is well suited for discussion on differential operators on manifolds.

2.2. Jets. Two smooth mappings g, $f\colon M \to N$ have the same *jet of order r* at $x \in M$ (r-jet briefly) if the values and partial derivatives up to the order r of f and g at x coincide in some local coordinates (equivalently in all local coordinates) at x and $f(x)$. We write $j^r_x f = j^r_x g$ and the corresponding equivalence class is called an r-jet with source x and target $f(x)$. The composition of jets is defined by the composition of the representatives, i.e. $j^r_{f(x)} g \circ j^r_x f = j^r_x (g \circ f)$.

One has to prove that this definition is correct (which is an easy exercise in analysis).[2]

The rule which associates the set $J^r(M,N)$ of all r-jets with source in M and target in N to each couple (M,N) of manifolds and the map $J^r(f,g)\colon J^r(M,N) \to J^r(M',N')$, to each couple $(f\colon M \to M', g\colon N \to N')$ of local diffeomorphisms defined by the obvious compositions (inverse to f on right, g on left), is a covariant functor from the category $\mathcal{M}f_m \times \mathcal{M}f_n$ with values in sets. The *local diffeomeorphisms* are globally defined and locally invertible maps and $\mathcal{M}f_m$ denotes the category of m-dimensional manifolds and local diffeomorphisms. We shall write also $J^r_x(M,N)$, $J^r(M,N)_y$ and $J^r_x(M,N)_y$ for spaces of jets with fixed source or

[1] In fact, it would be more precise to use the language of sheaves but I am sure we will not get any trouble when speaking about globally defined sections.

[2] The reader can find a much more geometric definition and a thorough treatment of all basic properties in [Kolář, Michor, Slovák, 93]. Roughly, we define the contact of order r for smooth functions $\mathbb{R} \to \mathbb{R}$ and then $j^r_x f = j^r_x g$ if and only if $h \circ f \circ c$ and $h \circ g \circ c$ have contact of order r at $0 \in \mathbb{R}$ for all smooth curves $c\colon \mathbb{R} \to M$, $c(0) = x$, and functions $h\colon N \to \mathbb{R}$, $h(f(x)) = 0$. In this setting, jets have a clear geometric meaning depending only on the structure of smooth functions on the real line.

target or both. Since $J^r(\mathbb{R}^m,\mathbb{R}^n)$ admits canonical representatives for the jets – the Taylor polynomials, there is a canonical structure of a fiber bundle over $\mathbb{R}^m\times\mathbb{R}^n$ on this jet space. The composition is the truncated composition of the polynomials by definition, hence smooth. Thus, the functoriality ensures that there is a uniquely defined structure of a fiber bundle on each $J^r(M,N)$ over $M\times N$ with standard fibers $J^r_0(\mathbb{R}^m,\mathbb{R}^n)_0$ and the composition is smooth. There are also the obvious bundle projections $\pi^r_k\colon J^r(M,N)\to J^k(M,N)$.

For each fibered manifold Y over M we define the k-th jet prolongation $J^kY\subset J^k(M,Y)$ over M as the subbundle of all jets of local sections. Clearly, $J^k(M,Y)$ can be defined as the quotient of $C^\infty Y$ and the smooth structure is the induced one. For each local section $s\in C^\infty Y$, there is its k-th prolongation $j^ks\in C^\infty(J^kY)$ defined by $j^ks(x)=j^k_xs$. If E is a vector bundle over M, then J^kE is a vector bundle with the operations defined on the representatives. Analogous constructions can be performed for $k=\infty$. We shall not need them (if then without any differentiable structure) and so the modifications are left to the reader.

2.3. The tangent and cotangent bundles. It is easy to verify that the tangent functor T equals to $J^1_0(\mathbb{R},\ \ )$ – the usual 'kinematic' definition of tangent vectors. Notice that the tangent maps are defined through composition of jets.

Similarly, $T^*=J^1(\ \ ,\mathbb{R})_0$. In this definition, T^*M always carries a natural vector bundle structure, TM is its dual bundle (with $\langle j^1_0c,j^1_{c(0)}f\rangle=j^1_0(f\circ c)\in\mathbb{R}$). More generally, $J^r(M,\mathbb{R})_0$ is a bundle of algebras.

2.4. Proposition. *The fiber $J^r_x(M,N)_y$ equals to the algebra homomorphisms $Hom(J^r_y(N,\mathbb{R})_0,J^r_x(M,\mathbb{R})_0)$.*

Proof. Given j^r_xf with target y we define $\varphi\colon J^r_y(N,\mathbb{R})_0\to J^r_x(M,\mathbb{R})_0$ by $\varphi(j^r_yg)=j^r_x(g\circ f)$. Since the algebra $J^r_y(N,\mathbb{R})_0$ is generated by the coordinate functions in arbitrary local coordinates, we can set the values on φ on the jets of these functions arbitrarily. This defines an element from the other algebra. □

Notice: If $r=1$ we get the identification of $J^1_x(M,N)_y$ with linear mappings $\mathrm{Hom}(T_xM,T_yN)$, since the multiplication on T^*_yN is zero and the latter claim is dual to the proposition above.

2.5. Differential operators. Let Y, Y' be two fibered manifolds with a common base M. We say that an operator $D\colon C^\infty Y\to C^\infty Y'$ is of order $0\le k\le\infty$ if the equality $j^r_xs=j^r_xq$ always implies $Ds(x)=Dq(x)$. Clearly, this is equivalent to the existence of a mapping $D_k\colon J^kY\to Y'$ which satisfies $Ds(x)=D_k(j^k_xs)$ for all $s\in C^\infty Y$ defined at x. Then $Ds=D_k\circ j^ks$ so that j^k plays the role of a universal operator of order k. *Differential operators* are the smooth operators $D\colon C^\infty Y\to C^\infty Y'$ of a finite order k. We shall often use the brief notation '$D\colon J^kY\to Y'$ is a differential operator'.

Now, we can reformulate Proposition 2.1 easily as follows: *Let E and E' be two vector bundles with a common compact base M. Then each local linear operator is a differential operator. Consequently, all local linear operators are expressed by smooth linear mappings $D_k\colon J^kE\to E'$.*

2.6. Invariant operators. Let Y, Y' be two bundles with a common base M and let G be a group, $\lambda\colon G\to\mathrm{Aut}(Y)$, $\lambda'\colon G\to\mathrm{Aut}(Y')$ be two group homomorphisms with values in the fiber bundle automorphisms. Let us write $\underline{\lambda}$, $\underline{\lambda}'$ for the induced actions on M. There is the canonical action of G on the spaces of sections defined by $(\lambda_g)_*(s)(x) = \lambda_g \circ s \circ \underline{\lambda}_g^{-1}(x)$ and similarly for λ'. An operator $D\colon C^\infty Y \to C^\infty Y'$ is said to be *G-invariant* if $D \circ (\lambda_g)_* = (\lambda'_g)_* \circ D$ for all $g \in G$. In fact the mapping D is G-equivariant (i.e. it intertwines the actions), but we use the traditional name invariant for operators. On the other hand, we shall use the word G-invariant for elements under invariant action of G and an invariant operator in the above sense is such an element in the space of all operators $C^\infty Y \to C^\infty Y'$ with the induced action of G.

The action of G on $C^\infty Y$ defines of course the canonical action of G on $J^k Y$, we shall use the same notation λ_* for both. We have $(\lambda_g)_*(j^k_x s) = j^k_{\underline{\lambda}_g(x)}((\lambda_g)_* s)$. A differential operator $D\colon C^\infty Y \to C^\infty Y'$ is G-invariant if and only if the corresponding mapping $D_k : J^k Y \to Y'$ is G-equivariant. The proof is evident.

Proposition. *Assume G is a Lie group, the action λ is smooth and the induced action $\underline{\lambda}$ on M is transitive. Then there is a bijection between smooth G-invariant differential operators $D\colon C^\infty Y \to C^\infty Y'$ and G_x-equivariant smooth mappings $J^k_x Y \to Y'_x$ where x is an arbitrary fixed point in M and G_x its isotropy group.*

Proof. If D is invariant, then the corresponding mapping D_k on the jet bundle must be G-equivariant. The isotropy group G_x respects the fiber $J^k_x Y$ and so the restriction of D_k to this fiber must be G_x-equivariant. On the other hand, each G_x-equivariant smooth mapping $J^k_x Y \to Y'$ gives rise to a smooth equivariant mapping $J^k Y \to Y'$ defined by the action of G and this defines a G-invariant differential operator. It is an easy exercise to work out more details. $\square$

2.7. Proposition. *Let $E \to M$ be a vector bundle. For each $k \in \mathbb{N}$ the following sequence is exact*

$$0 \to S^k T^* M \otimes E \xrightarrow{i} J^k E \xrightarrow{\pi^k_{k-1}} J^{k-1} E \to 0.$$

Proof. Consider $X = (j^1_x f_1 \bigcirc \cdots \bigcirc j^1_x f_k) \otimes e \in S^k T^* M \otimes E$ with $f_j\colon M \to \mathbb{R}$, $f(x) = 0$, $e \in E_x$. Let us choose some $q \in C^\infty E$ with $q(x) = e$ and define $s \in C^\infty E$ by $s(y) = f_1(y) f_2(y) \dots f_k(y) q(y)$. Then $j^{k-1}_x s = 0$ since at least one of the functions is not differentiated and hence zero at x and, for the same reason, the element $i(X \otimes e) := j^k_x s$ does not depend on our choice of q. Obviously, i is injective. Using local vector bundle coordinates at $0 \in E_x$, the jets of sections lying in the kernel of the jet projection are generated by those of the form of s and so the image of i coincides with the kernel. $\square$

2.8. The symbols. Let E and E' be two vector bundles with a common base M and let $D\colon J^k E \to E'$ be a differential operator. The composition $\sigma = D \circ i\colon S^k T^* M \otimes E \to E'$ is called the *symbol* of D.

$$\begin{tikzcd}
0 \arrow[r] & S^k T^* M \otimes E \arrow[r, "i"] \arrow[dr, "\sigma"'] & J^k E \arrow[r, "\pi^k_{k-1}"] \arrow[d, "D"] & J^{k-1} E \arrow[r] & 0 \\
& & E' & &
\end{tikzcd}$$

Two differential operators with the same symbol (so in particular of the same order) differ by an operator of a lower order.

If G is a (Lie) group acting on M, i.e. we have a homomorphism $\underline{\lambda}\colon G \to \mathrm{Diff}(M)$, then there is the induced action λ^{T^*} of G on T^*M defined by $\lambda_g^{T^*}(j_x^1 f) := (T^*\underline{\lambda}_g)(j_x^1 f) = j^1_{\underline{\lambda}_g(x)}(f \circ \underline{\lambda}_g^{-1})$ for each $g \in G$. This must be a correct definition since we have used the functoriality of T^* (the functor is covariant – hence the inverse involved!). The same procedure applies to a large class of functor on manifolds which involves e.g. all tensor bundles, cf. 2.12.

Now, given actions of G on E and E' we have a well defined action of G also on $S^kT^*M \otimes E$ and we get

Proposition. *If $D\colon J^kE \to E'$ is a G-invariant linear differential operator, then its symbol $\sigma\colon S^kT^*M \otimes E \to E'$ is G-equivariant.*

Proof. We have only to prove that i is G-equivariant but this is more or less evident. $\square$

This simple result is often very useful as it allows to exclude the existence of invariant operators. On the other hand, not every equivariant map $S^kT^*M \otimes E \to E'$ is a symbol of an invariant operator.

2.9. Examples. We start with the simplest example, the exterior differential on functions. So $EM = M \times \mathbb{R}$, $E' = T^*M$, $D\colon J^1(M \times \mathbb{R}) \to E'$ and $\sigma\colon T^*M \otimes \mathbb{R} = T^*M \to E'$. Consider $G = \mathrm{Diff}(M)$. If D is G-invariant, then σ must be G-equivariant, too. The action of $\mathrm{Diff}(M)$ on M is transitive and smooth if M is connected and the action of the isotropy group $\mathrm{Diff}_x(M)$ on T_x^*M factorizes through the well known linear action of $GL(m, \mathbb{R})$. We can restrict ourselves to $M = \mathbb{R}^m$, $x = 0$, for our operators are local.

Let us assume E' is not fixed but suppose that the action of $\mathrm{Diff}_0(\mathbb{R}^m)$ on E'_0 also factorizes through $GL(m)$ and is irreducible. Since the action on $T_0^*\mathbb{R}^m = \mathbb{R}^{m*}$ is also irreducible, σ is a multiple of the identity and, moreover, there is no other possibility for E' beside $E' = T^*M$. Thus, the only $\mathrm{Diff}(M)$-invariant local first order linear operator on functions is the exterior differential, up to the identity and scalar multiples. Notice, if the target space corresponds to a decomposable representation of the linear group, then the operator must be a sum of multiples of the exterior differentials and identities with values in the irreducible components.

The symbol of the exterior differential $d\colon \Lambda^kT^*M \to \Lambda^{k+1}T^*M$ is the alternation $\mathrm{Alt}\colon T^*M \otimes \Lambda^kT^*M \to \Lambda^{k+1}T^*M$. We shall see that this is the only $GL(m)$-equivariant map between these spaces and so d must be unique up to multiples (and lower order terms).

The mapping $\mathrm{id}\colon S^2T^*M \otimes \mathbb{R} \to S^2T^*M$ is of course $\mathrm{Diff}(M)$-invariant, but we shall see that this is not a symbol of an invariant differential operator.

2.10. Operators on homogeneous bundles. We have seen that the description of invariant operators reduces to the description of some equivariant mappings (between finite dimensional manifolds) if the action on the base manifold is transitive. The most common situation is, we are given a manifold M with a transitive action of a finite dimensional[3] Lie group G. Let $x \in M$ be fixed and write $B = G_x$ for its isotropy group. Hence $M = G/B$ and the projection $p\colon G \to M$ is a principal fiber bundle with structure group B. Given any Lie group homomorphism $\lambda\colon B \to \mathrm{Diff}(S)$, there is the associated bundle $Y = G \times_\lambda S$ over M with standard fiber S. We call this bundle a *homogeneous bundle* over the homogeneous space M. If S is a vector space and $\lambda\colon B \to GL(V)$, we get a vector bundle. This construction is functorial in the principal fiber bundle entry and so there is an induced action on Y to each action λ' with values in the principal bundle automorphisms of G. In particular, the Lie group G acts on itself via left translations, let us denote this action by a dot. We have $g.(h \times_\lambda t) = gh \times_\lambda t$ and there is the induced action $\bar\lambda$ on $C^\infty Y$.

Consider now the space $C^\infty(G,S)^B$ of all B-equivariant mappings which means $s(gh) = \lambda_{h^{-1}}(s(g))$. There is the obvious left action of G there, $g.s(h) = s(g^{-1}h)$.

Lemma. *We identify $C^\infty Y = C^\infty(G,S)^B$ as spaces with a left action of G via $s \simeq \tilde s$, $u \times_\lambda \tilde s(u) := s(p(u))$.*

Proof. The identification is well defined, for $u.b \times_\lambda \tilde s(u.b) = u.b \times_\lambda \lambda_{b^{-1}}(\tilde s(u)) = u \times_\lambda \tilde s(u)$, $b \in B$. Under this identification, the actions of G coincide: $(g.s)(p(u)) = gg^{-1}u \times_\lambda \tilde s(g^{-1}u) = u \times_\lambda \tilde s(g^{-1}u)$. $\square$

This simple lemma is very important since we can view the G-invariant operators on homogeneous bundles as operators on S-valued functions on the principal bundle $G \to G/B$ which are invariant with respect to the left translations.

2.11. The geometric structures. The r-th order *frame bundle* P^rM on an m-dimensional manifold M is defined as the bundle of all invertible jets $\mathrm{inv}J^r_0(\mathbb{R}^m, M)$ over M. This is a principal bundle with structure group $G^r_m := \mathrm{inv}J^r_0(\mathbb{R}^m,\mathbb{R}^m)_0$, the so called *jet group*, and the principal action defined by the composition of jets. This construction is functorial, i.e. we have the local principal fiber bundle isomorphism $Pf\colon P^rM \to P^rN$ for each local diffeomorphism $f\colon M \to N$ which is defined by the composition of jets. The elements in the frame bundles are 'local coordinate charts up to order r' and the elements in the jet groups are 'transformations of coordinates up to order r'.

Definition. Let $B \subset G^r_m$ be a closed Lie subgroup. A *B-structure on a manifold* M is a reduction $FM \hookrightarrow P^rM$ to the structure group B. The category $\mathcal{M}f_m(B)$ consists of m-dimensional manifolds with B-structures and local diffeomorphisms $f\colon M \to N$ satisfying $P^rf(FM) \subset FN$.

2.12. Geometric objects and operators. Let us consider a closed Lie subgroup $B \subset G^r_m$ and its action $\lambda\colon B \to \mathrm{Diff}(S)$ on a manifold S. This defines the functor $E\colon \mathcal{M}f_m(B) \to \mathcal{M}f$, $E(M,FM) := FM \times_\lambda S$ and $Ef := P^rf \times_\lambda \mathrm{id}_S|E(M,FM)$. These bundles are called *bundle functors* or *natural bundles* or *bundles of geometric objects*, their sections are called *geometric objects* on $\mathcal{M}f_m(B)$ (more precisely, the bundles functors are the functors, the geometric objects are sections of their values). If λ is a linear representation of B in a vector space V, then the corresponding

[3] We could certainly admit here infinite dimensional Lie groups as well, but our aim is to apply tools from finite dimensional representation theory to find all the equivariant mappings and so we have to proceed in another way if dealing with say $G = \mathrm{Diff}(M)$

geometric objects are sections of vector bundles. For each manifold with a B-structure (M, FM), there is the subgroup $\mathcal{B}_M \subset \mathrm{Diff}(M)$ of all local $\mathcal{M}f_m(B)$-isomorphisms and its action on all geometric objects on M (i.e. on sections of the bundles corresponding to representations of B). We shall denote the latter action corresponding to $\lambda\colon B \to \mathrm{Diff}(S)$ by λ_*. Let us remark that even if we deal with linear representation, we cannot restrict ourselves to the irreducible representations since the action of the nilpotent kernel of the jet projections to the first order must then act trivially.

Definition. Let E and E' be two arbitrary bundle functors on $\mathcal{M}f_m(B)$. A *natural operator* $D\colon E \to E'$ is a system of $\mathcal{B}_M$-invariant local smooth operators $D_{(M,FM)}$: $C^\infty(EM) \to C^\infty(E'M)$ invariant with respect to restrictions to open submanifolds (with the restricted B-structures). More precisely, all $D_{(M,FM)}$ are smooth, and for all local $\mathcal{M}f_m(B)$-isomorphisms $f\colon M \to N$ and sections $s_1 \in C^\infty(EM)$, $s_2 \in C^\infty(EN)$, the right-hand square commutes whenever the left-hand one does

$$\begin{tikzcd} EM \arrow[d, "Ef"'] & M \arrow[l, "s_1"'] \arrow[r, "Ds_1"] \arrow[d, "f"'] & E'M \arrow[d, "E'f"] \\ EN & N \arrow[l, "s_2"'] \arrow[r, "Ds_2"] & E'N \end{tikzcd}$$

Notice, that the latter definition involves both the locality of the operators and invariance of them with respect to restrictions to open submanifolds.[4]

2.13. Lemma. *Let $B \subset G^r_m$. The B-structures $i\colon FM \hookrightarrow P^rM$ correspond bijectively to smooth sections of P^r_M/B.*

Proof. Each reduction $i\colon FM \hookrightarrow P^rM$ induces a map $\bar{i}\colon FM \to P^rM/B$. If σ_α are local sections of FM with domains covering the base M, then the transition functions of their composition with $\bar{i}$ are identities and so they determine a global smooth section of P^rM/B. On the other hand, each global section of P^rM/B can be locally obtained as a projection of local smooth sections of P^rM. Their transition functions must have values in B, hence we get a reduction. □

2.14. The coverings of structure groups. Let $\tilde{B}$ be a covering of the Lie group $B \subset G^r_m$ and write $\mathcal{M}f_m(\tilde{B})$ for the category of m-dimensional manifolds M with a distinguished covering $\tilde{F}M$ of the reduction FM of the frame bundle P^rM to B, and distinguished coverings $\tilde{F}f$ of the values Ff on $\mathcal{M}f_m(B)$-morphisms f. Repeating the above construction of the associated spaces, each representation $\lambda\colon \tilde{B} \to \mathrm{Diff}(S)$ gives rise to the functor F_λ, the bundle functor corresponding to λ. A natural operator $D\colon F_\lambda \to F_\rho$ is a system of local operators $D_M : C^\infty(F_\lambda M) \to C^\infty(F_\rho M)$ which commute with the actions of the $\mathcal{M}f_m(\tilde{B})$-morphisms and behave well with

[4] The reader interested in axiomatic description of geometric objects and operators is advised to [Kolář, Michor, Slovák, 93]. Roughly, all functors on categories "similar" to $\mathcal{M}f_m(B)$ with some mild conditions are of the above form (the conditions do not involve regularity and the "dependence on jets", so that this description needs a long and involved analytical proof). One can also define more general operators which "extend" the base, but we shall not treat them in this text.

respect to restrictions to open submanifolds. The exact formulation mimics the above definition.[5]

2.15. Examples of linear natural operators.

(1) Take $B = GL(m, \mathbb{R})$. The irreducible representations are invariant subbundles of tensor bundles. The natural operators are Diff(M)-invariant operators, i.e. they have to commute with pullbacks of tensor fields. One can prove that all of them are constructed by means of the standard operations from the tensor algebra (cf. 3.6) and the exterior derivatives on exterior forms. In particular, all of them have order one, see [Kirillov, 77] or [Terng, 78]. We shall comment on this in more details later.

(2) Consider $B = SL(m, \mathbb{R}) = \{A \in GL(m, \mathbb{R}); \det A = 1\}$. We claim that the B-structures are fixed volume forms on the manifolds. Indeed, it is easy to verify $P^1M/SL(m, \mathbb{R}) = \Lambda^m T^*M \setminus \{0\}$. Hence, the local diffeomorphisms in $\mathcal{B}_{\mathbb{R}^m}$ are just the unimodular ones, i.e. those preserving the canonical volume form. In a similar way, we can describe the manifolds with a fixed tensor field of some given type in the terms of B-structures. For example $O(m)$ yields Riemannian manifolds and local isometries. Also in the case $B = SL(m)$ all operators are built from tensor algebra operations and exterior differentials. However, we have to take into account the natural equivalence $T \to \Lambda^{m-1}T^*$ and also $\Lambda^m T^* \to \Lambda^0 T^*$ (the functors are defined on $\mathcal{M}f_m(B)$). In this way, there also appears the second order operation $\Lambda^{m-1}T^* \xrightarrow{d} \Lambda^m T^* \to \Lambda^0 T^* \xrightarrow{d} T^*$. The first d in this composition also corresponds to the divergence of vector fields, the whole operation to the differential of divergence.[6]

2.16. The Riemannian case. If $B = O(m)$, we have the natural equivalence $T \to T^*$ and so there are many linear natural operators. Some of them can be easily obtained using the canonical Levi-Cività connection Γ on the tensor bundles E over Riemannian manifolds which can be viewed as a distinguished section of $\pi^1_0 \colon J^1E \to E$.

Thus, we get a splitting of the exact sequence from 2.7 in the special case $k = 1$. The induced splitting of on the left is just the well known Riemannian covariant derivative ∇ on E. In fact, the Riemannian covariant derivative is a first order natural operator available on each first order natural bundle. Since the values of the natural bundle are associated bundles to the linear frame bundles and the above values of ∇ are section of another tensor bundle, there is also the Riemannian covariant derivative. In this way, we can define the iterated covariant derivative $\nabla^k \colon J^kE \to \otimes^k T^*M \otimes E$. We claim that the symmetrization $\tilde{\nabla}^k$ of ∇^k is a splitting of the above mentioned exact sequence. Indeed, in coordinates, we express the iterated covariant derivative as the sum of the usual partial derivatives (which are symmetric) and a polynomial expression depending on $(k-1)$-jet of the connection

[5] In fact, we have used the concept of the so called gauge natural operators in the sense of [Eck, 81]. The gauge natural bundles are functors on principal fiber bundles with values fibered over the base manifolds, see [Kolář, Michor, Slovák, 93, Chapter XII] for detailed treatment.

[6] Exactly as in the example (1), the latter operators exhaust all linear natural operators in the category of manifolds with fixed volume forms, up to decompositions into irreducible components, identities and scalar multiples, for more comments see [Kolář, Michor, Slovák, 93], the proofs are in [Kirillov 77], [Rudakov, 74], [Rudakov, 75]

and $(k-1)$-jet of the section of E. Moreover, the degrees of the homogeneous components of this polynomial in the entry from E are non-zero. Hence on the image of $i\colon S^kT^*\otimes E\to J^kE$, we get just the inverse to i, c.f. 2.7. Since $\tilde\nabla^k$ is natural on Riemannian manifolds, we get a natural operator to each 'natural symbol' (i.e. a natural operator of order zero between the appropriate bundles):

$$\begin{tikzcd}
0 \arrow[r] & S^kT^*\otimes E \arrow[r, shift right, "i"'] \arrow[dr, "\sigma"'] & J^kE \arrow[l, shift right, "\tilde\nabla^k"'] \arrow[r, "\pi^k_{k-1}"] \arrow[d, "D"] & J^{k-1}E \arrow[r] & 0 \\
 & & E' & &
\end{tikzcd}$$

For example, the contraction $\mathrm{Tr}\colon S^2\mathbb{R}^{m*}\to\mathbb{R}$ corresponds to the well known Laplacian. Unfortunately, we cannot describe easily all operators in this way as we do not know explicitly, how the Riemannian connection may enter (their influence need not be linear or even polynomial a priori). So we have to solve the nonlinear problem of finding all 'natural symbols', if we want to describe all natural linear operators on Riemannian manifolds.

An important feature is that the group of all $\mathcal{M}f_m(O(m))$-morphisms on a Riemannian manifold is a finite dimensional Lie group. Unfortunately, its action is rarely transitive. But if this is the case, e.g. for the flat Riemannian manifold $\mathbb{R}^m$ or the spheres, then we can view the bundles of geometric objects as homogeneous bundles as described in 2.9.

2.17. Remark. Let us notice the importance of Lemma 2.13. Namely, the quotients P^rM/B form bundle functors on the whole category $\mathcal{M}f_m$ with the action of local diffeomorphisms defined on the representatives of the cosets. Thus if we want to discuss natural operators on some category $\mathcal{M}f_m(B)$ and if the arguments of the operators happen to be geometric objects on the whole $\mathcal{M}f_m$, then we can always add the B-structures to the arguments of the operators and solve the problem in the category $\mathcal{M}f_m$ (it is a nice exercise to verify that this is really equivalent – see [Kolář, Michor, Slovák, 93] for more details if neccessary). We shall see later on that all linear representations of $SL(m,\mathbb{R})$ and $O(m,n,\mathbb{R})$ live in tensor spaces. Since the invariant subspaces are always images of natural linear projections and they are naturally linearly embedded into the whole space, we can also overcome the fact that there are much more invariant subspaces in the tensor spaces in the unimodular or pseudo-Riemannian case there. Let us also notice, that we would be able to treat the unimodular case directly, but serious problems arise in the Riemannian one as the objects do not admit a transitive action of the local isometries and so we cannot reduce the classification problem to finding equivariant mappings between finite dimensional manifolds, cf. 2.6.

We need some additional work to incorporate also the spinor fields into this setting, we shall apply the approach from 2.14.

3. Invariant tensors

3.1. Definition. Let G be a Lie group with a representation φ on a finite dimensional vector space V (real or complex). Then the representation $\tilde\varphi$ on V^* is defined by $\tilde{\varphi}_g(v^*)(v) = v^*(\varphi_{g^{-1}}(v))$ and the tensor products of these representations yield representations on all $\otimes^p V^* \otimes \otimes^q V$. All the above actions of G will be often denoted simply by a dot. An *invariant tensor* in the latter space is an element t with $G.t = t$.

3.2. Lemma. *A linear mapping $f\colon \otimes^p V^* \otimes \otimes^q V \to \otimes^r V^* \otimes \otimes^s V$ is G-equivariant if and only if the corresponding element $f^{\otimes} \in \otimes^{p+s} V^* \otimes \otimes^{q+r} V$ is G-invariant. If f is polynomial and G-equivariant, then each homogeneous component of f is G-equivariant.*

Proof. It follows immediately from the definition of the tensor product of representations and the identification involved. □

3.3. The *total polarization* of a homogeneous polynomial $f\colon W_1 \to W_2$ of degree k between vector spaces (or affine spaces) is a linear mapping $Pf\colon S^k W_1 \to W_2$ defined as follows. The first order term in the (partial) Taylor polynomial $f(x + ty) = f(x) + tP_1 f(x, y) + \dots$ is a polynomial map of degree $k-1$ in x. The k-th iteration $P_k f = P_1(P_{k-1} f)$ is k-linear and symmetric in variables $y_1, \dots, y_k \in W_1$ and independent of x. Let Pf be the corresponding linear map. The original map f is obtained back through $f(x) = Pf(x \otimes \dots \otimes x)$.

Lemma. *A polynomial mapping $f\colon \otimes^p V^* \otimes \otimes^q V \to \otimes^r V^* \otimes \otimes^s V$ is G-equivariant and homogeneous if and only if its total polarization is G-equivariant.*

Proof. Notice that the actions are linear. □

The aim of this section is to describe all G-invariant tensors for some of the classical subgroups of $GL(m, \mathbb{C})$ or $GL(m, \mathbb{R})$. If we shall not specialize the field $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, the arguments and results will apply to both cases. In the view of the above lemmas, this will describe G-equivariant polynomial maps.

Let us start with $GL^+(m, \mathbb{R})$, the group of real invertible matrices with positive determinant, or the full linear group $GL(m, \mathbb{C})$. As before, we shall use the 'Penrose abstract index notation', i.e. usual indices denote a kind of abstract labels and if they should be concrete numbers, they are indicated by underlined letters, cf. 1.2.

3.4. Definition. Let us denote by δ^i_j the identity tensor in $V^* \otimes V$, i.e. the trace (evaluation) $\mathrm{Tr}\colon V^* \otimes V \to \mathbb{K}$. For every permutation $\sigma \in \mathcal{S}_r$, $r \in \mathbb{N}$, we define the *elementary invariant tensor* $I^\sigma \in \otimes^r V^* \otimes \otimes^r V$ *of degree* r, $I^\sigma = \delta^{j_1}_{i_{\sigma(1)}} \dots \delta^{j_r}_{i_{\sigma(r)}}$.

Evidently, all I^σ are $GL(m)$-invariant tensors, hence also $GL^+(m, \mathbb{R})$-invariant tensors.

3.5. Theorem. *All $GL^+(m, \mathbb{R})$-invariant tensors are linear combinations of the elementary invariant tensors. In particular, a non-zero invariant tensor lies in a tensor space $\otimes^q V \otimes \otimes^r V^*$ with $q = r$.*

All $GL(m, \mathbb{C})$-invariant tensors are linear combinations of elementary invariant tensors.

Proof. Let $G = GL^+(m, \mathbb{R})$ or $G = GL(m, \mathbb{C})$ and let V be real or complex, respectively. The elements $a \in G$ and their inverses $\tilde{a}$ are identified with $a^i_j, \tilde{a}^i_j \in V^* \otimes V$ and the invariance of $t \in \otimes^p V^* \otimes \otimes^q V$ is expressed through a system of tensor equations

$$a^{j_1}_{k_1} \dots a^{j_q}_{k_q} t^{k_1 \dots k_q}_{l_1 \dots l_p} \tilde{a}^{l_1}_{i_1} \dots \tilde{a}^{l_p}_{i_p} = t^{j_1 \dots j_q}_{i_1 \dots i_p} \tag{1}$$

where the a^i_j's are copies of an arbitrary element in G. In particular, if we substitute $a^i_j = c\delta^i_j$, $c \in \mathbb{R}$, $c > 0$, then we get $c^{q-p} t^{\underline{j}_1 \dots \underline{j}_q}_{\underline{i}_1 \dots \underline{i}_p} = t^{\underline{j}_1 \dots \underline{j}_q}_{\underline{i}_1 \dots \underline{i}_p}$ for all concrete indices. Hence either $t = 0$ or $p = q$. So let us assume $p = q$.

Evaluating for concrete values of indices we see that (1) is equivalent to

$$a^{j_1}_{k_1} \dots a^{j_p}_{k_p} t^{k_1 \dots k_p}_{l_1 \dots l_p} = a^{k_1}_{l_1} \dots a^{k_p}_{l_p} t^{j_1 \dots j_p}_{k_1 \dots k_p}$$

and this is further equivalent to

$$a^{i_1}_{k_1} \dots a^{i_p}_{k_p} \delta^{j_1}_{i_1} \dots \delta^{j_p}_{i_p} t^{k_1 \dots k_p}_{l_1 \dots l_p} = a^{i_1}_{k_1} \dots a^{i_p}_{k_p} t^{j_1 \dots j_p}_{i_1 \dots i_p} \delta^{k_1}_{l_1} \dots \delta^{k_p}_{l_p}. \tag{2}$$

Since the 'variables' $a^{\underline{i}}_{\underline{j}}$ run through an open subset of a Euclidean space ($\mathbb{R}^{m^2}$ or $\mathbb{C}^{m^2}$), the 'coefficients' of the same expressions in a's must coincide on both sides. Taken into account that the concrete values of the monomials in a's are symmetric in the simultaneous permutations of superscripts and subscripts, we get the equivalent form of (2)[7]

$$\sum_{\sigma \in \mathcal{S}_p} \delta^{j_1}_{i_{\sigma(1)}} \dots \delta^{j_p}_{i_{\sigma(p)}} t^{k_{\sigma(1)} \dots k_{\sigma(p)}}_{l_1 \dots l_p} = \sum_{\sigma \in \mathcal{S}_p} \delta^{k_{\sigma(1)}}_{l_1} \dots \delta^{k_{\sigma(p)}}_{l_p} t^{j_1 \dots j_p}_{i_{\sigma(1)} \dots i_{\sigma(p)}}. \tag{3}$$

Assume first $m \geq p$ and let us define scalar coefficients $c_\sigma := t^{\underline{1} \dots \underline{p}}_{\underline{\sigma(1)} \dots \underline{\sigma(p)}}$. Consider the equations (3) with concrete indices $\underline{j}_1 = \underline{i}_1 = 1, \dots, \underline{j}_p = \underline{i}_p = p$. Then only the term with $\sigma = \mathrm{id}$ remains on the left hand side, and so we get

$$t^{\underline{k}_1 \dots \underline{k}_p}_{\underline{l}_1 \dots \underline{l}_p} = \sum_{\sigma \in \mathcal{S}_p} c_\sigma \delta^{\underline{k}_{\sigma(1)}}_{\underline{l}_1} \dots \delta^{\underline{k}_{\sigma(p)}}_{\underline{l}_p}.$$

Thus, the theorem is proved for $m \geq p$.

If $m < p$, then we would still like to view (3) as a system of equations for t's on the left hand side, while those on the right should be known.[8] But the rank of this system is not maximal and we have to add some suitable equations.

Consider the homogeneous system in $p!$ tensorial variables $X_\sigma = (X_\sigma)^{\underline{k}_1 \dots \underline{k}_p}_{\underline{l}_1 \dots \underline{l}_p}$ corresponding to (3)

$$\sum_{\sigma \in \mathcal{S}_p} \delta^{\underline{j}_1}_{\underline{i}_{\sigma(1)}} \dots \delta^{\underline{j}_p}_{\underline{i}_{\sigma(p)}} X_\sigma = 0 \tag{4}$$

and let $Z^\alpha_\sigma = (Z^\alpha_\sigma)^{k_1 \dots k_p}_{l_1 \dots l_p}$, $\alpha = 1, \dots, r$, be a fundamental system of its solutions. Let us further consider the system of r equations (with the same variables as in (4))

$$\sum_{\sigma \in \mathcal{S}_p} Z^\alpha_\sigma X_\sigma = \sum_{\sigma \in \mathcal{S}_p} (Z^\alpha_\sigma)^{\underline{i}_1 \dots \underline{i}_p}_{\underline{j}_1 \dots \underline{j}_p} (X_\sigma)^{\underline{k}_1 \dots \underline{k}_p}_{\underline{l}_1 \dots \underline{l}_p} = 0. \tag{5}$$

For each tensor $X \in \otimes^p V^* \otimes \otimes^p V$ let us write $\sigma.X$ for the action by permutation of superscripts.

[7] The same system of equations is obtained by differentiating (1) with respect to a^i_j at the unit and collecting the corresponding terms together (evaluated at $\delta^i_j \in \mathfrak{gl}(m)$).

[8] H. Weyl presents a general tool for the reduction of the considerations to $m \geq p$, the so called Capelli identity, see [Weyl, 39, Chapter II, section 4]. We shall proceed more elementary following [Gurevich, 48] and [Kolář, Michor, Slovák, 93].

Sublemma. *For each X the system of tensors $X_\sigma = \sigma.X$, $\sigma \in \mathcal{S}_p$, is a solution of (5).*

Proof. We have $(X_\sigma)^{i_1\dots i_p}_{j_1\dots j_p} = X^{i_{\sigma^{-1}(1)}\dots i_{\sigma^{-1}(p)}}_{j_1\dots j_p} = \delta^{i_1}_{k_{\sigma(1)}}\dots\delta^{i_p}_{k_{\sigma(p)}} X^{k_1\dots k_p}_{j_1\dots j_p}$ so that the lemma is obvious. $\square$

Now, it remains to notice that the rank of the system (4) and (5) (considered with concrete indices as a linear system for $p!$ tensorial variables) is maximal. This is shown easily: if X_σ is a solution of both systems, then, in particular, $X_\sigma = \sum_\alpha c_\alpha Z^\alpha_\sigma$, $c_\alpha \in \mathbb{K}$. But then (5) yields

$$\begin{aligned} 0 = \sum_\alpha c_\alpha (\textstyle\sum_\sigma Z^\alpha_\sigma X_\sigma) &= \sum_\alpha\sum_\sigma c_\alpha (Z^\alpha_\sigma)^{\underline{i}_1\dots\underline{i}_p}_{\underline{j}_1\dots\underline{j}_p}(X_\sigma)^{\underline{k}_1\dots\underline{k}_p}_{\underline{l}_1\dots\underline{l}_p} \\ &= \sum_\sigma (X_\sigma)^{\underline{i}_1\dots\underline{i}_p}_{\underline{j}_1\dots\underline{j}_p}(X_\sigma)^{\underline{k}_1\dots\underline{k}_p}_{\underline{l}_1\dots\underline{l}_p}. \end{aligned}$$

In particular, $\sum_\sigma \big((X_\sigma)^{\underline{i}_1\dots\underline{i}_p}_{\underline{j}_1\dots\underline{j}_p}\big)^2 = 0$ and so all X_σ are zero.

Sublemma. *Let $c \in \otimes^p V^* \otimes \otimes^p V$ be a fixed tensor and let $r!$ tensors X_σ satisfy the system*

$$\begin{aligned} \sum_{\sigma\in\mathcal{S}_p} \delta^{i_1}_{j_{\sigma(1)}}\dots\delta^{i_p}_{j_{\sigma(p)}} X_\sigma &= \sum_{\sigma\in\mathcal{S}_p} c^{i_1\dots i_r}_{j_{\sigma(1)}\dots j_{\sigma(p)}} I^\sigma \\ \sum_{\sigma\in\mathcal{S}_p} Z^\alpha_\sigma X_\sigma &= 0 \qquad\qquad \alpha = 1,\dots,r. \end{aligned}$$

Then every X_σ is a linear combination of the elementary invariant tensors.

Proof. Since the system (4) and (5) has full rank, there must be a subsystem (4') in (4) such that the system (4') and (5) is linearly independent and has still full rank. Consider the corresponding subsystem in the statement of the lemma and apply the Cramer rule for modules. $\square$

Now Theorem 3.5 follows easily: if t is invariant, it satisfies (3) and the system of tensors $X_\sigma = \sigma.t$ is a solution of the system from the above lemma with $c = t$. $\square$

Theorem 3.5 and Lemma 3.2 imply the following implicit description of all linear equivariant mappings between tensor spaces.

3.6. Corollary. *All $GL^+(m,\mathbb{R})$-equivariant or $GL(m)$-equivariant linear mappings between tensor spaces are obtained through a finite iteration of the following steps*

(a) *permutation of indices*
(b) *tensor product with invariant tensors*
(c) *trace with respect to one subscript and one superscript*
(d) *linear combinations.*

3.7. Polynomial equivariant mappings. Let W_1, W_2 be $GL(m)$-invariant subspaces in tensor spaces $V_1 = \otimes^p V^* \otimes \otimes^q V$, $V_2 = \otimes^r V^* \otimes \otimes^s V$. There are equivariant projections $p_1 \colon V_1 \to W_1$, $p_2 \colon V_2 \to W_2$ and the equivariant inclusions $j_1 \colon W_1 \to V_1$, $j_2 \colon W_2 \to V_2$ (the former can be defined by extending a fixed basis of the invariant subspace). Since p_i are the left inverses to j_i, all $GL(m)$-equivariant linear maps $f \colon W_1 \to W_2$ are also described by 3.6. Thus, we know all polynomial $GL(m)$-equivariant mappings $f \colon W_1 \to W_2$, cf. Lemma 3.3.

$$\begin{tikzcd} W_1 \arrow[r, "f"] \arrow[d, shift left, "j_1"] & W_2 \arrow[d, shift right, "j_2"'] \\ V_1 \arrow[r] \arrow[u, shift left, "p_1"] & V_2 \arrow[u, shift right, "p_2"'] \end{tikzcd}$$

3.8. Examples. Let us take $V_1 = V_2 = \otimes^r V$, and p_i be either the alternation or the symmetrization. A polynomial mapping $f \colon W_1 \to W_2$ commutes with the action of the center of $GL(m)$ and therefore f is linear. Hence *all polynomial $GL(m)$-equivariant mappings*

(1) $S^r V \to S^r V$ *are the constant multiples of the identity*
(2) $\Lambda^r V \to \Lambda^r V$ *are the constant multiples of the identity*
(3) $\otimes^r V \to S^r V$ *are the constant multiples of the symmetrization*
(4) $\otimes^r V \to \Lambda^r V$ *are the constant multiples of the alternation*
(5) $S^r V \to \otimes^r V$ *and* $\Lambda^r V \to \otimes^r V$ *are the constant multiples of the inclusion.*

3.9. $SL(m)$-invariant tensors. Next we shall restrict our group G to $SL(m, \mathbb{R})$ or $SL(m, \mathbb{C})$. Let us write also $G^{\pm} = \{A \in GL(m, \mathbb{R}); \det A = \pm 1\}$. We shall not need to modify the proof of 3.5 for these groups since we shall be able to reduce this case to Theorem 3.5.

First of all we have to notice the existence of the invariant tensor $\nu \in \Lambda^m V^*$, the canonical volume form, and its dual contravariant tensor $\eta \in \Lambda^m V$. Further, there are the linear isomorphisms $\alpha \colon V \to \Lambda^{m-1} V^*$, $\beta \colon V^* \to \Lambda^{m-1} V$ defined by $\alpha(v) = i_v(\nu)$, $\beta(v^*) = i_{v^*}\eta$. Thus, we may restrict ourselves to invariant tensors in $\otimes^p V^*$, i.e. to invariant linear mappings $f \colon \otimes^p V \to \mathbb{K}$. Let us denote $W = \Lambda^m V^* \setminus \{0\}$, the space of volume forms with the restriction of the action of $G^{\pm}$. Given f, we define

$$\bar\varphi \colon W \times \otimes^p V \to \mathbb{K}, \qquad \bar\varphi((\det A)\nu, t) = f(A.t).$$

Lemma. *If f is G- or $G^{\pm}$-invariant, then $\bar\varphi$ is well defined and $GL(m)$-invariant.*

Proof. Since the action of G or $G^{\pm}$ on W is $A.\nu = (\det A)^{-1}\nu$, this follows directly from the definition of $\bar\varphi$.

We would like to extend $\bar\varphi$ to a polynomial $GL(m)$-invariant map on $\Lambda^m V^* \times \otimes^p V$, for then we can apply directly Theorem 3.5. The mapping $\bar\varphi$ gives rise to a mapping $\varphi \colon GL(m) \times \otimes^p V \to \mathbb{K}$, $\varphi(A, t) = \bar\varphi(A.\nu, t)$ and for each $t \in \otimes^p V$ we get the restriction $\varphi_t \colon GL(m) \to \mathbb{K}$ which is polynomial and G- or $G^{\pm}$-invariant.

3.10. Lemma. *If $\psi\colon GL(m) \to \mathbb{C}$ is polynomial and $SL(m,\mathbb{C})$-invariant, then there is a polynomial $\tilde\psi\colon \mathbb{C}\to\mathbb{C}$ such that $\psi(A) = \tilde\psi(\det A)$. If ψ is $G^{\pm}$-invariant, then we can find $\tilde\psi$ with $\psi(A) = \tilde\psi((\det A)^2)$. If $\psi\colon GL^+(m) \to \mathbb{R}$ is $SL(m,\mathbb{R})$-invariant, then there is polynomial $\tilde\psi$ with $\psi(A) = \tilde\psi(\det A)$ defined on positive real numbers.*

Proof. Let us define $i\colon \mathbb{K}\to\mathfrak{gl}(m,\mathbb{K})$, $\quad i(a) = \begin{pmatrix} a & 0 \\ 0 & \mathbb{I}_{m-1}\end{pmatrix}$. Then we get

$$\begin{tikzcd} GL(m) \arrow[r, "\psi"] \arrow[d, shift left, "\det"] & \mathbb{K} \\ \mathbb{K}\setminus\{0\} \arrow[u, shift left, "i"] \arrow[ur, "\tilde\psi"'] & \end{tikzcd}$$

If $\mathbb{K}=\mathbb{C}$, then $A = i(\det A) \bmod SL(m)$. If $\mathbb{K}=\mathbb{R}$ and $\det A > 0$ then $A = i(\det A)$ mod $SL(m)$. If $\det A < 0$ then $A = i(-\det A) \bmod SL(m) = i(\det A) \bmod G^{\pm}$ and so $\tilde\psi(a) = \tilde\psi(-a)$. $\square$

3.11. Theorem. *If $t \in \otimes^p V^* \otimes \otimes^q V$ is $SL(m)$ invariant and non-zero, then $p - q = km$, $k\in\mathbb{Z}$. All $SL(m)$-invariant maps between $SL(m)$-invariant subspaces of tensor spaces are exhausted by those obtained by iterating a finite number of steps 3.6.(a)–(d) and*

(i) *the tensors ν and η are invariant.*

If f is $G^{\pm}$ invariant, we have to replace (i) by

(i') *the tensors $\nu\otimes\nu$, $\nu\otimes\eta$, $\eta\otimes\eta$ are invariant.*

Proof. As discussed in 3.9, we may restrict ourselves to G-invariant mappings $f : \otimes^p V \to \mathbb{K}$ and we have constructed the polynomial $GL(m)$-invariant mapping $\varphi\colon GL(m)\times\otimes^p V\to\mathbb{K}$. For each $t\in\otimes^p V$, the restricted map $\varphi_t\colon GL(m)\to\mathbb{K}$ satisfies the assumptions of Lemma 3.10. Assume first $\mathbb{K}=\mathbb{C}$. Then we can extend the map φ_t to the whole space $\Lambda^m V^*$ and we obtain a homogeneous polynomial mapping $\varphi\colon \Lambda^m V^*\times\otimes^p V$ linear in the second entry. The total polarization of φ is a $GL(m)$-invariant mapping $S^k(\Lambda^m V^*)\otimes\otimes^p V\to\mathbb{C}$. Thus it results from finite iteration of the steps 3.6.(a)–(d). The original mapping is $f(t) = \varphi(\mathbb{I}_m, t) = \bar\varphi(\nu, t)$. Thus, taking into account $V = \Lambda^{m-1}V^*$, this proves the theorem in the complex case. Indeed, if $t\in\otimes^r V^*\otimes\otimes^s V$ is invariant, then t is viewed as an invariant linear mapping $\otimes^{r+(m-1)s}V\to\mathbb{C}$ and so $r + (m-1)s = km$, i.e. $r - s = (k-s)m$.

Consider now the real case and an $SL(m,\mathbb{R})$-invariant linear map $f\colon \otimes^p V\to\mathbb{R}$. Since the description of all $GL^+(m,\mathbb{R})$-invariant tensors coincides with that of $GL(m,\mathbb{R})$-invariant ones, we can repeat step by step the above proof on replacing $GL(m)$ by $GL^+(m,\mathbb{R})$.

If f is $G^{\pm}$-invariant, then we turn back to $GL(m,\mathbb{R})$ invariant φ_t, but the total polarization will happen to be a map $S^{2k}(\Lambda^m V^*)\otimes\otimes^p V\to\mathbb{C}$. Hence the number ν treats in f must be even.[9] $\square$

[9] The description can also be deduced by a standard trick which might be useful at another occasion as well: we could describe only the $G^{\pm}$-invariant tensors. On the space $\mathcal{I}$ of all $SL(m,\mathbb{R})$-invariant tensors, there is an action of $\mathbb{Z}_2 = G^{\pm}/SL(m,\mathbb{R})$ which splits $\mathcal{I}$ into the ± 1 eigen spaces $\mathcal{I}_+$, $\mathcal{I}_-$. Then notice, if $t\in\mathcal{I}_-$, then $\nu\otimes t\in\mathcal{I}_+$.

3.12. Relative invariant tensors. A tensor $t \in \otimes^p V^* \otimes \otimes^q V$ is called *relative $GL(m)$-invariant* if there is a $\chi\colon GL(m) \to \mathbb{K}$ with $A.t = \chi(A)t$ for all $A \in GL(m)$. Clearly χ must be a continuous character of $GL(m)$.

Theorem. *The relative invariant tensors are exactly the $SL(m)$-invariant tensors. In particular, if $t \in \otimes^p V^* \otimes \otimes^q V$ is relative $GL(m)$-invariant and non-zero, then $p - q = km$, $k \in \mathbb{Z}$, and the corresponding character χ is $(\det A)^k$.*

Proof. First, let us notice how easily we can find all continuous characters of $GL(m)$ using Theorem 3.5. Since χ must be a continuous Lie group homomorphism, it must be smooth. The corresponding Lie algebra homomorphism $\chi'\colon V^* \otimes V \to \mathbb{K}$ is Ad-invariant ($\chi' \circ \mathrm{Ad} = T_e\chi \circ T_e\mathrm{Conj} = T_e(\chi \circ \mathrm{Conj}) = \chi'$ since the action on $\mathbb{C}$ is trivial). Further, $\mathrm{Ad}(A)$ is exactly the standard representation of $GL(m)$ on $V^* \otimes V$. Thus, χ' is a scalar multiple of the trace and so $\chi|GL^+(A) = (\det A)^k$, $k \in \mathbb{R}$, in the real case while $\chi(A) = (\det A)^k$, $k \in \mathbb{C}$, in the complex case. This shows t is $SL(m)$-invariant and the description of all such t from Theorem 3.11 finishes the proof. $\square$

3.13. Irreducible representations. Later on we shall often treat only irreducible representations and we shall also need to pass from the real to the complex situation or back. The latter is usually denoted as the 'complexification' and 'realification'. Let us describe briefly the irreducible representations of $GL(m)$ and $SL(m)$, the (pseudo-)orthogonal groups will follow later, the details can be found e.g. in [Boerner, 67]. In the case $G = GL(m)$ this is a problem closed to 3.5. Indeed, as discussed in 3.7, each invariant subspace is an image of a G-equivariant projection and for contravariant tensors of degree r all such projections are obtained through actions of the permutation group $\mathcal{S}_r$. Let $\mathcal{D}_r$ be the group ring of $\mathcal{S}_r$ which acts obviously on the (contravariant) tensors. The idempotents $e \in \mathcal{D}_r$ which represent the irreducible representations $\mathcal{D}_r.e \subset \mathcal{D}_r$ of $\mathcal{S}_r$ (and these correspond to polynomial irreducible representations of $GL(m)$) are described with the help of the so called standard Young diagram. The latter is given by a system $n_1 \ge n_2 \ge \cdots \ge n_p > 0$ of natural numbers with $n_1 + \cdots + n_p = r$ which is graphically described by

$$\begin{array}{ll} n_1 & \square\square\square\square \\ n_2 & \square\square\square \\ \vdots & \\ n_p & \square \end{array}$$

with numbers $1, \dots, m$ written inside the individual boxes in such a way that they increase in the columns and do not decrease in the rows. Labeling the boxes by numbers $1, \dots, r$, such a diagram determines an element $e \in \mathcal{D}_r$ defined as the composition of the sum of all permutations indicated in the rows and the alternated sum of the permutations in the columns. Hence the image of the corresponding projection $\otimes^p V \to W$ is obtained by numbering the indices of the tensors and applying the corresponding symmetrizations and alternations. For example, given the diagram $\begin{array}{l}\square\square \\ \square\end{array}$ with $(1,2)$ in the row and $(1,3)$ in the column, we get the projection $\otimes^3 V \to W$, $t^{ijk} \mapsto (t^{ijk} + t^{jik}) \mapsto t^{ijk} + t^{jik} - t^{kji} - t^{jki}$. Notice, we permute the tensors according to the original numbering of indices, the permutation leading to t^{kij} in the last term lies in $S^2V \otimes V$ which is further reducible!

Each such diagram (without the numbers inside) with the number of rows less than or equal to m determines in this way an irreducible polynomial representation of $GL(m, \mathbb{C})$ with dimension equal to the number of the possible standard Young diagrams of the same shape. Two diagrams with different shapes correspond to inequivalent representations. We shall denote the representation corresponding to the diagram described by $n_1, \dots, n_p$ by the symbol $C^m_{(n_1,\dots,n_p)}$ or $C^m_{(n_1,\dots,n_m)}$ where we set $n_j = 0$ for all $j > p$. Let us remark that in view of Theorem 3.5 the proof of the irreducibility is a combinatorial problem in the representation theory of the symmetric group.[10] The representations $C^m_{(n_1,\dots,n_p)}$ exhaust all irreducible polynomial representations of $GL(m, \mathbb{C})$, see [Boerner, 67, Chapter V, section 5]. These representations remain irreducible if we restrict the group to $SL(m, \mathbb{C})$ but some of them coincide (notice that this follows from Theorem 3.11). On the other hand, they exhaust all rational representations of $SL(m, \mathbb{C})$ and each continuous representation of $SL(m, \mathbb{C})$ is rational and completely reducible, hence polynomial (remember $\mathbb{C}^{m*} = \Lambda^{m-1}\mathbb{C}^m$). This is not true for $GL(m, \mathbb{C})$ where only all rational representations are completely reducible and there are some reducible but not completely reducible ones[11]. All rational representation of $GL(m, \mathbb{C})$ are of the form $(\det A)^k C^m_{(n_1,\dots,n_p)}$, $k \in \mathbb{Z}$. For the proofs we refer to [Boerner, 67, Chapter V, section 8].

Given a real Lie group G and its linear action on a real vector space W, there is the induced action of G on $W \otimes \mathbb{C}$ and if the action on W is reducible, then also the action on $W \otimes \mathbb{C}$ is reducible. If G is one of the matrix groups discussed above or $O(m, \mathbb{R})$, $SO(m, \mathbb{R})$, $SO(m, n, \mathbb{R})$, then the latter action is extended to an action of the corresponding complex group. In [Boerner, 67, p.164] we find the following statement

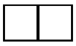
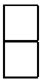
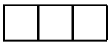
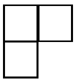
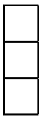
Theorem. *The irreducible rational representations of the groups $GL(m, \mathbb{C})$ remain irreducible if restricted to the subgroups $GL(m, \mathbb{R})$, $SL(m, \mathbb{C})$, $SL(m, \mathbb{R})$, $U(m)$, $SU(m)$.*[12]

---

[10] See [Boerner, 67, Chapter IV]). Roughly speaking, any further permutations can be built from those concerning only indices lying either in rows or in columns and so a composition with further permutations yields some 'conjugated elements'.

[11] As well known, a representation of a semisimple complex Lie group is completely reducible and a representation of a general complex Lie group is completely reducible if and only if its restriction to the radical is. In our case, the radical is the one-dimensional center of $GL(m, \mathbb{C})$, while the semisimple part is $SL(m, \mathbb{C})$.

[12] The proof is surprisingly elementary. Roughly speaking, if the restricted representation were reducible, then there would be a 'common null box' in all the matrices of the representation in a suitable basis. Hence there are non-zero linear forms on $\mathfrak{gl}(W)$ which annihilate all the matrices of the representation. The composition of these forms with the representation matrices yields rational functions on $GL(m)$ which are zero on all matrices from the subgroups in question. These subgroups have enough points to assure that these rational functions are zero identically and, thus, the original representation must be reducible. In the case of $SU(m)$, there is a similar trick available using the representation of the Lie algebra and for $U(m)$ we apply the results for $SU(m)$ and $SL(m, \mathbb{C})$. A similar reasoning will become an important step in our discussion on (pseudo-)orthogonal groups below.

3.14. Examples. The standard representation on $\mathbb{K}^m$ is the $C^m_{(1,0,\dots,0)}$ (there is only one index, no symmetry), $\mathbb{K}^*$ corresponds to $(\det A)^{-1}C^m_{(1,\dots,1,0)}$. The space $\otimes^2\mathbb{K}$ admits two indices and so its decomposition must correspond to the diagrams  and , i.e. the symmetrization and alternation. In the decomposition of $\otimes^3\mathbb{K}$, there appear only the diagrams ,  and . We should notice that we have not discussed at all the multiplicities of these representations!

3.15. $O(m)$-invariant tensors. We shall proceed in a way similar to 3.11. Recall $O(m) = \{A \in GL(m); A.g_0 = g_0\}$ where $g_0 \in S^2V^*$ is the canonical Euclidean metric (or its complex analog). Given g in the space $S^2_+V^*$ of positive definite non-degenerate 2-forms (non-degenerate in the complex case), this defines an isomorphism $g\colon V \to V^*$ and its inverse $\tilde{g}\colon V^* \to V$. Clearly g_0 and $\tilde{g}_0$ are $O(m)$-equivariant and therefore we do not have to consider both covariant and contravariant entries of the tensors. Thus, we have to describe all $O(m)$-invariant linear mappings $f\colon \otimes^p V \to \mathbb{K}$. Given such f we define $\bar{\varphi}\colon S^2_+V^* \times \otimes^p V \to \mathbb{K}$ by $\bar{\varphi}(A.g_0, t) := f(A.t)$. Since $GL(m).g_0 = S^2_+V^*$ and f is $O(m)$-invariant, $\bar{\varphi}$ is well defined and $GL(m)$-invariant. Similarly to 3.11, we need to extend $\bar{\varphi}$ to a polynomial $GL(m)$-invariant mapping on the whole space $S^2V^* \times \otimes^p V$. This will be possible using the next two lemmas which are interesting for themselves.[13]

Since we want to treat at the same time metrics with arbitrary signature (in the real case – in complex situation they are all equivalent), we need some more notation. We write $O(m,n) = \{A \in GL(m); AJA^T = J\}$ where $J = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$, so that the matrices from $O(m,n)$ preserve the canonical pseudo-metric of signature m on $\mathbb{K}^{m+n}$. This definition makes sense not only for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ but for any other extension $\mathbb{L}$ of $\mathbb{R}$ as well. Further, the Zariski connected components of $O(m,n,\mathbb{L})$ are always algebraic varieties in $\mathbb{L}^{(m+n)^2}$ and there is the canonical inclusion $O(m,n,\mathbb{K}) \subset O(m,n,\mathbb{L})$.

3.16. Lemma. *Let $\mathbb{L}$ be any algebraic extension of $\mathbb{R}$ and let $f\colon O(m,n,\mathbb{L}) \to \mathbb{L}$ be a rational function. If f vanishes on $O(m,n,\mathbb{R})$, then f is zero.*

Proof. We shall write $\mathfrak{o}(m,n) = \{A \in \mathfrak{gl}(m+n,\mathbb{R}); AJ + JA^T = 0\}$ for the real Lie algebra of the pseudo-orthogonal group. Let us consider the Caley map

$$C\colon \mathfrak{o}(m,n) \to GL(m), \quad C(S) = (1+S)(1-S)^{-1}$$

defined for all S with $\det(1-S) \neq 0$. This is injective and rational. Further we claim that the image lies in $O(m,n)$ and C admits a rational inverse $C^{-1}\colon O(m,n,\mathbb{R}) \to$

[13] The proofs of these lemmas (and also that one of the description of the invariant tensors) follow [Atiyah, Bott, Patodi, 73, p. 323], where the positive definite real case is presented.

$\mathfrak{o}(m,n)$. Indeed, we have $J = J^{-1}$, $J = J^T$, $SJ + JS^T = 0$ and so

$$\begin{aligned} C(S)J(C(S))^T &= (1+S)(1-S)^{-1}J(1-S^T)^{-1}(1+S^T) \\ &= (1+S)(J - JS - S^TJ + S^TJS)^{-1}(1+S^T) \\ &= (1+S)(J(1-S)(1+S))^{-1}(1+S^T) \\ &= (J-JS)^{-1}(1+S^T) = J(1+S^T)^{-1}(1+S^T) = J. \end{aligned}$$

Further, if $Z = (1+S)(1-S)^{-1}$ then $S = (Z-1)(1+Z)^{-1}$ whenever both expressions are defined and it remains only to verify $S = (Z-1)(1+Z)^{-1} \in \mathfrak{o}(m,n)$ if $ZJZ^T = J$. The latter means $(Z-1)(1+Z)^{-1}J + J(1+Z^T)^{-1}(Z^T-1) = 0$, but in order to see that the left hand side is zero we can multiply it by invertible matrices. Let us multiply by $(1+Z^T)J$ on the left and by $J(1+Z)$ on the right. This yields $(1+Z^T)J(Z-1) + (Z^T-1)J(1+Z)$ which is zero.

Hence we have proved: *the connected component of the unit in $O(m,n)$ is birationally isomorphic to the real affine space $\mathfrak{o}(m,n)$.*

Thus, if f vanishes at all real points, then the composition with this isomorphism is a zero rational map on an affine space and hence all coefficients of the representing polynomials vanish. This proves the lemma for the connected component of the unit.

It remains to know that $O(m,n,\mathbb{R})$ consists of four connected components determined by the signs of the two subdeterminants along the diagonal corresponding to the matrices $\mathbb{I}_m$ and $-\mathbb{I}_n$ in J, see [Boerner, 67, p.297]. Hence we can compose the mapping $C(S)$ with multiplication by one of the four matrices $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mathbb{I}_{m-1} \end{pmatrix}$, $B = \begin{pmatrix} \pm 1 & 0 \\ 0 & \mathbb{I}_{n-1} \end{pmatrix}$. This yields the result for all connected components of $O(m,n,\mathbb{R})$. $\square$

3.17. Lemma. *Let $h\colon GL(m+n) \to \mathbb{K}$ be a polynomial or rational $O(m,n)$-invariant mapping. Then there is a polynomial or rational mapping F defined on the space of all symmetric matrices such that $h(A) = F(A^TJA)$ for all $A \in GL(m)$, respectively.*

Proof. In dimension one, we deal with the well known assertion that each even polynomial h, i.e. $h(x) = h(-x)$, is a polynomial in x^2 and analogously for rational mappings. However in higher dimensions, the proof is quite non trivial.

We shall prove the polynomial case, the rational one follows by omitting some extensions. If we were in the real situation, then h extends to a complex valued function $h\colon GL(m,\mathbb{C}) \to \mathbb{C}$ which is $O(m,n,\mathbb{C})$-invariant by virtue of Lemma 3.16. Indeed, consider $\bar h_A\colon O(m,n,\mathbb{C}) \to \mathbb{C}$, $\bar h_A(B) := h(BA) - h(A)$. This is polynomial for each $A \in GL(m,\mathbb{C})$ and it vanishes on real matrices, hence also on the complex ones and this is the invariance we require. Thus we can restrict ourselves to the complex case.

First notice that if $A^TJA = P$ with P non singular and if there is a symmetric Q with $QJQ = P$, then A lies in the $O(m,n)$-orbit of Q. Indeed, Q is also non singular and $B = AQ^{-1}$ satisfies $B^TJB = Q^{-1}A^TJAQ^{-1} = Q^{-1}PQ^{-1} = J$. Each symmetric matrix P admits a symmetric square root in the complex domain. Let us construct this as follows. Take $P = B^T DB$ with D diagonal, $B^T JB = J$. Write $\sqrt{D}$ for the diagonal matrix consisting of square roots of the eigen values of D and take the matrix $D' = B^T \sqrt{J}\sqrt{D}B$. This satisfies ${D'}^T JD' = B^T DB = P$ and is well defined. So it suffices to restrict ourselves to symmetric matrices.

Finally, since $O(m, n, \mathbb{C})$ is isomorphic to $O(m + n, \mathbb{C})$, it is sufficient to restrict ourselves to $O(m, \mathbb{C})$. Notice, the isomorphism is induced by constant multiplication of first m coordinates in $\mathbb{C}$ by i. Hence the corresponding isomorphism on non-degenerate symmetric matrices is $A^T A \mapsto A^T JA$ which is well defined, see above. Hence we want to find a polynomial map g satisfying $h(Q) = g(Q^2)$ for all symmetric matrices.

As already mentioned, there is the square root $\sqrt{P} = Q$ for each symmetric $P = B^T DB$, $\sqrt{P} = B^T\sqrt{D}B$. But we should express Q as a universal polynomial in the elements p_{ij} of the matrix P. If all eigen values λ_i of P are different, then we can write

$$Q = \sum_{i=1}^{m} \sqrt{\lambda_i} \prod_{j\neq i} \frac{P - \lambda_j}{\lambda_i - \lambda_j}.$$

In order to make this to a polynomial expression, we have first to extend the field of complex numbers to the field $\mathbb{L}$ of rational functions (i.e. the elements are ratios of polynomials in p_{ij}'s). So for matrices with entries from $\mathbb{L}$, all eigen values depend polynomially on p_{ij}'s. We also need their square roots to express Q, but we shall see that after inserting $Q = \sqrt{P}$ into $h(Q)$ all square roots will factor out. For any fixed P, let us consider the splitting field $\tilde{\mathbb{L}}$ over $\mathbb{L}$ with respect to the roots of the equation $\det(P - \lambda^2) = 0$. So $\sqrt{P}$ is polynomial over $\tilde{\mathbb{L}}$. Now the basic fact is, that for any automorphism $\sigma\colon \tilde{\mathbb{L}} \to \tilde{\mathbb{L}}$ from the Galois group of $\tilde{\mathbb{L}}$ over $\mathbb{L}$ we have $(\sigma Q)^2 = \sigma P = P$ and since both Q and σQ are symmetric, $B = \sigma Q Q^{-1}$ is orthogonal.

Using Lemma 3.16, we get $\sigma h(Q) = h(\sigma Q) = h(BQ) = h(Q)$. Since this holds for all σ, $h(Q)$ lies in $\mathbb{L}$ and so $h(Q) = g(Q^2)$ for a rational function g.

The latter equality remains true if $P = Q^2$ is a real or complex symmetric matrix such that all its eigen values are distinct and the denominator of $g(P)$ is non zero. If $g = F/G$ for two polynomials F and G, we get $F(A^T A) = h(A)G(A^T A)$. If we choose A so that $G(A^T A) = 0$ and h is a polynomial, we get $F(A^T A) = 0$. Hence if h is polynomial, then g is a globally defined rational function without poles and so a polynomial.

Thus, we have found a rational function (a polynomial in the polynomial case) F on the space of symmetric matrices such that $h(A) = F(A^T A)$ holds for a Zariski open set in $\mathfrak{gl}(m)$. □

3.18. Theorem[14]. *All $O(m, n)$-equivariant mappings between invariant subspaces of tensor spaces are constructed by a finite iteration of steps 3.6.(a)–(d) and*

(i) *the tensors g_0 and $\tilde{g}_0$ are invariant.*

Proof. Let us continue the discussion from 3.15 and denote for a moment $S^2_+ V^*$, $V = \mathbb{K}^{m+n}$ the space of metrics of some fixed signature m. Thus, we want to

[14] This is the famous Weyl's theorem, [Weyl, 39]

describe all $O(m,n)$-invariant linear maps $f\colon \otimes^p V \to \mathbb{K}$. It suffices to prove that all such maps are complete contractions over permuted indices (this means there is an even number of indices there and we choose a half of them, shift them to the other position using g_0 and then apply some complete contraction). If we are in the real situation, then f extends to the complexified spaces and becomes $O(m,n,\mathbb{C})$-invariant, cf. Lemma 3.16. Hence we shall restrict ourselves to $\mathbb{K}=\mathbb{C}$. (We could also stick to $O(m)$, for all signatures are equivalent now).

The mapping f defines an $O(m,n)$-invariant mapping $\varphi\colon GL(m+n)\times\otimes^p V\to\mathbb{C}$, $\varphi(A,t)=f(A.t)$. By Lemma 3.17, every restricted map $\varphi_t\colon GL(m+n)\to\mathbb{C}$ satisfies $\varphi_t(A)=h_t(A^TJA)$ for certain polynomial h_t and so we get a polynomial mapping $h\colon S^2V^*\times\otimes^pV\to\mathbb{C}$ linear in the second entry. For all $B,A\in GL(m+n)$ we have $h((B^{-1})^TA^TJAB^{-1},B.t)=f(AB^{-1}B.t)=f(A.t)=h(A^TJA,t)$ and so $h\colon S^2V^*\times\otimes^pV\to\mathbb{K}$ is $GL(m+n)$-invariant. Then the composition of h with the symmetrization yields a polynomial $GL(m+n)$-invariant map $\otimes^2V^*\times\otimes^pV\to\mathbb{C}$, linear in the second entry. Each homogeneous component of degree $s+1$ is also $GL(m)$-invariant and so its total polarization is a linear $GL(m)$-invariant map $\Phi\colon\otimes^{2s}V^*\otimes\otimes^pV\to\mathbb{C}$. Hence, by Theorem 3.5, $p=2s$ and Φ is a sum of complete contractions over possible permutations of indices. Since the original mapping f is given by $f(t)=h(J,t)$, the Weyl's theorem follows. □

3.19. Special (pseudo-) orthogonal group. This is the case we shall be most interested in later on.

Theorem. *All $SO(m,n)$-equivariant linear mappings between $SO(m,n)$-invariant subspaces in tensor spaces are obtained through a finite iteration of steps 3.6.(a)–(d) and*

(i) *$g_0\in V^*\otimes V^*$ and $\tilde g_0\in V\otimes V$ are invariant (the pseudo-metric and its inverse)*
(ii) *$\nu\in\Lambda^mV^*$ is invariant (the canonical volume form).*

Proof. The theorem follows from 3.18 by means of the trick mentioned as a footnote in 3.11. Indeed, the $SO(m,n)$-invariant tensors split into the ±1-eigenspaces for the induced action of $\mathbb{Z}_2=O(m,n)/SO(m,n)$ and once a tensor appears in the -1-eigenspace, its tensor product with ν belongs to the other one, i.e. it is $O(m,n)$-invariant. Since the canonical volume element has components $\nu_{\underline{i}_1\cdots\underline{i}_{m+n}}=((-1)^n\det(g_{\underline{i}\underline{j}}))^{1/2}\varepsilon_{\underline{i}_1\cdots\underline{i}_{m+n}}$ where $\varepsilon_{\underline{i}_1\cdots\underline{i}_{m+n}}$ are the components of the Levi-Cività tensor, this proves the theorem. □

3.20. Remark. In dimension $m=1$, every polynomial can be expressed as a sum of an odd polynomial and an even one. We generalized the description of the even polynomials in Lemma 3.16, but there is also an analogy to the above splitting of polynomials:

Lemma. *Let $h\colon GL(m+n)\to\mathbb{K}$ be a rational or polynomial $SO(m,n)$-invariant mapping, then there are rational or polynomial mappings F, G defined on the space of all symmetric matrices such that $h(A)=F(A^TJA)+(\det A)^{-1}G(A^TJA)$ for all $A\in GL(m)$, respectively.*

Proof. Let $\mathcal{I}$ be the space of all $SO(m,n)$-invariant rational functions $h\colon GL(m+n)\to\mathbb{K}$. There is the action of $\mathbb{Z}^2=O(m,n)/SO(m,n)$ on $\mathcal{I}$. Hence $\mathcal{I}$ is splitted into the eigen spaces $\mathcal{I}_+$ and $\mathcal{I}_-$. If $h_- \in \mathcal{I}_-$, then the map $\tilde{h}_-$ given by $A \mapsto (\det A) h_-(A)$ lies in $\mathcal{I}_+$, i.e. is $O(m,n)$-invariant. Now we can split h as a sum of elements from $\mathcal{I}_\pm$, $h = h_+ + h_-$, and apply Lemma 3.15 to both h_+ and $\tilde{h}_-$. Consequently, $h_-(A) = (\det A)^{-1}\tilde{h}_-$ has the desired form. The polynomial case is completely analogous. $\square$

As a consequence of this lemma, we can identify the ring of rational functions on the 'space of all (pseudo-) metrics' $GL(m+n)/O(m,n)$ with the ring $\mathbb{K}[g_{ij}, (\det g_{ij})^{-\frac{1}{2}}]$ (notice, the metric corresponding to $[A] \in GL(m+n)/O(m,n)$ is $(AA^T, \operatorname{sign}(\det A))$).

Let us also remark, the analogous statements to Lemmas 3.17 and 3.20 are available for the right actions of the orthogonal subgroups. Indeed, we have only to consider $g(A) = h(A^{-1})$, to apply the lemmas and then to notice that in the polynomial case we get polynomials.

We shall end this section with an analytical proposition which is often useful to avoid the polynomiality assumption, i.e. to describe all smooth equivariant mappings.

Consider a product $V_1 \times \ldots \times V_n$ of finite dimensional vector spaces and write $x_i \in V_i$, $i = 1, \ldots, n$.

3.21 Proposition[15]**.** *Let $f\colon V_1 \times \ldots \times V_n \to \mathbb{K}$ be a smooth function and let $a_i > 0$, b be real numbers such that*

$$k^b f(x_1, \ldots, x_n) = f(k^{a_1} x_1, \ldots, k^{a_n} x_n) \tag{1}$$

holds for every real number $k > 0$. Then f is a sum of homogeneous polynomials of degrees d_i in x_i satisfying the relation

$$a_1 d_1 + \cdots + a_n d_n = b. \tag{2}$$

If there are no non-negative integers $d_1, \ldots, d_n$ with the property (2), then f is the zero function.

Proof. Assume first $b < 0$. If there were $f(x_1, \ldots, x_n) \neq 0$, then the limit of the right-hand side of (1) for $k \to 0_+$ would be $f(0, \ldots, 0)$, while the limit of the left-hand side would be improper. Hence f is zero identically.

In the case $b \geq 0$ we write $a = \min(a_1, \ldots, a_n)$ and $r = \left[\frac{b}{a}\right]$(=the integer part of the ratio $\frac{b}{a}$). We claim that all partial derivatives of the order $r+1$ of every function f satisfying (1) vanish identically. Differentiating (1) with respect to x^j, we obtain

$$k^b \frac{\partial f(x_1, \ldots, x_n)}{\partial x^j} = k^{a_i} \frac{\partial f(k^{a_1} x_1, \ldots, k^{a_n} x_n)}{\partial x^j}.$$

Hence for $\frac{\partial f}{\partial x^j}$ we have (1) with b replaced by $b - a_i$. This implies that every partial derivative of the order $r+1$ of f satisfies (1) with a negative exponent on the left-hand side, so that it is the zero function by the first part of the proof.

[15] The so called Homogeneous function theorem, cf. [Kolář, Michor, Slovák, 93, Theorem 24.1].

Since all the partial derivatives of f of order $r+1$ vanish identically, the remainder in the r-th order Taylor expansion of f at the origin vanishes identically as well, so that f is a polynomial of order at most r. For every monomial $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ of degree $|\alpha_i|$ in x_i, we have

$$(k^{a_1} x_1)^{\alpha_1} \dots (k^{a_n} x_n)^{\alpha_n} = k^{a_1|\alpha_1| + \dots + a_n|\alpha_n|} x_1^{\alpha_1} \dots x_n^{\alpha_n}.$$

Since k is an arbitrary positive real number, a non-zero polynomial satisfies (1) if and only if (2) holds. □

Let us remark that the assumption $a_i > 0$, $i = 1, \dots, n$, is essential. For example, all smooth functions $f(x, y)$ of two independent variables satisfying $f(kx, k^{-1}y) = f(x, y)$ for all $k \neq 0$ are of the form $\varphi(xy)$, where $\varphi(t)$ is any smooth function of one variable. In this case we have $a_1 = 1$, $a_2 = -1$, $b = 0$.

4. Operators on (pseudo-) Riemannian manifolds

4.1. Our next problem is: Let us consider two representations λ_F, λ_E of $G_m^1 = GL(m, \mathbb{R})$ in Diff$S$, DiffS' and the corresponding bundle functors F and E, see 2.12. We shall consider them as functors on the category $\mathcal{M}f_m(O(m', n, \mathbb{R}))$ of $(m' + n)$-dimensional pseudo-Riemannian manifolds with signature m', n and local isometries. Find all natural operators $D \colon F \to E$ on the category $\mathcal{M}f_m(O(m', n, \mathbb{R}))$!

The most common examples for the functors are the identity action on $\mathbb{R}^m$ (corresponds to the tangent functor T), its contragredient action on $\mathbb{R}^{m*}$ (yields T^*) and their tensor products. We shall denote $T^{(q,p)}$ the natural bundle of p-times covariant and q-times contravariant tensors. Hence $C^\infty(T^{(q,p)}M)$ are local tensor fields on the manifold M. In particular, we shall study in detail the operations on exterior forms.

It does not seem to be satisfactory that we restrict ourselves to bundle functors on the whole category $\mathcal{M}f_{m'+n}$. But this has two good reasons: all (univalued) linear representations of $O(m', n)$ are invariant subspaces of some tensor spaces (with the restricted usual action), see the Appendix, and dealing with the whole tensor spaces we can add the metrics themselves to the arguments of the operators as discussed in 2.17. So we shall deal with natural operators $S^2_+ T^* \times F \to E$ where $S^2_+ T^*$ stands for the bundle functor of pseudo-Riemannian metrics with some fixed signature (m', n) and the cross denotes the product in the category of bundle functors and their natural transformations (i.e. the values are the fibred products over the base manifolds). The only disadvantage, namely we cannot treat directly the $O(m', n)$-invariant subspaces is not serious, see 2.17 and 3.7.

A more detailed explanation of the technical tools sketched below in 4.2 – 4.9 can be found in [Kolář, Michor, Slovák, 93, Sections 28, 33], the exposition follows [Slovák, 92a], [Slovák, 92b].

4.2. The orbit reduction. In our situation, all the manifolds are locally isomorphic to $\mathbb{R}^{m'+n}$ and the action of Diff$(\mathbb{R}^{m'+n})$ is transitive. Let us assume first that the operators are of finite order k and so we can use 2.6 and the whole classification problem reduces to the finding of all G_m^{k+1}-equivariant mappings

$f\colon \mathcal{A} := J_0^k(S_+^2\mathbb{R}^{m*} \times \otimes^p\mathbb{R}^{m*} \otimes \otimes^q\mathbb{R}^m) \to \otimes^r\mathbb{R}^{m*} \otimes \otimes^s\mathbb{R}^m$ (the latter mappings correspond to natural operators $T^{(q,p)} \to T^{(s,r)}$ on pseudo-Riemannian manifolds which are of order k in both the tensors and the metrics). We see immediately that on the target of the equivariant mapping f, the action is of order one, i.e. the whole kernel $K = \ker\pi_1^{k+1}$ in G_m^{k+1} acts trivially. This shows that f must be constant on the orbits of K in the domain. If we succeed in the description of the corresponding orbit space $\mathcal{A}/K$ with the canonical action of G_m^1, then all our equivariant mappings f with values in some G_m^1-space $\mathcal{Z}$ will factor through G_m^1-equivariant mappings $g\colon \mathcal{A}/K \to \mathcal{Z}$.

First we shall present such a procedure for operations depending on connections. Since there is the canonical Levi-Cività connection on each pseudo-Riemannian manifold this will be helpful even in the Riemannian case.

There is a simple criterion for such descriptions: Let $\varphi\colon G \to H$ be a Lie group homomorphism with kernel K, M be a G-space, Q be an H-space and let $p\colon M \to Q$ be a φ-equivariant surjective submersion, i.e. $p(gx) = \varphi(g)p(x)$ for all $x \in M$, $g \in G$. We can consider every H-space N as a G-space via φ.

Lemma. *If each $p^{-1}(q)$, $q \in Q$, is a K-orbit in M, then $Q = M/K$. Consequently, there is a bijection between the G-maps $f\colon M \to N$ and the H-maps $g\colon Q \to N$ given by $f = g \circ p$.* $\square$

4.3. Operations on manifolds with connection. The linear connections on m-dimensional manifolds are sections of the natural bundle $QP^1 = J^1P^1/G_m^1$. This expresses the definition of principal connections as right invariant horizontal distributions. The bundle of symmetric connections (i.e. without torsion) will be denoted by $Q_\tau P^1$.

A classical observation due probably to Veblen or Schouten claims that the natural operators of order k on tensor fields depending on connections factorize through the covariant derivatives of the arguments up to the order k and through the curvature and its covariant derivatives up to the order $k-1$. Several authors derived more precise formulations involving some further assumptions, see e.g. [Lubczonok, 72], [Atiyah, Bott, Patodi, 73], [Epstein, 75], [Krupka, Janyška, 90]. A (rather technical) verification of such reduction without any additional assumption is presented in the framework of natural operators by Kolář in [Kolář, Michor, Slovák, 93, Section 28]. The proof is based on the above orbit reduction principle. On the set-theoretical level, this is a more or less classical technical computation, but the subtle point is the smoothness.

Let F be a first order bundle functor on $\mathcal{M}f_m$, E be an open natural subbundle of a natural vector bundle $\bar E$ on $\mathcal{M}f_m$. The curvature and its covariant derivatives are natural operators $\rho_k : Q_\tau P^1 \to R_k$, with values in tensor bundles R_k, $R_k\mathbb{R}^m = \mathbb{R}^m \times W_k$, $W_0 = \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \Lambda^2\mathbb{R}^{m*}$, $W_{k+1} = W_k \otimes \mathbb{R}^{m*}$. Similarly, the covariant differentiation of sections of E forms natural operators $d_k : Q_\tau P^1 \times E \to E_k$, where $E_0 = \bar E$, $E_0\mathbb{R}^m = \mathbb{R}^m \times V_0$, d_0 is the inclusion, $E_k\mathbb{R}^m = \mathbb{R}^m \times V_k$, $V_{k+1} = V_k \otimes \mathbb{R}^{m*}$. Let us write $D^k = (\rho_0, \dots, \rho_{k-2}, d_0, \dots, d_k)\colon Q_\tau P^1 \times E \to R^{k-2} \times E^k$, where $R^l = R_0 \times \ldots \times R_l$, $E^l = E_0 \times \ldots \times E_l$. All D^k are natural operators.

In view of the lemma above, the next assertion shows that there are bundle functors $Z^k(E)$ such that all k-th order natural operators $Q_\tau P^1 \times E \to F$ factor through natural transformations $Z^k \to F$.

Lemma. *There are sub bundle functors $Z^k \subset R^{k-2} \times E^k$ such that $D^k : Q_\tau P^1 \times E \to Z^k$ and the associated maps $\mathcal{D}^k : J_0^{k-1}(Q_\tau P^1\mathbb{R}^m) \times J_0^k(E\mathbb{R}^m) \to Z_0^k\mathbb{R}^m$ are surjective submersions for all k. Furthermore, for each point $z \in Z_0^k\mathbb{R}^m$ the preimage $(\mathcal{D}^k)^{-1}(z)$ forms one orbit under the action of the kernel B_1^{k+1} of the projection $\pi_1^{k+1} : G_m^{k+1} \to G_m^1$.*

4.4. Proposition. *For every natural operator $D: Q_\tau P^1 \times E \to F$ which depends on k-jets of sections of the bundles EM and on $(k-1)$-jets of the connections, there is a unique natural transformation (i.e. a zero order natural operator) $\tilde{D}: Z^k \to F$ such that $D = \tilde{D} \circ D^k$.*

Furthermore, D is polynomial if and only if $\tilde{D}$ is polynomial, and D is polynomial in all variables except those from V_0 with smooth real functions on V_0 as coefficients if and only if $\tilde{D}$ is polynomial with smooth real functions on V_0 as coefficients.

Proof. We have only to prove to polynomiality.

Let us write S_i for the tensor space $\mathbb{R}^m \otimes S^{i+2}\mathbb{R}^{m*}$, $Q = S_0$ for the standard fiber of the bundle of symmetric connections and

$$\mathcal{S}: J_0^{k-1}(\mathbb{R}^m, Q) = J_0^{k-1}(Q_\tau P^1\mathbb{R}^m) \to S_0 \times \ldots \times S_{k-1}$$

be the 'symmetrization of the derivatives of the Christoffel symbols' (i.e. we express the jet space $J_0^{k-1}(Q_\tau P^1\mathbb{R}^m)$ as the sum of the tensor spaces corresponding to the individual degrees of derivatives and apply the symmetrization to the individual summands). A more or less classical construction in local coordinates leads to a polynomial mapping

$$\psi: W_0 \times \ldots \times W_{k-2} \times V_0 \times \ldots \times V_k \times (S_0 \times \ldots \times S_{k-1}) \to J_0^{k-1}(\mathbb{R}^m, Q) \times J_0^k(\mathbb{R}^m, V)$$

such that $\psi \circ (\mathcal{D}^k \times \mathcal{S})$ is the identity on $J_0^{k-1}(\mathbb{R}^m, Q) \times J_0^k(\mathbb{R}^m, V)$.

Since the standard fiber V_0 of the bundle $E_0\mathbb{R}^m$ is embedded identically into $Z^k(E)_0\mathbb{R}^m$ by D^k, we get also the last statement. □

4.5. The finiteness of the order. Even if we have no estimate on the order, we can get an analogous result. The way is paved by the non-linear version of the Peetre theorem proved in [Slovák, 88]. The general result is rather technical and so we formulate a special case which we shall need.

Proposition. *Let $Y \to M$ and $Y' \to M$ be fibered manifolds and let $D: C^\infty(Y) \to C^\infty(Y')$ be a smooth local operator. Then for every fixed section $s \in C^\infty(Y)$ and for every compact set $K \subset M$, there is an order $r \in \mathbb{N}$ and a neighborhood V of s in the compact open C^∞-topology such that for every $x \in K$ and s_1, $s_2 \in V$ the condition $j_x^r s_1 = j_x^r s_2$ implies $Ds_1(x) = Ds_2(x)$.*

As a direct consequence of this result, we see that each natural operator $D: F \to E$ is of order $k = \infty$ and so D is determined by the associated G_m^∞-equivariant map $\mathcal{D}: J_0^\infty(F\mathbb{R}^m) \to E_0\mathbb{R}^m$.

Let us remark that a stronger version of the above proposition (without the smoothness assumption) is also proved in [Kolář, Michor, Slovák, 93, Theorem 19.7] and it is applied there in an alternative proof of the regularity and the finiteness of the order of bundle functors which avoids the original manipulation with infinite dimensional Lie groups G_m^∞, cf. [Epstein, Thurston, 79].

4.6. Lemma. *Let $F\colon \mathcal{M}f_m \to \mathcal{FM}$ be an arbitrary bundle functor and $p > q$ be non-negative integers. Then every natural operator $D\colon Q_\tau P^1 \times T^{(q,p)} \to F$ has finite order.*

Proof. Let us write $E = Q_\tau P^1 \times T^{(q,p)}$. By 4.5, D is determined by the associated map $\mathcal{D}\colon J_0^\infty(E\mathbb{R}^m) \to F_0\mathbb{R}^m$ induced by $D_{\mathbb{R}^m}$. Furthermore, for every jet $j_0^\infty s \in J_0^\infty(E\mathbb{R}^m)$ there is an order $r < \infty$, a neighborhood U_r of $j_0^r s$ in $J_0^r(E\mathbb{R}^m)$ and a smooth mapping $\mathcal{D}_r\colon U_r \subset J_0^r(E\mathbb{R}^m) \to F_0\mathbb{R}^m$ such that for all $j_0^\infty q \in V_r := (\pi_r^\infty)^{-1}U_r$ we have $\mathcal{D}(j_0^\infty q) = \mathcal{D}_r(j_0^r q)$. The naturality of D implies that if the open neighborhood U_r is the maximal one with this property, then V_r is G_m^∞-invariant. The induced action of G_m^1 turns $J_0^k(E\mathbb{R}^m)$ into a sum of G_m^1-invariant linear subspaces in the tensor spaces $(\mathbb{R}^m \otimes \otimes^{\ell+2}\mathbb{R}^{m*}) \oplus (\otimes^s \mathbb{R}^m \otimes \otimes^{r+\ell}\mathbb{R}^{m*})$, $\ell \le k$. Since $r > s$, the action of the homotheties (i.e. the center) in G_m^1 shows, that the orbit of any neighborhood of the jet $j_0^k 0$ of the zero section under the action of G_m^∞ coincides with the whole space $J_0^k(E\mathbb{R}^m)$. $\square$

4.7. Now, we come back to our natural operators $Q_\tau P^1 \times E \to F$ without any assumption on the order. Proceeding as in the proof of 4.6, we obtain an open filtration of the whole fiber $J_0^\infty((Q_\tau P^1 \times E)\mathbb{R}^m)$ consisting of maximal G_m^∞-invariant open subsets V_k where the associated mapping $\mathcal{D}$ factorizes through $\mathcal{D}_k\colon \pi_k^\infty(V_k) \subset J_0^k((Q_\tau P^1 \times E)\mathbb{R}^m) \to F_0\mathbb{R}^m$. Now, we can apply the same procedure as in the finite order case to this invariant open submanifolds $\pi_k^\infty(V_k)$.

Let us define the functor Z^∞ as the inverse limit of Z^k, $k \in \mathbb{N}$, with respect to the obvious natural transformations (projections) $\pi_\ell^k\colon Z^k \to Z^\ell$, $k > \ell$, and similarly $D^\infty\colon Q_\tau P^1 \times E \to Z^\infty$.

Theorem. *For every natural operator $D\colon Q_\tau P^1 \times E \to F$ there is a unique natural transformation $\tilde{D}\colon Z^\infty \to F$ such that $D = \tilde{D} \circ D^\infty$. Furthermore, for every m-dimensional compact manifold M and every section $s \in C^\infty(Q_\tau P^1 M \times_M EM)$, there is a finite order k and a neighborhood V of s in the C^k-topology such that*

$$\tilde{D}_M|(D^\infty)_M(V) = (\pi_k^\infty)^*(\tilde{D}_k)_M, \qquad \text{for some } (\tilde{D}_k)_M\colon (D^k)_M(V) \to C^\infty(Z^k M)$$
$$D_M|V = (\tilde{D}_k)_M \circ (D^k)_M|V.$$

In words, a natural operator $D\colon Q_\tau \times E \to F$ is determined in all coordinate charts of an arbitrary m-dimensional manifold M by a universal smooth mapping defined on the curvatures and all their covariant derivatives and on the sections of EM and all their covariant derivatives, which depends 'locally' only on finite number of these arguments.

4.8. The pseudo-Riemannian case. Let us write $S^2_+T^*$ for the bundle functor of pseudo-Riemannian metrics with some fixed signature on m-dimensional manifolds.

On pseudo-Riemannian manifolds, there is the natural operator $\Gamma\colon S^2_+T^* \to Q_\tau P^1$ defined by the Levi-Cività connection. Every operator $S^2_+T^* \times E \to F$ can be viewed as an operator $Q_\tau P^1 \times S^2_+T^* \times E \to F$, independent of the first argument. Since $S^2_+T^* \subset S^2T^*$ is an open sub bundle functor, we can consider the compositions $D^k \circ (\Gamma, \mathrm{id})\colon S^2_+T^* \times E \to Q_\tau P^1 \times S^2_+T^* \times E \to R^{k-2} \times (S^2_+T^* \times E)^k$ and apply Proposition 4.4. Since all covariant derivatives of the metric with respect to the metric connection are zero (the parallel transport consists of isometries), the covariant derivatives of the metric will not appear in the codomain of the operators D^k after the composition. Hence we get

Proposition. *There are sub bundle functors $Z^k \subset R^{k-2} \times E^k$ such that $D^k \circ (\Gamma, \mathrm{id})\colon S^2_+T^* \times E \to S^2_+T^* \times Z^k$ and the associated mappings $\mathcal{D}^k$ on the jet spaces are surjective submersions with the preimages $(\mathcal{D}^k)^{-1}(z)$ forming one orbit under the action of the kernel B_1^{k+1} of the projection $\pi_1^{k+1}\colon G_m^{k+1} \to G_m^1$. Hence for all k, and for every k-th order natural operator $D\colon S^2_+T^* \times E \to F$, there is a natural transformation $\tilde{D}\colon S^2_+T^* \times Z^k \to F$ such that $D = \tilde{D} \circ D^k \circ (\Gamma \times \mathrm{id})$.*

For the proof see [Slovak, 92a] or [Kolář, Michor, Slovák, 93, Section 33]. Let us notice that the bundles Z^kM involve the curvature of the Riemannian connection on M, its covariant derivatives, and the covariant derivatives of the sections of EM. Similarly as above, we define the inverse limits Z^∞ and D^∞ and we get

Corollary. *For every natural operator $D\colon S^2_+T^* \times E \to F$ there is a natural transformation $\tilde{D}\colon S^2_+T^* \times Z^\infty \to F$ such that $D = \tilde{D} \circ D^\infty \circ (\Gamma, \mathrm{id})$. Furthermore, for every m-dimensional compact manifold M and every section $s \in C^\infty(S^2_+T^*M \times_M EM)$, there is a finite order k and a neighborhood V of s in the C^k-topology such that*

$$\begin{gathered}\tilde{D}_M|(D^\infty \circ (\Gamma, \mathrm{id}))_M(V) = (\pi_k^\infty)^*(\tilde{D}_k)_M, \\ \textit{where} \quad (\tilde{D}_k)_M\colon (D^k \circ (\Gamma, \mathrm{id}))_M(V) \to C^\infty(Z^kM) \\ D_M|V = (\tilde{D}_k)_M \circ (D^k)_M \circ (\Gamma, \mathrm{id})_M|V.\end{gathered}$$

4.9. The polynomial operations. We call a natural operator $D\colon S^2_+T^* \times E \to F$ a *polynomial operator on (pseudo-) Riemannian manifolds* if the associated map $\mathcal{D}\colon J_0^\infty(S^2_+\mathbb{R}^m) \times J_0^\infty(E\mathbb{R}^m) \to F_0\mathbb{R}^m$ depends polynomially on k-jets of sections of $E\mathbb{R}^m$ for some k.

By the nonlinear Peetre theorem, this means that for each Riemannian manifold (M, g) the operator D_M is given by a universal polynomial expression depending on the derivatives of the sections of EM but the coefficients are functions depending on (locally finitely many) derivatives of the metric.

Let us consider now a k-th order operator D and the natural transformation $\tilde{D}$ corresponding to D, see 4.8. In the center of normal coordinates, each metric has the canonical pseudo-Euclidean form g_0 and so the whole transformation $\tilde{D}$ is determined by the restriction of the associated map $\tilde{\mathcal{D}}$ to $\{g_0\} \times Z_0^k\mathbb{R}^m$. This restriction is polynomial if and only if $\tilde{\mathcal{D}}$ depends polynomially on elements from $Z_0^k\mathbb{R}^m$, the metric g_{ij} and the square root of the inverse of its determinant $\det(g_{ij})$. Indeed, in order to find the transformation of coordinates which maps the canonical pseudo-euclidean metric to g_{ij} we need to decompose $g_{ij} = AJA^T$ with $A \in GL(m, \mathbb{R})$, cf. 3.17. The same applies to $\mathcal{D}$: if this G_m^∞-equivariant map depends polynomially on the derivatives of the metric and the jets of sections of $E\mathbb{R}^m$, then the values of the metric appear in $\mathcal{D}$ polynomially through g_{ij} and the square root of the inverse of its determinant $\det(g_{ij})$.

Now, let us fix g_{ij}. Since Γ depends polynomially on the 1-jet of the metric and the values of the inverse metric, it follows that $\tilde{\mathcal{D}}$ *depends polynomially on the elements from* $Z_0^k\mathbb{R}^m$ *if and only if* $\mathcal{D}$ *depends polynomially on the derivatives of the metric* g_{ij} *and on the jets of the sections of* E (with functions of g_{ij} as coefficients), and this happens if and only if $\mathcal{D}$ depends polynomially on the jets of the metrics, the jets of the sections of E and on the square root of the inverse of the determinant of (g_{ij}).

Let us remark that such operations were introduced in [Atiyah, Bott, Patodi, 73] under the name *regular operators*, a reason why they should be distinguished can be also found in 3.20.

4.10. Before studying the (pseudo-) Riemannian case, we shall treat the operations depending on connections. On the way we shall prepare all necessary tools for solving our initial problem .

Let us first discuss the natural operators $D\colon Q_\tau P^1 \times T^{(s,r)} \to T^{(q,p)}$ with $r > s$.

Proposition. *All natural operators* $Q_\tau P^1 \times T^{(s,r)} \to T^{(q,p)}$ *are obtained by a finite iteration of the following steps:*

(a) *the tensor field and its covariant derivatives with respect to the connection are invariant*
(b) *the curvature of the connection and its covariant derivatives are invariant*
(c) *tensor multiplication is invariant*
(d) $GL(m,\mathbb{R})$*-equivariant operations on the tensors determine invariant operations (i.e. trace, permutations of indices)*
(e) *linear combinations (over* $\mathbb{R}$*) of invariant operators are invariant*

In particular, they are all polynomial.

Proof. By 4.6, every such operator has some finite order k and so it is determined by a smooth G_m^{k+2}-equivariant map $f = (f^{i_1\dots i_q}_{j_1\dots j_p})\colon J_0^k(\mathbb{R}^m, Q) \times J_0^k(\mathbb{R}^m, V) \to S$, where Q is the standard fiber of the connection bundle, $V = \otimes^s\mathbb{R}^m \otimes \otimes^r\mathbb{R}^{m*}$ and $S = \otimes^q\mathbb{R}^m \otimes \otimes^p\mathbb{R}^{m*}$. Let us assume, we have chosen k in such a way that f depends on $(k-1)$-jets of the connections only. If we apply the equivariance of f with respect to the transformation $x \mapsto c^{-1}x$, $c \in \mathbb{R}$ positive, from the center of G_m^1, we get

$$\begin{aligned} c^{p-q} f^{i_1\dots i_q}_{j_1\dots j_p}(\Gamma^\ell_{ij}, \dots, \Gamma^\ell_{ij,\ell_1\dots\ell_{k-1}}, v^{i_1\dots i_s}_{j_1\dots j_r}, \dots, v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}) = \\ = f^{i_1\dots i_q}_{j_1\dots j_p}(c\Gamma^\ell_{ij}, \dots, c^k\Gamma^\ell_{ij,\ell_1\dots\ell_{k-1}}, c^{r-s}v^{i_1\dots i_s}_{j_1\dots j_r}, \dots, c^{r-s+k}v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}) \end{aligned}$$

where the subscripts ℓ_j denote the usual derivatives. By 3.21 $f^{i_1\dots i_q}_{j_1\dots j_p}$ must be sums of homogeneous polynomials.

Now, 4.4 and 2.6 imply that there is a unique smooth G_m^1-equivariant map g on $Z_0^k\mathbb{R}^m$ which is a restriction of a polynomial map $\bar{g} = (g^{i_1\dots i_q}_{j_1\dots j_p})\colon W_0 \times \dots \times W_{k-2} \times V_0 \times \dots \times V_k \to S$ and satisfies $f = g \circ \mathcal{D}^k$. Therefore the coordinate expression of our operator is given by polynomial mappings

$$g^{i_1\dots i_q}_{j_1\dots j_p}(R^i_{jkl}, \dots, R^i_{jklm_1\dots m_{k-2}}, v^{i_1\dots i_s}_{j_1\dots j_r}, \dots, v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_k})$$

where the subscripts m_j denote the covariant derivatives. If we apply once more the equivariance with respect to the homotheties $c^{-1}\delta^i_j \in G^1_m$, we get

$$c^{p-q} g^{i_1\dots i_q}_{j_1\dots j_p}(R^i_{jkl},\dots,R^i_{jklm_1\dots m_{k-2}},v^{i_1\dots i_s}_{j_1\dots j_r},\dots,v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_k}) =$$
$$= g^{i_1\dots i_q}_{j_1\dots j_p}(c^2 R^i_{jkl},\dots,c^k R^i_{jklm_1\dots m_{k-2}},c^{r-s}v^{i_1\dots i_s}_{j_1\dots j_r},\dots,c^{k+r-s}v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_k}).$$

This homogeneity implies that the g's must be sums of homogeneous polynomials of degrees a_ℓ and b_ℓ in the variables $R^i_{jklm_1\dots m_\ell}$ and $v^{i_1\dots i_s}_{i_1\dots i_r m_1\dots m_\ell}$, satisfying

$$2a_0 + \dots + ka_{k-2} + (r-s)b_0 + \dots + (k+r-s)b_k = p-q. \tag{1}$$

Now we consider the total polarization of each multi homogeneous component to obtain linear mappings

$$S^{a_0}W \otimes \dots \otimes S^{a_{k-2}}W_{k-2} \otimes S^{b_0}V \otimes \dots \otimes S^{b_k}V_k \to S.$$

The description of all invariant tensors (see 3.7) implies that the polynomials in question are linearly generated by monomials obtained by multiplying an appropriate number of variables $R^i_{jklm_1\dots m_\ell}$, $v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_\ell}$ and applying $GL(m)$-equivariant operations. This yields the statement of the proposition. $\square$

If $q = p$, then the polynomials must be of degree zero, and so only the $GL(m)$-invariant tensors can appear. If $q - p < 0$, there are no non negative integers solving (1) and so all natural operators in question are the zero operators only.

4.11. In order to determine all natural operators $D\colon Q_\tau P^1 \times T^{(0,r)} \to \Lambda T^*$ we have to consider the case $s = 0$ in the above construction, to contract all superscripts and to apply the alternation on all remaining subscripts at the very end.

Every $GL(m,\mathbb{R})$-invariant polynomial P defined on $\mathbb{R}^m \otimes \mathbb{R}^{m*}$ determines via the Chern-Weil construction a natural form, i.e. a natural operator of our type independent of the second argument. In particular, the homogeneous components of the invariant polynomial $\det(\mathbb{I}_m + A)$ give rise to the Chern forms c_q. The wedge product of exterior forms defines the algebra structure on the space of all operators in question.

Theorem. *The algebra of all natural operators $D\colon Q_\tau P^1 \times T^{(0,r)} \to \Lambda T^*$ is generated by the alternation, the exterior derivative d and the Chern forms c_q. The operators which do not depend on the second argument are generated by the Chern forms only.*

In particular, we see that all natural forms have even degrees. Since the exterior differential is natural, they must be closed.

4.12. In the proof of this result, we shall need several lemmas. The most of the covariant derivatives of the curvature and of the forms which are involved in the general construction from 4.10 are disabled by some of their symmetries during the final alternation. Let us first recall the antisymmetry of the curvature form, the first and the second Bianchi identity. We have

$$R^i_{jkl} = -R^i_{jlk} \tag{1}$$

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0 \tag{2}$$

$$R^i_{jklm} + R^i_{jlmk} + R^i_{jmkl} = 0. \tag{3}$$

Lemma. *The alternation of $R^i_{jklm_1\dots m_s}$ over any 3 indices among the first three or four subscripts is zero.*

Proof. Since the covariant derivative commutes with the tensor operations like the permutation of indices, it suffices to discuss the variables R^i_{jkl} and R^i_{jklm}. By (2), the alternation over the subscripts in R^i_{jkl} is zero and (3) yields the same for the alternation on k, l, m in R^i_{jklm}. In view of (1), it remains to discuss the alternation of R^i_{jklm} on j, l, m. Then (1) implies $R^i_{jkml} = -R^i_{jmkl}$ and so we can rewrite this alternation as follows

$$\begin{aligned} &R^i_{jklm} + R^i_{jmkl} + R^i_{jlmk} - R^i_{jlmk} \\ +&R^i_{mkjl} + R^i_{mlkj} + R^i_{mjlk} - R^i_{mjlk} \\ +&R^i_{lkmj} + R^i_{ljkm} + R^i_{lmjk} - R^i_{lmjk}. \end{aligned}$$

The first three entries on each row form a cyclic permutation and hence give zero. The same applies to the last column. □

4.13. Lemma. *For every tensor field $t = (t_{i_1\dots i_q})$, the alternation of its second covariant derivative $\nabla^2 t = (t_{i_1\dots i_q i_{q+1} i_{q+2}})$ over all indices is zero.*

Proof. Every linear connection Γ^i_{jk} determines a connection Γ with curvature R on each vector bundle associated to the linear frame bundle. The components of R are easily evaluated from R^i_{jkl} using the action of $\mathfrak{gl}(m)$ on the tensor space in question. In our case, $(a^i_j) \in GL(m)$ acts on a tensor $\omega_{i_1\dots i_q}$ by $(a^i_j)\omega_{i_1\dots i_q} = \tilde a^{j_1}_{i_1}\dots \tilde a^{j_q}_{i_q}\omega_{i_1\dots i_q}$ where $\tilde{}$ denotes the components of the inverse matrix, and so given a tensor field t we get the expression of the contraction $\langle R, t\rangle = -\sum_{s=1}^q R^m_{i_s i_{q+1} i_{q+2}} t_{i_1\dots m\dots i_q}$. If the connection is symmetric, then the Ricci identity yields $\mathrm{Alt}(\nabla^2 t) = \langle R, t\rangle$, where the alternation concerns only the last two indices. Hence we can apply our alternation to this expression. Up to a constant multiple, we get

$$\sum_{\sigma\in\Sigma} \mathrm{sgn}\sigma t_{i_{\sigma(1)}\dots i_{\sigma(q+2)}} = -\sum_s\sum_m\sum_\sigma \mathrm{sgn}\sigma R^m_{i_{\sigma(s)} i_{\sigma(q+1)} i_{\sigma(q+2)}} t_{i_{\sigma(1)}\dots m\dots i_{\sigma(q)}}.$$

Let us decompose this sum into summands with fixed m, s and all $\sigma(j)$ with $j \neq s$, $j \neq q+1$, $j \neq q+2$. These summands have the form

$$\pm\Big(\sum_{\bar\sigma\in\Sigma_3} \mathrm{sgn}\bar\sigma R^m_{i_{\bar\sigma(s)} i_{\bar\sigma(q+1)} i_{\bar\sigma_{q+2}}}\Big) t_{i_{\sigma(1)}\dots m\dots i_{\sigma(q)}}.$$

Now the first Bianchi identity implies that all these summands vanish. □

4.14. Lemma. *For every tensor $t = (t_{i_1\dots i_q})$, the alternation of the first covariant derivative ∇t coincides with the exterior differential $d(\mathrm{Alt}(t))$.*

Proof. Whenever the coordinate expressions of two natural operators coincide in one coordinate chart, the operators are equal. The first covariant derivative is of order zero in the connection argument, and at a fixed point the Christoffel symbols are zero in a suitable coordinate system. But then the formula for the alternation of the covariant derivative of the tensor t coincides with that for the exterior differential of the alternated tensor at this point. □

Proof of Theorem 4.11. Let us continue in the discussion from 4.10 and consider first a monomial in R's and v's containing at least one quantity $R^i_{jklm_1\dots m_s}$ with $s>0$. Then there exists one term among the R's in the product with three free subscripts among the first four ones, or one term R^i_{jkl} with all free subscripts, so that the monomial vanishes after alternation. Further, 4.12.(1) and (2) imply $R^i_{jkl}-R^i_{lkj}=-R^i_{klj}$. Hence we can restrict ourselves to contractions on the first two subscripts in the R's. Obviously, no subscript in the v's can be contracted since otherwise the alternation would kill one of the R's. So in view of Lemma 4.13, only the first order covariant derivatives can appear, and they yield the exterior derivatives of the alternated tensor v by Lemma 4.14. Hence all the possible operators are generated by the expressions $R^{k_q}_{k_1ab}R^{k_1}_{k_2cd}\dots R^{k_{q-1}}_{k_qef}$ where the indices $a,\dots,f$ remain free for the alternation, $v_{i_1\dots i_r}$ and $\mathrm{Alt}(v_{i_1\dots i_ri_{r+1}})$. This is a coordinate expression of the theorem. □

4.15. Operations on functions. Up to now, we have assumed $r>s\geq 0$, so that the case $r=0$ was excluded. In this case we cannot use 4.6 and so we must apply Theorem 4.7 instead of 4.4, but the codomain of the operations in question will still ensure the polynomiality of the operations. By 4.7, each jet $(j^\infty_0\Gamma, j^\infty_0 v)$ lies in some G^∞_m-invariant open subset (in the inverse limit topology) $V_k\subset J^\infty_0(Q_\tau P^1\mathbb{R}^m\times\mathbb{R})$ such that the restriction of the associated mapping $\mathcal{D}$ of the operator to V_k is determined by a (locally defined) G^{k+2}_m-equivariant mapping $f\colon J^k_0(\mathbb{R}^m,Q)\times J^k_0(\mathbb{R}^m\mathbb{R})\to S$. Taking k large enough we can assume that the jet of the zero section lies in V_k. Now, proceeding as in 4.6 and 4.10 we get for every positive $c\in\mathbb{R}$ the homogeneity condition

$$\begin{aligned}c^{p-q}f^{i_1\dots i_q}_{j_1\dots j_p}(\Gamma^\ell_{ij},\dots,\Gamma^\ell_{ij,\ell_1\dots\ell_{k-1}},v,\dots,v_{\ell_1\dots\ell_k})&=\\ &=f^{i_1\dots i_q}_{j_1\dots j_p}(c\Gamma^\ell_{ij},\dots,c^k\Gamma^\ell_{ij,\ell_1\dots\ell_{k-1}},v,\dots,c^kv_{\ell_1\dots\ell_k}).\end{aligned}$$

Thus, f is a polynomial mapping in all variables except v with functions of v as coefficients.

Using 4.4 and 4.7, we pass to G^1_m-equivariant mappings

$$g^{i_1\dots i_q}_{j_1\dots j_p}(R^i_{jkl},\dots,R^i_{jklm_1\dots m_{k-2}},v,\dots,v_{m_1\dots m_k})$$

with the homogeneity

$$\begin{aligned}c^{p-q}g^{i_1\dots i_q}_{j_1\dots j_p}(R^i_{jkl},\dots,R^i_{jklm_1\dots m_{k-2}},v,\dots,v_{m_1\dots m_k})&=\\ &=g^{i_1\dots i_p}_{j_1\dots j_q}(c^2R^i_{jkl},\dots,c^kR^i_{jklm_1\dots m_{k-2}},v,\dots,c^kv_{m_1\dots m_k}).\end{aligned}$$

Hence g is polynomial with smooth functions in one real variable v as coefficients and the degrees of its monomials satisfy 4.10.(1) with $r=s=0$. Now we can repeat the arguments from the end of 4.10 and we get

Lemma. *All natural operators $D\colon Q_\tau P^1 \times T^{(0,0)} \to T^{(q,p)}$ are obtained by iterating the following steps. Given a function, we can compose the function with arbitrary smooth function of one real variable, we can take covariant derivatives of the function and the covariant derivatives of the curvature, we can tensorize, we can apply any $GL(m,\mathbb{R})$-equivariant operation, and we can take linear combinations.*

The arguments from the proof of 4.11 are also valid now and so we can extend this theorem to the case of functions.

Theorem. *The algebra of all natural operators $D\colon Q_\tau P^1 \times T^{(0,0)} \to \Lambda T^*$ is generated by the compositions with arbitrary smooth functions of one real variable, the exterior derivative d and the Chern forms c_q.*

4.16. There are many natural operators on pseudo-Riemannian manifolds. In particular, using the inverse metric we can contract on any couple of indices and the complete contractions of suitable covariant derivatives of the curvature of the Levi-Cività connection give rise to natural functions of all even orders greater then one. Composing k natural functions with any fixed smooth function $\mathbb{R}^k \to \mathbb{R}$, we get a new natural function. Since every natural form can be multiplied by any natural function, we see that there is no hope to describe at least all natural forms in a way similar to the above characterization of the Chern forms. However, in Riemannian geometry we meet operations with a sort of homogeneity with respect to the change of the scale of the metric and these can be described in more details.

Definition. Let E and F be natural bundles over m-dimensional manifolds. We say that a natural operator $D\colon S^2_+T^* \times E \to F$ is *possibly-conformal*, if $D(c^2g,s) = D(g,s)$ for all metrics g, sections s, and all positive $c \in \mathbb{R}$. If F is a natural vector bundle and D satisfies $D(c^2g,s) = c^\lambda D(g,s)$, then D is said to be *homogeneous with weight* λ.

Let us notice that the weight of the metric g_{ij} is 2 (we consider the inclusion $g\colon S^2_+T^* \to S^2T^*$), that of its inverse g^{ij} is -2, while the curvature and all its covariant derivatives are conformal. The regular operators on Riemannian manifolds (cf. 4.9) homogeneous in the weight were studied extensively, see e.g. [Atiyah, Bott, Patodi, 73], [Epstein, 75], [Stredder, 75]. Using the above approach, we shall recover and generalize some of their results.

4.17. Recall $S^2_+T^*$ means the bundle functor of pseudo-Riemannian metrics on m-dimensional manifolds with some fixed signature. We shall discuss first the natural operators $D\colon S^2_+T^* \times T^{(s,r)} \to T^{(q,p)}$ with $s<r$. Similarly to 4.15, we use 4.8 to find G^∞_m-invariant open subsets V_k in $J^\infty_0((S^2_+T^* \times T^{(s,r)})\mathbb{R}^m)$ forming a filtration of the whole jet space. On these subsets $\mathcal{D}$ factorizes through smooth G^{k+1}_m-equivariant mappings

$$f^{i_1\dots i_q}_{j_1\dots j_p} = f^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},\dots,g_{ij\ell_1\dots\ell_k},v^{i_1\dots i_s}_{j_1\dots j_r},\dots,v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}).$$

defined on $\pi^\infty_k V_k$. Using the action of the homotheties $c^{-1}\delta^i_j$ for large k's, we get

$$\begin{aligned}(1)\quad & c^{p-q} f^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},\dots,g_{ij\ell_1\dots\ell_k},v^{i_1\dots i_s}_{j_1\dots j_r},\dots,v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}) = \\ &= f^{i_1\dots i_q}_{j_1\dots j_p}(c^2 g_{ij},\dots,c^{2+k}g_{ij\ell_1\dots\ell_k},c^{r-s}v^{i_1\dots i_s}_{j_1\dots j_r},\dots,c^{r-s+k}v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}).\end{aligned}$$

Now, let us add the assumption that D is homogeneous with weight λ, choose the change $g \mapsto c^{-2}g$ of the scale of the metric and insert this new metric into (1). We get

$$\begin{aligned} c^{p-q-\lambda} f^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},\dots,g_{ij\ell_1\dots\ell_k},v^{i_1\dots i_s}_{j_1\dots j_r},\dots,v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}) = \\ = f^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},\dots,c^k g_{ij\ell_1\dots\ell_k},c^{r-s}v^{i_1\dots i_s}_{j_1\dots j_r},\dots,c^{r-s+k}v^{i_1\dots i_s}_{j_1\dots j_r\ell_1\dots\ell_k}). \end{aligned}$$

This formula shows that the mappings $f^{i_1\dots i_q}_{j_1\dots j_p}$ are polynomials in all variables except g_{ij} with functions in g_{ij} as coefficients.

According to 4.8 and 4.9, the map $\mathcal{D}$ is on V_k determined by a polynomial mapping

$$\omega = (\omega^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},W^i_{jkl},\dots,W^i_{jklm_1\dots m_{k-2}},v^{i_1\dots i_s}_{j_1\dots j_r},\dots,v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_k}))$$

which is G^1_m-equivariant on the values of the covariant derivatives of the curvatures and the sections. If we apply once more the equivariance with respect to the homothety $x \mapsto c^{-1}x$ and at the same time the change of the scale of the metric $g \mapsto c^{-2}g$, we get

$$\begin{aligned} c^{p-q-\lambda}\omega^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},R^i_{jkl},\dots,R^i_{jklm_1\dots m_{k-2}},v^{i_1\dots i_s}_{j_1\dots j_r},\dots,v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_k}) = \\ \omega^{i_1\dots i_q}_{j_1\dots j_p}(g_{ij},c^2R^i_{jkl},\dots,c^kR^i_{jklm_1\dots m_{k-2}},c^{r-s}v^{i_1\dots i_s}_{j_1\dots j_r},\dots,c^{r-s+k}v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_k}). \end{aligned}$$

This homogeneity shows that the polynomial functions $\omega^{i_1\dots i_q}_{j_1\dots j_p}$ must be sums of homogeneous polynomials with degrees a_ℓ and b_ℓ in the variables $R^i_{jklm_1\dots m_\ell}$ and $v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_\ell}$ satisfying

$$(2) \qquad 2a_0 + \dots + ka_{k-2} + (r-s)b_0 + \dots + (k+r-s)b_k = p-q-\lambda$$

and their coefficients are functions depending on g_{ij}'s (in fact polynomials depending on g_{ij} and on the square root of the inverse of the determinant of g_{ij}, cf. 4.9).

Now, we shall fix $g_{ij} = g_0$ and use the $O(m',n,\mathbb{R})$-equivariance of the homogeneous components of the polynomial mapping ω. For this reason, we shall switch to the variables $R_{ijklm_1\dots m_s} = g_{ia}R^a_{jklm_1\dots m_s}$ (the v's remain). Using the standard polarization technique and Theorem 3.18, we get that each multi homogeneous component in question results from multiplication of variables $R_{ijklm_1,\dots,m_s}$, $v^{i_1\dots i_s}_{j_1\dots j_r m_1\dots m_s}$, $s = 0,1,\dots,r$, and application of some $O(m',n)$-equivariant tensor operations on the target space. Hence we have proved:

Theorem. *All natural operators $D\colon T^{(s,r)} \to T^{(q,p)}$, $s<r$, on pseudo-Riemannian manifolds which are homogeneous in weight result from a finite number of the following steps:*

(a) *take tensor product of arbitrary covariant derivatives of the curvature tensor or the covariant derivatives of the tensor fields form the domain*
(b) *tensorize by the metric or by its inverse*
(c) *apply arbitrary $GL(m)$-equivariant operation*
(d) *take linear combinations.*

4.18. If the codomain of the operator is ΛT^*, then all indices which were not contracted must be alternated at the end of the above procedure. Since the metric is a symmetric tensor, we get zero whenever using the above step (b) and alternating on both indices. But contracting over any of them has no proper effect, for $\delta_{ij}t_{jj_2,\dots,j_s} = t_{ij_2,\dots,j_s}$. So we can omit the step (b) at all.

Surprisingly enough we shall prove that among the homogeneous natural operators $D\colon S^2_+T^* \times T^{(0,r)} \to \Lambda T^*$ with non-negative weights, there are no other ones then those obtained by the evaluation of the operators from Theorem 4.11 using the Levi-Cività connection.

It is more suitable to discuss the curvatures and their covariant derivatives in the form $R_{ijklm_1\dots m_s}$. These are all of weight two. The Riemannian curvature is a two-form with values in the algebra of skew-symmetric matrices, so we have the symmetry $R^i_{\underline{jkl}} = -R^j_{\underline{ikl}}$ in the positive definite case. The pseudo-Riemannian curvature has values in 'pseudo-skew-symmetric' matrices, but after shifting the index down, we get always the same symmetry, i.e.

$$R_{\underline{ijkl}} = -R_{\underline{jikl}}. \tag{1}$$

Therefore, the evaluation of the Chern forms using the pseudo-Riemannian connection yields zero in degrees not divisible by four and the Pontrjagin forms in degrees 4ℓ.

Theorem. *There are no non-zero homogeneous natural operators $D\colon S^2_+T^* \times T^{(0,r)} \to \Lambda T^*$ with a positive weight. The algebra of all possibly-conformal natural operators $D\colon S^2_+T^* \times T^{(0,r)} \to \Lambda T^*$ is generated by the Pontrjagin forms p_q, the alternation and the exterior differential. The operators which do not depend on the second argument are generated by the Pontrjagin forms.*[16]

Proof. The theorem will follow quite easily from the above proposition using Lemmas 4.12–4.14 concerning the symmetries of the curvature of arbitrary torsion-free connections and the one more symmetry specific for the pseudo-Riemannian curvatures:

Sublemma. *The alternation of $R_{ijklm_1\dots m_s}$ on arbitrary 3 indices among the first four or five ones is zero.*

Proof. Since the pseudo-Riemannian connections satisfy $R_{ijkl} = R_{klij}$ (this is a consequence of (1) and Bianchi identity), Lemma 4.12 and (1) yield this lemma. $\square$

Consider a monomial P with degrees a_s in $R_{ijklm_1\dots m_s}$ and b_s in $v^{i_1\dots i_s}_{i_1\dots i_r m_1\dots m_s}$. In view of the above lemma, if P remains non zero after all alternations, then we must contract on at least two indices in each $R_{ijklm_1\dots m_s}$ with $s>0$ and so we can alternate over at most $2a_0 + \dots + ka_{k-2} + pb_0 + \dots (p+k)b_k$ indices. This means

[16] This generalizes the famous Gilkey theorem on the uniqueness of the Pontrjagin forms, see [Gilkey, 73], [Atiyah, Bott, Patodi, 73]. The Gilkey theorem describes the regular possibly-conformal natural forms in the Riemannian case, while we use no assumptions on the order or polynomiality or regularity, only the smoothness. In [Gilkey, 75], the uniqueness of the Pontrjagin forms is proved in the pseudo-Riemannian case as well. Let us remark, Gilkey proves his theorems directly discussing the derivatives of the metric.

$p \le 2a_0 + \dots + ka_{k-2} + rb_0 + \dots (r+k)b_k = p - \lambda$. Consequently $\lambda \le 0$ if there is a non-zero natural form with weight λ. This proves the first assertion of the theorem.

Let $\lambda = 0$. Since the weight of g^{ij} is -2, any contraction on two indices in the monomial decreases the weight of the operator by 2. Every covariant derivative $R_{ijklm_1\dots m_s}$ of the curvature has weight 2. So we must contract on exactly two indices in each $R_{ijklm_1\dots m_s}$ which implies, there are $s+2$ of them under alternation. But then there must appear three alternated indices among the first five if $s \neq 0$. This proves $a_1 = \dots = a_k = 0$. Moreover, there is no further contraction for our disposal, and so any covariant derivative of the tensors of order greater then one kills the whole monomial after alternation. Hence all the natural operators are generated by the forms p_q, the alternation and the exterior differential. This completes the proof. □

4.19. Exactly in the same way as in 4.15, we can modify the proof of Theorem 4.18 for the case $r = 0$. In the implicit description of all operators $D\colon S^2_+T^* \times T^{(0,0)} \to T^{(q,p)}$ in 3.3, we have to add the compositions with smooth real functions and we get

Theorem. *There are no non-zero homogeneous natural operators $D\colon S^2_+T^* \times T^{(0,0)} \to \Lambda T^*$ with a positive weight. The algebra of all possibly-conformal natural operators $D\colon S^2_+T^* \times T^{(0,0)} \to \Lambda T^*$ is generated by the Pontrjagin forms p_q, the compositions with arbitrary smooth functions of one real variable and the exterior differential.* □

4.20. Linear operations homogeneous in weight. The discussion from the proof of the Theorem 4.18 can be continued for any fixed negative weight. In particular, the situation is interesting for $\lambda = -2$. Beside the well known codifferential $\delta\colon \Lambda^p \to \Lambda^{p-1}$, the compositions $d \circ \delta$ and $\delta \circ d$ (the Laplace-Beltrami operator is $\Delta = \delta \circ d + d \circ \delta$), and the multiplication by the scalar curvature, there appear some other simple operators. Let us describe this case in more detail for the linear operators $\Lambda^p T^* \to \Lambda^p T^*$ (in the Riemannian case and under stronger assumptions this can be also found in [Stredder, 75]).

If compared with the proof of 4.18, we have exactly one more contraction for our disposal in each monomial. Hence we might involve also more covariant derivatives. But once there appears $R_{ijklm_1\dots m_s}$ with $s > 0$, we have never enough contractions to kill a necessary number of indices. If R_{ijkl} appears, then no covariant derivative of the argument can be involved for the same reason. So there are only the following possibilities:

$$(1) \qquad R_{abab}v_{i_1\dots i_p},\ R_{aba[i_1}v_{i_2\dots i_p]b},\ R_{ab[i_1i_2}v_{i_3\dots i_p]ab},\ v_{i_1\dots i_paa},\ v_{[i_1\dots i_{p-1}aai_p]}$$

Here $[\dots]$ denotes the alternation of the indicated indices and all natural operators in question result from a linear combination of these five ones.

The *codifferential* δ is defined as the formal adjoint to d, i.e. we require

$$\int_U \langle d\omega, \eta\rangle \nu = \int_U \langle \omega, \delta\eta\rangle \nu$$

for all forms $\omega \in \Omega^{p-1}$, $\eta \in \Omega^p$ with compact supports in U. Here $\langle\ ,\ \rangle$ is the induced (pseudo-) Riemannian metric defined by $\langle v_{i_1\dots i_p}, w_{j_1\dots j_p}\rangle = \frac{1}{p!}v_{i_1\dots i_p}w^{i_1\dots i_p}$, and ν is the local Riemannian volume form on U. (U is small enough to allow the existence of ν, $\nu_{\underline{i}_1\dots\underline{i}_m} = ((-1)^n \det(g_{\underline{i}\underline{j}}))^{1/2}\varepsilon_{\underline{i}_1\dots\underline{i}_m}$, where $\varepsilon_{i_1\dots i_m}$ is the Levi-Cività tensor.) Clearly, the definition does not depend on the orientation (i.e. on the choice of ν) and δ is a local linear operator $\Omega^p \to \Omega^{p-1}$. Once we have chosen ν, we can define the *Hodge star operator* $*\colon \Omega^p \to \Omega^{m-p}$ by the equality $\langle\omega,\eta\rangle\nu = \omega\wedge *\eta$. This yields for $\omega = v_{i_1\dots i_p}$ the expression

$$*\omega = \frac{1}{p!} v_{i_1\dots i_p}\nu^{i_1\dots i_p}{}_{i_{p+1}\dots i_m}.$$

Now, we compute easily for $\omega\in\Omega^p$

$$**\omega = \frac{1}{p!(m-p)!} vj_1\dots j_p \nu^{j_1\dots j_p}{}_{j_{p+1}\dots j_m}\nu^{j_{p+1}\dots j_m} i_1\dots i_p = (-1)^{(m-p)p}\omega.$$

Further we get

$$\begin{aligned}(*d*)(v_{i_1\dots i_p}) &= *(\frac{1}{p!} v_{j_1\dots j_p[i_{m+1}}\nu^{j_1\dots j_p}{}_{i_{p+1}\dots i_m]}) = \\ &= \frac{1}{p!(m-p)!} v_{j_1\dots j_p[k_{m+1}}\nu^{j_1\dots j_p}{}_{k_{p+1}\dots k_m]}\nu^{k_{p+1}\dots k_m k_{m+1}}{}_{i_1\dots i_{p-1}} = \\ &\qquad\qquad = (-1)^p(m-p+1)v_{i_1\dots i_{p-1}aa}.\end{aligned}$$

Let us choose $\omega\in\Omega^{p-1}$, $\eta\in\Omega^{m-p}$ and write the equation for δ with ω and $*\delta$:

$$0 = \int_U (-1)^{p(m-p)} d\omega\wedge\eta - \int_U \omega\wedge(*\delta*\eta) = \int_U (-1)^{p(m-p)} d(\omega\wedge\eta).$$

Since this holds for all ω and η, we get $*\delta* = (-1^{p(m-p)+p}d)$ and so, finally,

$$\delta = (-1)^{pm+m+1}(*d*) = (-1)^{(p+1)(m+1)}(m-p+1)v_{i_1\dots i_{p-1}aa}.$$

Now, we are ready to write down the generators from (1) (up to constant multiples).

4.21. Proposition. *All linear operators $\Lambda^pT^*\to\Lambda^pT^*$ on pseudo-Riemannian manifolds which are homogeneous with weight -2 are linearly generated by the following generators: the multiplication by scalar curvature, the contraction with the Ricci curvature, the contraction with the full pseudo-Riemannian curvature , the compositions $\delta\circ d$ and $d\circ\delta$.*

4.22. Operations on oriented pseudo-Riemannian manifolds. Let us notice that in the description of natural operators $S^2_+T^*\times E\to F$ we used the $O(m',n)$-invariance as late as at the very end of 4.17 and that the whole proof of 4.18 uses only the discussion on the steps from Proposition 4.17. Therefore, we can prove easily:

Theorem. *All natural operators $D\colon T^{(s,r)} \to T^{(q,p)}$, $s < r$, on oriented pseudo-Riemannian manifolds which are homogeneous in weight result from a finite number of the following steps:*

(a) *take tensor product of arbitrary covariant derivatives of the curvature tensor or the covariant derivatives of the tensor fields from the domain*
(b) *tensorize by the metric or by its inverse*
(c) *tensorize by the (pseudo-) Riemannian volume form ν*
(d) *apply arbitrary $GL(m)$-equivariant operation*
(e) *take linear combinations.*

Proof. It remains to prove that the covariant derivatives of the volume form ν cannot be involved. But the latter are zero, for the covariant derivative is defined through the parallel transport which consists of isometries. □

Let us remark that the latter theorem, as well as Theorems 4.17 and 4.10 are valid also without the requirement $s < r$ if we add the polynomiality assumption.

4.23. Possibly-conformal linear operators on forms. At the end of this section, we prepare some technical results which shall be of fundamental importance in our description of all conformally invariant operators on conformally flat manifolds in Section 8.

As we have mentioned, the volume form ν is defined by the expression $\nu_{\underline{i}_1\dots\underline{i}_m} = ((-1)^n \det(g_{\underline{i}\underline{j}}))^{1/2}\varepsilon_{\underline{i}_1\dots\underline{i}_m}$ (the signature is (m', n)) and so it is evidently homogeneous with weight m. Thus, the homogeneous weight of $*\colon \Omega^p \to \Omega^{m-p}$ is $m-2p$. In general, there exist more possibly-conformal natural operators in the oriented case. First of all, if the dimension $m = 2p$ is even, then $**\colon \Omega^p \to \Omega^p$ is identity up to sign and we can split the space of p-forms, $\Omega^p = \Omega^p_+ \oplus \Omega^p_-$, where $\Omega_\pm$ are the two eigen spaces for $*$. We shall see later that these spaces are not only $O(m', n)$-invariant but even irreducible. If we compose the exterior differential d with the projections, we get the operators $d = d_+ + d_-$ and the compositions $d \circ d_\pm$ are not more zero. Further, it might happen that composing enough d's and $*$'s together, we get a possibly-conformal operator. Let us write $\delta_q = *d*\dots d*\colon \Omega^{q+1} \to \Omega^{m-q-1}$, $q < p$, with $m - 2q - 1$ stars involved, and $D_q = d \circ \delta_q \circ d\colon \Omega^q \to \Omega^{m-q}$.

Proposition. *If the dimension $m = 2p$ is even, then each operator D defined by $D = D_q = d\circ\delta_q\circ d$ or $D = \delta_q \circ d$ or $D = \delta_q$ is a possibly-conformal natural operator on oriented pseudo-Riemannian manifolds. In particular, $D_{p-1}\colon \Omega^{p-1} \to \Omega^{p+1}$ equals to $d * d = d \circ d_+ - d \circ d_-$. Up to constant multiples and up to terms involving the curvature and its covariant derivatives, the operators D are the only non-zero possibly-conformal linear natural operator on forms on (oriented) pseudo-Riemannian manifolds beside the exterior differentials d, $d_\pm$ and the identities.*

If the dimension m is odd, then up to constant multiples and up to terms involving the curvature and its covariant derivatives, the only non-zero possibly-conformal linear natural operators on forms on (oriented) pseudo-Riemannian manifolds are the exterior differentials and the identities.

Proof. Clearly each operator D is natural. If we start in Ω^{q+1} and apply $*d*$, then the mappings go: $\Omega^{q+1} \mapsto \Omega^{m-q-1} \mapsto \Omega^{m-q} \mapsto \Omega^q$ while the weights which are added are: $0 \mapsto m - 2q - 2 \mapsto m - 2q - 2 \mapsto -2$ (the total is obvious – the weight of δ). Hence if $m = 2p$, $q < p$ and if we start at Ω^{q+1} we reach weight zero exactly after composing $(m - 2q - 2)$-times $d*$ and applying $*$ at the very end. In all other cases we never get weight zero, for each turn around decreases the weight by 2 and once we get back to the initial position with a negative weight in all three last positions the hope is lost.

Let us now perform the discussion from 4.18 in this special situation and let us restrict ourselves to the natural operators on the whole category of (not oriented) pseudo-Riemannian manifolds. If we want to get a linear operator $D\colon \Omega^q \to \Omega^{q'}$ which is non-zero on flat manifolds, then the only monomials which make sense are of the form $v_{i_1\dots i_q l_1\dots l_s}$. Since we do not admit the curvatures, we may restrict ourselves to the flat case and so the covariant derivatives l_k are symmetric. Thus at most one index among the l's may remain uncontracted and at most one can be contracted with some of the i's. Hence what we only can do is to involve $2s$ or $2s+1$ or $2s+2$ derivatives, to choose s pairs, to contract them and to contract one of the remaining indices (if any) with some of the i's. Hence, up to constant multiples and linear combinations, $D = d\circ\delta\dots\circ d$ or $D = \delta\circ d\dots\circ d$ or $D = d\circ\delta\dots\circ\delta$ or $D = \delta\circ d\dots\circ\delta$ and we get $q' - q = 1$ or 0 or 0 or -1, respectively.

On the space of all natural operators $D\colon \Omega^q \to \Omega^{q'}$, there is the canonical action of $O(m', n)/SO(m', n) = \mathbb{Z}_2$ and so each such operator is a sum $D = D_+ + D_-$ where D_+ is invariant with respect to the change of orientation while D_- changes the sign. If D is natural and possibly-conformal, then also both D_+ and D_- are natural and possibly-conformal. Now, notice that $*\circ D_-$ is invariant with respect to the change of orientation and $D_- = \pm **D_-$. Thus, $*D_-\colon \Omega^q \to \Omega^{m-q'}$ and, up to constant multiples and linear combinations, either $m - q' - q = 1$ and $**D_- = *d\circ\delta\dots\circ d$, or $m - q' - q = 0$ and $**D_- = *\delta\circ d\dots\circ d$ or $**D_- = *d\circ\delta\dots\circ\delta$, or $m - q' - q = -1$ and $**D_- = *\delta\circ d\dots\circ\delta$. The last Hodge star in these operators acts on $\Omega^{m-q'}$ and so its weight is $2q' - m$. If m is odd then this can never kill the even negative weight appearing through δ's. Thus, there is no codifferential involved in the expression, $D_- = 0$ and D is either exterior differential or identity (up to constant multiples). This proves the last statement of the proposition.

If $m = 2p$ is even and $2q' - m < 0$, then the weight of $*$ is negative and we get the same result as in the odd-dimensional case. If $2q' - m \geq 0$, then a simple discussion shows that the only possible operators are those listed in the proposition. $\square$

We can describe the most interesting operators by the following two diagrams, separately for the even and odd dimension m.

The even case $m = 2p$:

$$\begin{tikzcd}
 & & & & \Omega^p_+ \arrow[dr, "d"] & & & & \\
\Omega^0 \arrow[r, "d"] \arrow[rrrrrrrr, bend right=40, "D_0=d\circ(*d)^{m-1}"'] & \Omega^1 \arrow[r, "d"] \arrow[rrrrrr, bend right=35, "D_1=d\circ(*d)^{m-3}"'] & \cdots \arrow[r, "d"] & \Omega^{p-1} \arrow[ur, "d_+"] \arrow[dr, "d_-"'] \arrow[rr, bend right=60, "D_{p-1}=d*d=d\circ d_+ - d\circ d_-"'] & & \Omega^{p+1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{m-1} \arrow[r, "d"] & \Omega^m \\
 & & & & \Omega^p_- \arrow[ur, "d"'] & & & &
\end{tikzcd}$$

The diagram is not commutative! The horizontal line is exact, but not the arrows in the central diamond. On the other hand, all three operators $\Omega^{p-1} \to \Omega^{p+1}$

differ by constant multiples. The diagram does not exhaust all operators from the proposition, but notice that the operators indicated on the arrows are unique, up to multiples.

The odd-dimensional case coincides with the de Rham resolvent:

$$0 \longrightarrow \Omega^0 \xrightarrow{\ d\ } \Omega^1 \xrightarrow{\ d\ } \dots \xrightarrow{\ d\ } \Omega^{m-1} \xrightarrow{\ d\ } \Omega^m \longrightarrow 0$$

We shall see later on, that the arrows in the above diagrams correspond exactly to the conformal operators on forms on conformally flat manifolds.

4.24. Linear operators on functions. Another important information is the description of all homogenous linear operators on functions with values in functions. For each even number $2k \in \mathbb{N}$ we define the operator $A_k \colon \Omega^0 \to \Omega^0$ by $A(v) = v_{b_1 b_1 \dots b_k b_k}$, i.e. we take the $2k$-th covariant derivative of v and contract all indices. Notice that if we change our choice of the contracted couples of indices then the result differs by some expression built of curvatures and its covariant derivatives, cf. 4.13. In particular, on the flat manifolds we get no difference. The operator A_k is a homogeneous natural linear operator with weight $-2k$. In view of the above discussion there is no other possibility for homogenous linear operators beside those involving the curvature or its covariant derivatives. Thus we have proved:

Proposition. *Up to constant multiples and up to terms involving the curvature and its covariant derivatives, the operators A_k are the only linear homogeneous natural operators defined on functions with values in functions. In particular, there are no homogeneous operators with an odd weight.*

5. Conformally flat manifolds

5.1. Conformal structures. A conformal (pseudo-Riemannian) manifold M is an m-dimensional manifold with a $CO(m', n, \mathbb{R})$-structure, $m = m' + n$, see 2.11 for the definition. Hence M is a base manifold of a principal fiber bundle $FM \subset P^1M$ with structure group $CO(m', n, \mathbb{R})$. By 2.13, the latter bundles correspond to sections of $P^1M/CO(m', n, \mathbb{R})$. Whenever we choose representing local sections with values in the cosets of $P^1M/CO(m', n, \mathbb{R})$, we get an induced reduction of P^1M to $O(m', n)$ and, moreover, if we take another representing local sections, then the corresponding metric will be deformed by multiplication by a smooth real function. Analogously we define the complex conformal structures on complex manifolds.

Further, by the definition, a mapping $f \colon M \to N$ between conformal manifolds is a morphism in $\mathcal{M}f_m(O(m', n, \mathbb{R}))$ if and only if $P^1 f(FM) \subset FN$, and the latter happens if and only if f preserves each metric from the conformal class up to multiplication by a function. Thus, we can study the conformal local isomorphisms by fixing an arbitrary metric from the given conformal class.

In this section, we shall deal mostly with the real manifolds.

5.2. Definition. The *flat conformal structure* $\bar{F}\mathbb{R}^m$ *on* $\mathbb{R}^m$ is determined by the canonical (pseudo-) Euclidean metric. A conformal manifold (M, FM) is called *locally flat* if each point $x \in M$ admits a local conformal isomorphism $(M, FM) \to (\mathbb{R}^m, \bar{F}\mathbb{R}^m)$ defined at x.

5.3. Local conformal transformations on the flat $\mathbb{R}^m$. We shall write $g = g_{ij}$ for the canonical (pseudo-) metric and its evaluation on vectors will be denoted by $\langle\ ,\ \rangle$. Further we write $|x|^2$ for the value $\langle x, x\rangle$. Each local conformal isomorphism f determines the positive (locally defined) function λ^2 defined by $f_* g = \lambda^2 g$. There are four types of evident local conformal isomorphisms on $\mathbb{R}^m$:

(a) the transformations from $O(m', n, \mathbb{R})$ are defined globally, $\lambda(x) = 1$
(b) the translations $x \mapsto x + a$ are defined globally, $\lambda(x) = 1$
(c) the homotheties (dilatations) $x \mapsto \lambda x$ are defined globally, $\lambda(x) = \pm\lambda$ is constant
(d) the inversions $x \mapsto |x - x_0|^{-2}(x - x_0)$ are defined for all x with $|x - x_0|^2 \neq 0$, $\lambda(x)$ is a constant multiple of $|x - x_0|^{-2}$.

To see that the inversions are really conformal, let us write down the tangent of the inversion f with $x_0 = 0$ at x evaluated on ξ. We get $T_x f.\xi = |x|^{-2}\xi - 2\langle\xi, x\rangle|x|^{-4}x$, so that $|x|^4 |T_x f.\xi|^2 = \langle \xi - \frac{2\langle\xi,x\rangle}{\langle x,x\rangle}x, \xi - \frac{2\langle\xi,x\rangle}{\langle x,x\rangle}x\rangle = \langle\xi,\xi\rangle$. This yields the λ as stated in (d).

These four types of mappings generate a pseudogroup of local conformal transformations. If the dimension $m = 2$, then there is a plenty of other locally defined conformal transformations, for each complex analytic function is conformal. We shall restrict ourselves to the case $m \geq 3$ in the rest of this section.

5.4. The Liouville theorem. *All smooth local conformal transformations on the pseudo-Euclidean space $\mathbb{R}^{m'+n}$, $m' + n \geq 3$, are generated by the mappings 5.3.(a)–(d).*

Proof. The indices in this proof will be always concrete (no 'Penrose abstract index notation'). Let us consider a locally defined conformal mapping $f : \mathbb{R}^m \to \mathbb{R}^m$. This means, the Jacobi matrix $D = D(x) = (\frac{\partial f^i(x)}{\partial x^j})$ is an element of $CO(m', n, \mathbb{R})$ for each x from the domain. Equivalently, for each tangent vector ξ at x we have $|D(x).\xi|^2 = (\lambda(x))^2|\xi|^2$ for some fixed smooth positive function λ. We shall use the brief notation $D\xi(x) := Tf \circ \xi(x)$ for an arbitrary vector field ξ. Consider a local frame $\xi_1, \dots, \xi_m$ at x belonging to the flat $O(m', n, \mathbb{R})$-structure, e.g. we may identify $\xi_1, \dots, \xi_m$ with the standard basis of $\mathbb{R}^m$. We shall view ξ as constant vector fields on $\mathbb{R}^m$. Then we have (globally)

$$0 = \langle\xi_i, \xi_j\rangle = \langle D\xi_i, D\xi_j\rangle, \qquad i \neq j. \tag{1}$$

If we differentiate the latter equality in the direction of a third vector field ξ_k, we get

$$0 = \partial_{\xi_k}\langle D\xi_i, D\xi_j\rangle = \langle\partial_{\xi_k} D\xi_i, D\xi_j\rangle + \langle D\xi_i, \partial_{\xi_k} D\xi_j\rangle. \tag{2}$$

Now, we fix three different indices i, j, k (recall $m \geq 3$) and we write down (2) three times with a cyclic permutation of these indices. Since our choice of the ξ's is a very special one, we have $\partial_{\xi_i} D\xi_j = \partial_{\xi_j} D\xi_i$ (in fact $D\xi_k(x) = \sum_i (\partial f_i/\partial x^k)(x)\xi_i(x)$ and ξ_i is constant, and so the latter claim follows from the symmetry of the second partial derivatives). Thus, if we add the first two equalities and subtract the third one, we obtain

$$\langle\partial_{\xi_i} D\xi_j, D\xi_k\rangle = 0.$$

Since this holds for each k with $k \neq i$, $k \neq j$, there are functions μ_{ij} and ν_{ij} such that

$$\partial_{\xi_i} D\xi_j = \mu_{ij} D\xi_i + \nu_{ij} D\xi_j . \tag{3}$$

By the definition, these functions satisfy

$$\begin{aligned} \mu_{ij} &= \frac{1}{|D\xi_i|^2}\langle \partial_{\xi_i} D\xi_j , D\xi_i\rangle = \frac{1}{2\lambda^2|\xi_i|^2}\partial_{\xi_j}\langle D\xi_i , D\xi_i\rangle = \frac{1}{\lambda}\partial_{\xi_j}\lambda \\ \nu_{ij} &= \frac{1}{\lambda}\partial_{\xi_i}\lambda. \end{aligned} \tag{4}$$

Let us denote $\rho(x) = \frac{1}{\lambda(x)}$. The Hessian $H = \frac{\partial^2\rho(x)}{\partial x^i \partial x^j}$ is a bilinear form at each x from the domain.

Sublemma. *It holds $H(x) = \sigma g(x)$ with σ constant.*

Proof. We shall write $y = f(x)$. Using (3) and (4) we express $\partial_{\xi_i}\partial_{\xi_j}(\rho y)$:

$$\begin{aligned} &\partial_{\xi_i}\partial_{\xi_j}(\rho y) = (\partial_{\xi_i}\partial_{\xi_j}\rho)y + (\partial_{\xi_j}\rho)D\xi_i + (\partial_{\xi_i}\rho)D\xi_j + \rho(\partial_{\xi_i}\partial_{\xi_j} y) = \\ &= (\partial_{\xi_i}\partial_{\xi_j}\rho)y + \rho(\mu_{ij}D\xi_i + \nu_{ij}D\xi_j) - \frac{1}{\lambda^2}(\partial_{\xi_j}\lambda)D\xi_i - \frac{1}{\lambda^2}(\partial_{\xi_i}\lambda)D\xi_j = (\partial_{\xi_i}\partial_{\xi_j}\rho)y. \end{aligned} \tag{5}$$

If we differentiate (5) with respect to ξ_k, we get

$$\partial_{\xi_k}\partial_{\xi_i}\partial_{\xi_j}(\rho y) = (\partial_{\xi_i}\partial_{\xi_j}\rho)D\xi_k + (\partial_{\xi_k}\partial_{\xi_i}\partial_{\xi_j}\rho)y.$$

Since two of the three terms commute in i, j, k, the third one must commute as well. Hence we have for two linear independent vectors $D\xi_k$ and $D\xi_i$ the equality $(\partial_{\xi_i}\partial_{\xi_j}\rho)D\xi_k = (\partial_{\xi_k}\partial_{\xi_j}\rho)D\xi_i$. This implies $H(\xi_i, \xi_j) = 0$ for all $i \neq j$. Since the vectors satisfy $\langle \xi_i, \xi_j\rangle = 0$, the latter means $H_{ij}(x) = \sigma(x)g_{ij} = 0$ for all different indices i, j. Since the function ρ is invariant with respect to isometries, $H_{ii}(x)$ are determined by $H_{11} =: \sigma(x)$ and $H_{ij}(x) = \sigma(x)g_{ij}$ for all indices i and j. We choose now an arbitrary third index k and differentiate

$$\partial_{\xi_k}\partial_{\xi_i}\partial_{\xi_j}\rho = (\partial_{\xi_k}\sigma)\langle \xi_i, \xi_j\rangle.$$

The left hand side is commutative in i and k, so we get

$$\langle (\partial_{\xi_k}\sigma)\xi_i - (\partial_{\xi_i}\sigma)\xi_k, \xi_j\rangle = 0.$$

Since all the three vectors are linearly independent, the latter implies $\partial_{\xi_k}\sigma = 0$ and so σ is constant. $\square$

The sublemma yields the system of partial differential equations for ρ which is easy to solve:

$$\begin{gathered} \frac{\partial^2 \rho}{\partial x^i \partial x^j} = \sigma g_{ij} \\ \rho(x) = \frac{1}{\lambda(x)} = a_1|x - x_0|^2 + b_1, \qquad a_1, b_1 \in \mathbb{R}. \end{gathered}$$

If we apply the same procedure to the inverse mapping $x = f^{-1}(y)$ we get

$$\lambda(y) = \frac{1}{\rho(y)} = \frac{1}{\rho} = a_2|y - y_0|^2 + b_2$$

and so the relation $\lambda(x)\rho(y) = 1$ yields the implicit description of f

$$(a_1|x - x_0|^2 + b_1)(a_2|y - y_0|^2 + b_2) = 1. \tag{6}$$

Composing with translations we can arrange $x_0 = y_0 = f(0) = 0$. The implicit expression (6) shows that f transforms spheres into spheres. Let us fix x with $|x|^2 > 0$ (if $|x|^2 \leq 0$ for all x, we can go through the whole proof with $-g$ instead of g) and let us define a curve $[0, \infty) \to \mathbb{R}^m$, $t \mapsto x(t) = \frac{t}{|x|}x$. This curve is transformed into a curve $y(t) = f(x(t))$ and we can evaluate the value $|y| = |f(x)|$ as follows (notice $|y|^2 > 0$ as $|x|^2 > 0$)

$$|y| = \int_0^{|x|} \frac{d|y(t)|}{dt} dt = \int_0^{|x|} \lambda(x(t)) dt = \int_0^{|x|} \frac{1}{a_1 t^2 + b_1} dt.$$

The integral on the right-hand side is a transcendent function in $|x|$, except $a_1 b_1 = 0$. In view of (6), either $a_1 = 0$ or $b_1 = 0$.

Assume $a_1 = 0$. Hence both ρ and λ are constant and so (5) shows that $y = f(x)$ is linear. Consequently f must be an element from $CO(m', n)$.

If $b_1 = 0$, then the composition of f with the inversion reduces the situation to the previous case and the Liouville theorem is proved. $\square$

5.5. Stereographic projections. We would like to define the conformal transformations globally on a suitable conformally flat manifold since then they will form a finite dimensional Lie group and the conformal invariance of operators will be better understood. For this reason we have to pass from the pseudo-Euclidean spaces to pseudo-spheres. Consider the pseudo-Euclidean space $\mathbb{R}^m$ with the canonical pseudo-metric described by the matrix $\mathbb{J} = \begin{pmatrix} \mathbb{I}_{m'} & 0 \\ 0 & -\mathbb{I}_n \end{pmatrix}$ and the space $\mathbb{R}^{m+2} = \mathbb{R} \times \mathbb{R}^{m'+n} \times \mathbb{R}$ equipped with the form

$$S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{J} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

This is a realization of the pseudo-Euclidean space with signature $(m' + 1, n + 1)$ and the 'light cone' $x^T S x = 0$ of all vectors with $|x|^2 = 0$ describes a quadric in the projective space $P_{m+1}(\mathbb{R})$. This quadric is called the Möbius space $S_{(m',n)}$. We shall identify the Möbius space with the pseudo-sphere $S^{(m',n)}$, at least locally.

Consider a ('finite') point $(z, y) = (z, y^1, \ldots, y^m) \in \mathbb{R}^{m+1}$, $|y|^2 + z^2 = 1$, on the (pseudo-) sphere $y^T \mathbb{J} y + z^2 = 1$. Let us define a point in $P_{m+1}(\mathbb{R})$ with homogeneous coordinates $(x^0, \ldots, x^{m+1})$

$$x^0 = \frac{1}{\sqrt{2}}(z - 1), x^{m+1} = \frac{1}{\sqrt{2}}(z + 1), x^1 = y^1, \ldots, x^m = y^m.$$

Clearly $x^T S x = z^2 - 1 + y^T \mathbb{J} y = 0$, hence we have defined a mapping Φ transforming the 'finite part' of the pseudo-sphere into $S_{(m',n)}$. We claim that this mapping is injective and in the positive definite case even bijective.

Indeed, let us take $x = (1, p, q) = (1, p^1, \dots, p^m, q) \in S_{(m',n)}$ and try to find some suitable multiple of these homogeneous coordinates to obtain the corresponding point $\Phi^{-1}(x) = (z, y)$ on the pseudo-sphere. So let us consider a multiple of the first and the last coordinates and try to find the factor so that the first two relations in the definition of Φ are satisfied: $c = \frac{1}{\sqrt{2}}(z-1)$, $cq = \frac{1}{\sqrt{2}}(z+1)$, i.e. $z = \sqrt{2}c + 1 = \sqrt{2}cq - 1$. So a good possibility seems to be $c = \frac{\sqrt{2}}{q-1}$. Since $x \in S_{(m',n)}$, we have $2q = -|p|^2$. Hence $c = \frac{-\sqrt{2}}{1+\frac{1}{2}|p|^2}$ and $z = \frac{-2}{\frac{1}{2}|p|^2+1} + 1 = \frac{\frac{1}{2}|p|^2-1}{\frac{1}{2}|p|^2+1}$. A direct evaluation shows $z^2 + c^2|p|^2 = 1$ so that we really get a point of the pseudo-sphere and we had no free choice. If the signature is $(m, 0)$, we have a global bijection (the point with $z = 1$ is obtained if we replace the roles of the first and the last homogeneous coordinate), but if the metric is indefinite, we need $\frac{1}{2}|p|^2 \neq -1$.

Every vector $p \in \mathbb{R}^m$ defines the matrix $P = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & -p^T\mathbb{J} & 0 \end{pmatrix}$ which lies in the Lie algebra $\mathfrak{o}(m'+1, n+1)$ (i.e. $P^T S + SP = 0$). Applying the exponential mapping, we obtain a matrix in $O(m'+1, n+1)$

$$\exp P = \begin{pmatrix} 1 & 0 & 0 \\ p & \mathbb{I}_m & 0 \\ -\frac{1}{2}|p|^2 & -p^T\mathbb{J} & 1 \end{pmatrix} \tag{1}$$

In this way we get a mapping $\psi\colon \mathbb{R}^m \to S_{(m',n)}$

$$p \mapsto \exp \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & -p^T\mathbb{J} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} \tag{2}$$

For all points with $|p|^2 \neq -2$ we can compose this mapping with the inverse of the above injection Φ of the pseudo-sphere and we get the so called *stereographic projection* $\varphi\colon \mathbb{R}^m \to S^{(m',n)}$

$$p \mapsto (z, y) = \left(\frac{\frac{1}{2}|p|^2 - 1}{\frac{1}{2}|p^2| + 1}, \frac{-\sqrt{2}p}{\frac{1}{2}|p|^2 + 1}\right) \in S^{(m',n)} \subset \mathbb{R}^{m+1}. \tag{3}$$

$$\begin{tikzcd} & S_{(m',n)} \\ \mathbb{R}^m \arrow[ur, hook, "\psi"] \arrow[dr, dashed, "\varphi"'] & \\ & S^{(m',n)} \arrow[uu, hook, "\Phi"'] \end{tikzcd}$$

Lemma. *The mapping φ is conformal with the corresponding 'dilatation function'* $(\lambda(p))^2 = \frac{2}{(\frac{1}{2}|p|^2+1)^2}$.

Proof. Let us write $\mathbb{R}^{m'+n+1} = \mathbb{R} \times \mathbb{R}^{m'+n}$ where the second term in the product is the pseudo-Euclidean space with signature (m', n) while the first one is the usual $\mathbb{R}$. Let a be the vector $(1, 0, \dots, 0) \in \mathbb{R} \times \mathbb{R}^{m'+n}$, i.e. $\langle a, x\rangle = 0$ if and only if $x \in \{0\} \times \mathbb{R}^{m'+n}$.

If we compose our sterographic projection with multiplication $p \mapsto -\sqrt{2}p$ we get the more usual formula for the stereographic projection. This composition corresponds to the translation of the whole 'projection hyper-plane' in $\mathbb{R}^{m+1}$ to the point $(1-\sqrt{2})a$ and taking the symmetry with respect to the origin. Both these maps are conformal, so we can work with the more usual formula

$$\bar{\varphi}(p) = \frac{2}{|p|^2+1}p + \frac{|p|^2-1}{|p|^2+1}a \tag{4}$$

in our proof. In order to prove that (4) is conform, we have to evaluate $|T_p\varphi.\xi|$ for a tangent vector ξ at a point from the domain of $\bar{\varphi}$. We have

$$\begin{aligned} T_p\bar{\varphi}.\xi &= \frac{2\xi(|p|^2+1) - 4p\langle\xi,p\rangle + 4a\langle\xi,p\rangle}{(|p|^2+1)^2} \\ |(|p|^2+1)^2 T_p\bar{\varphi}\xi|^2 &= |2\xi(|p|^2+1) - 4p\langle\xi,p\rangle + 4a\langle\xi,p\rangle|^2 = 4(|p|^2+1)^2\langle\xi,\xi\rangle. \end{aligned}$$

The dilatation for the φ in the statement of the proposition is obtained by inserting $\frac{1}{\sqrt{2}}p$ into the latter formula.

5.6. The group of conformal transformations. The Lie group $O(m'+1, n+1)$ acts transitively on the Möbius space $S_{(m',n)}$. We shall use 5.5.(2) for a representation of all local conformal transformations on the pseudo-Euclidean $\mathbb{R}^{m'+n}$ as global transformations of $S_{(m',n)}$.

(a) $A \in O(m', n)$, i.e. $|Ap|^2 = |p|^2$, yields

$$\begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ Ap \\ -\frac{1}{2}|p|^2 \end{pmatrix}$$

(b) the translation $p \mapsto p + q$ corresponds to the action of exp$Q$, cf. 5.5.(1)

$$\begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ q & \mathbb{I}_m & 0 \\ -\frac{1}{2}|q|^2 & -q^T\mathbb{J} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ p+q \\ -\frac{1}{2}(|p|^2+|q|^2+2\langle q,p\rangle) \end{pmatrix}$$

(c) the dilatation $p \mapsto \lambda p$, $\lambda \neq 0$, is expressed by

$$\begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} \mapsto \begin{pmatrix} \lambda^{-1} & 0 & 0 \\ 0 & \mathbb{I}_m & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} = \begin{pmatrix} \lambda^{-1} \\ p \\ -\frac{1}{2}\lambda|p|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \lambda p \\ -\frac{1}{2}|\lambda p|^2 \end{pmatrix}$$

(d) the inversion $p \mapsto \frac{1}{|p|^2}p$ is not defined at $|p| = 0$, but the corresponding transformation on $S_{(m',n)}$ is defined globally by

$$\begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & -2 \\ 0 & \mathbb{I}_m & 0 \\ -\frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ p \\ -\frac{1}{2}|p|^2 \end{pmatrix} = \begin{pmatrix} |p|^2 \\ p \\ -\frac{1}{2} \end{pmatrix}$$

If $|p| \neq 0$ the value equals to $x^T = (1, \frac{1}{|p|^2}p, -\frac{1}{2}|p|^{-2})$.

Using the inversion, we see that all these transformations are well defined also in the points with homogeneous coordinates starting with $x^0 = 0$.

5.7. Spheres as homogeneous spaces. The transformations 5.6.(a)–(d) generate the whole group $O(m'+1, n+1)$ and, together with the conformal sterographic projections, they define a smooth atlas and a conformal structure on $S_{(m',n)}$. Since all conformal transformations of $S_{(m',n)}$ must be locally generated by those from $O(m'+1, n+1)$, the elements from $O(m'+1, n+1)$ exhaust exactly all conformal transformations on $S_{(m',n)}$. Let us fix the point $x = (1, 0. \dots, 0) \in S_{(m',n)}$. Its isotropy group B consists of matrices of the form

$$\begin{pmatrix} a^{-1} & q & * \\ 0 & A & * \\ 0 & 0 & a \end{pmatrix}$$

where $A \in O(m', n)$, $q \in \mathbb{R}^m$, $a \in \mathbb{R}$, $a \neq 0$, and the stars indicate expressions determined by A, and q. This subgroup is called the *Poincaré conformal group.* Consequently, we have identified the pseudo-spheres (or, more precise, the Möbius spaces) with the homogeneous spaces $O(m'+1, n+1)/B$ and the canonical left actions of $O(m'+1, n+1)$ on them exhausts just all conformal transformations.

The same description with complex orthogonal groups applies to the complex confomal spheres.

5.8. The conformal structure on $S_{(m',n)}$. If we employ the stereographic projection, we can identify elements h from the Poincaré conformal group with locally defined diffeomorphisms $\alpha(h)$ on $R^{m'+n}$. By our construction and by the Liouville theorem, $\alpha(h) = \mathrm{id}_{\mathbb{R}^m}$ if and only if $j^2_0(\alpha(h)) = j^2_0 \mathrm{id}_{\mathbb{R}^m}$ and we can identify the Lie group B with a subgroup of $\mathrm{inv} J^2_x(S_{(m',n)}, S_{(m',n)})_x$, and via the stereographic projection with a subgroup in the jet group G^2_m. The situation can be described by a diagram

$$\begin{tikzcd}
O(m'+1, n+1) \arrow[rr, "?"] \arrow[dr, dashed] \arrow[ddd] & & P^2 S_{(m',n)} \arrow[ddd] \\
& P^2\mathbb{R}^m \arrow[ur, "P^2\varphi"'] \arrow[d] & \\
& \mathbb{R}^m \arrow[dl, "\varphi"'] \arrow[dr, "\varphi"] & \\
S_{(m',n)} \arrow[rr, "\mathrm{id}"'] & & S_{(m',n)}
\end{tikzcd}$$

The stereographic projection determines a locally defined map $O(m'+1, n+1) \to P^2\mathbb{R}^m$ indicated in the diagram. This map is equivariant with respect to the principal action of the Poincaré conformal group and extends to a global map. Consequently, $O(m'+1, n+1)$ can be viewed as a reduction of the second frame bundle $P^2S_{(m',n)}$. This will be of basic importance later on for linking the results obtained in the flat case with the conformal invariance on curved conformal manifolds.

5.9. Proposition. *The Lie algebra $\mathfrak{b}$ of the group $B \subset O(m'+1, n+1)$ decomposes as a sum of $\mathfrak{b}_0 = \mathfrak{co}(m', n)$ and $\mathfrak{b}_1 = \mathbb{R}^{m*}$ with the projections*

$$\begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \mapsto A + a\mathbb{I}_m \qquad \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & -\mathbb{J}q^T \\ 0 & 0 & 0 \end{pmatrix} \mapsto q$$

where $A \in \mathfrak{o}(m'n)$, $a \in \mathbb{R}$, $q \in \mathbb{R}^{m*}$.

The whole algebra decomposes as $\mathfrak{o}(m'+1, n+1) = \mathfrak{b}_{-1} + \mathfrak{b}_0 + \mathfrak{b}_1$, where $\mathfrak{b}_{-1} = \mathbb{R}^m$ corresponds to the Abelian group of the 'translations', see 5.5.(2), and this decomposition is a grading. All three summands are subalgebras, $\mathfrak{b}_{\pm 1}$ are Abelian. The remaining non-trivial commutators are $[A, A'] = AA' - A'A$, $[A, p] = Ap$, $[q, A] = qA$ and $[p, q] = pq - \mathbb{J}(pq)^T\mathbb{J} + (qp)\mathbb{I}_m$ with $A, A' \in \mathfrak{co}(m', n)$, $p \in \mathfrak{b}_{-1}$, $q \in \mathfrak{b}_1$. Further,

(1) *There is the distinguished element $E = -\mathbb{I}_m \in \mathfrak{b}_0$ satisfying*

$$\mathfrak{b}_i = \{X \in \mathfrak{o}(m'+1, n+1);\ [E, X] = iX\}, \quad i = -1, 0, 1.$$

(2) *The linear endomorphism $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ defined for all $X_i \in \mathfrak{b}_i$ by*

$$\alpha(X_{-1} + X_0 + X_1) = -X_{-1} + X_0 - X_1$$

is an ivolutive automorphism of $\mathfrak{g}$

(3) *$\langle \mathfrak{b}_{-1} + \mathfrak{b}_1, \mathfrak{b}_0 \rangle = 0$, i.e. $\mathfrak{b}_{-1}$ and $\mathfrak{b}_1$ are orthogonal to $\mathfrak{b}_0$ with respect to the Killing form*
(4) *the Killing form is zero on $\mathfrak{b}_{-1}$ and $\mathfrak{b}_1$*
(5) *$\mathfrak{b}_1$ and $\mathfrak{b}_{-1}$ are dual spaces with respect to the Killing form*
(6) *the adjoint representations of $\mathfrak{b}_0$ on $\mathfrak{b}_{-1}$ and $\mathfrak{b}_1$ are contragredient representations on the dual spaces*

Proof. The proof of the first part consists in obvious computations of the commutators and verifications that the values are in the proper subspaces. Let us show at least one case. Given $q \in \mathbb{R}^{m*}$ and $p \in \mathbb{R}^m$ we have

$$\begin{aligned} \left[\begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & -p^T\mathbb{J} & 0 \end{pmatrix}, \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & -\mathbb{J}q^T \\ 0 & 0 & 0 \end{pmatrix} \right] &= \begin{pmatrix} -qp & 0 & 0 \\ 0 & pq - \mathbb{J}q^Tp^T\mathbb{J} & 0 \\ 0 & 0 & p^Tq^T \end{pmatrix} \\ &= (pq - \mathbb{J}q^Tp^T\mathbb{J} + qp\mathbb{I}_m) \end{aligned}$$

The other cases are even easier.

Knowing the commutator relations, (1) and (2) are obvious. Since the Killing form is invariant with respect to α, we get

$$\langle X_{-1}+X_1, X_0\rangle = \langle \alpha(X_{-1}+X_1), \alpha(X_0)\rangle = -\langle X_{-1}+X_1, X_0\rangle$$

and (3) follows. We have $\mathrm{ad}X_{-1}\mathrm{ad}Y_{-1} = 0$ on $\mathfrak{b}_{-1}$ (since the value would be in $\mathfrak{b}_{-3}$) and so the Killing form must be zero on $\mathfrak{b}_{-1}$. Similarly for $\mathfrak{b}_1$.

In order to prove (5), let us assume $\langle X_{-1}, \mathfrak{b}_1\rangle = 0$. Then (3) and (4) imply $\langle X_{-1}, \mathfrak{g}\rangle = 0$ and so $X_{-1} = 0$. Analogously we proceed for $\langle X_1, \mathfrak{g}\rangle$ and this proves (5).

Since the Killing form is invariant under the action of $\mathrm{ad}X_0$, we have

$$\langle \mathrm{ad}(X_0)X_{-1}, X_1\rangle = -\langle X_{-1}, \mathrm{ad}(X_0)X_1\rangle,$$

$X_{\pm 1}\in\mathfrak{b}_{\pm 1}$, $X_0\in\mathfrak{b}_0$. This verifies (6). □

5.10. The Lie subalgebra $\mathfrak{b}_1\subset\mathfrak{b}$ corresponds in the jet picture to the kernel of the projection $G^2_m\to G^1_m$. The Lie algebras of the jet groups are the algebras of jets of formal vector fields with the bracket being the negative of the jets of the Lie brackets of the formal fields, see [Kolář, Michor, Slovák, 93, Section 13].

The Lie subgroup B_1 in G^2_m corresponding to $\mathfrak{b}_1$ is described easily using our identifications of the generators of the conformal mappings. Notice that the inversion 5.6.(d) exchanges the subgroups corresponding to $\mathfrak{b}_{\pm 1}$. Since we know that $\mathfrak{b}_{-1}$ corresponds to translations, see 5.6.(b), we get the mapping $\mathbb{R}^m\to\mathbb{R}^m$ determined by $\exp q$, $q\in\mathbb{R}^{m*}$, by composing two inversions with the appropriate translation by q:

$$x\mapsto \frac{1}{|x|^2}x\mapsto \frac{1}{|x|^2}x+q\mapsto \frac{\frac{1}{|x|^2}x+q}{|\frac{1}{|x|^2}x+q|^2} = \frac{x+|x^2|q}{1+2\langle x,q\rangle+|x|^2|q|^2}.$$

A tedious but elementary calculation shows that the first derivative at the origin is the identity while the second derivative at the origin evaluated at vectors ξ and η is $D^2(0)(\xi,\eta) = 2(\langle\eta,\xi\rangle q - \langle\eta,q\rangle\xi - \langle\xi,q\rangle\eta)$. In the usual coordinates (a^i_j, a^i_{jk}) on G^2_m this means $B_1 = \{(\delta^i_j, a^i_{jk}); a^i_{jk} = q_a g^{ai}g_{jk} - q_a g^a_k\delta^i_j - q_a g^a_j\delta^i_k,\ q_a\in\mathbb{R}^{m*}\}$ where g is the pseudo-metric in question.

Now, $B/B_1 = CO(m',n,\mathbb{R})$ and so $O(m'+1,n+1,\mathbb{R})/B_1\subset P^1S_{(m',n)}$ is the conformal structure on the pseudo-sphere in the proper sense of Definition 5.1. The above reduction of $P^2S_{(m',n)}$ to $O(m'+1,n+1,\mathbb{R})$ is the so called first prolongation of the first order $CO(m',n)$-structure, we shall give more details on this construction at the beginning of Section 9.

5.11. Remark. All the previous development can be repeated with the connected component of the unit, the subgroup $SO_0(m'+1,n+1)$, instead of $O(m'+1,n+1)$ without any essential difference.[17] Hence the oriented pseudo-spheres are the homogeneous spaces $SO_0(m'+1,n+1)/B$ (with a smaller B then above, the connected component of the unit). On the Lie algebra level, everything remains unchanged.

The above discussion on the homogeneous spaces remains also unchanged in the complex case where we do not have to distinguish the signatures. So the complex m-dimensional sphere is the homogeneous space $SO_0(m+2,\mathbb{C})/B$ or $O(m+2,\mathbb{C})/B$ where the B's are the complex conformal Poincaré subgroups.

[17] In the not positive definite case, there are four connected components, two of them form the special pseudo-orthogonal group $SO(m',n,\mathbb{R})$.

6. The first order natural operators

First of all we have to describe the natural bundles on conformal manifolds. So let us discuss briefly the linear representations of the Poincaré conformal group, i.e. the natural vector bundles in the category of conformal manifolds, cf. 2.12. Roughly speaking, the natural bundles on a manifold M with a B-structure are bundles equipped with an action of the group $\mathcal{B}_M$ of the local $\mathcal{M}f_m(B)$-isomorphisms. In our case, the Poincaré conformal group B is the group of all conformal transformations fixing a point of the sphere. In 5.1, we defined the conformal structure as the reductions of the first order frame bundles to the group $CO(m', n, \mathbb{R})$. If we define a reduction PM of the second order frame bundle P^2M to the Poincaré conformal group B, then the quotient $PM/B_1 \subset P^1M$ is a reduction to $CO(m', n, \mathbb{R})$ and the same is valid for the connected components of the units. On the other hand, the general theory of prolongations of G-structures yields that P is just the first prolongation of the latter conformal structure, see Section 9.[18] We shall see, there is a naturally defined subbundle $PM \subset P^2M$ with structure group B on each conformal manifold and so, given a representation of B, there are the corresponding bundles (associated to PM) on all conformal manifolds M.

Our general problem is *to find all linear operators transforming sections of such bundles which intertwine the actions of the conformal transformations, i.e. which are natural.*

We want also to involve the so called two-valued representations, i.e. the representations of the double covering of the Poincaré group. Of course, there is a topological obstruction to the existence of the corresponding vector bundles, but since the classification problem of natural operators is a local one, we can always restrict ourselves to manifolds with a distinguished covering of the reduction of the frame bundle, the so called spin structure. The spheres are always spin manifolds and so we can use the global formulation in the terms of homogeneous vector bundles on spheres, cf. 2.10. But having a representation of a double covering of the jet group in question has another, more unpleasant consequence. We cannot use directly our definition of the natural operators, for there is no canonical action of the conformal transformations on the sections of the bundles. Thus we have to use the definition from 2.14 which does apply. In a large extent, the latter difficulty will be avoided using the infinitesimal version of naturality.

In this section, we shall employ the classical structure theory of semisimple Lie algebras and their representations in order to describe the first order operators. Our main reference is the thin introduction [Samelson, 89], where the reader can learn quickly all necessary topics. A brief survey of some elementary concepts and results is also involved in Section 10.

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[18] The first order structures give direct access to all first jets of mappings belonging to the structure. The prolongations describe directly higher order jets of the morphisms. The conformal structures form one of very few examples where only finitely many non-trivial prolongations are available. In fact already the second prolongation is trivial which reflects the global dependence of conformal morphisms on 2-jets at a single point. The $O(m', n, \mathbb{R})$-structures have no non-trivial prolongation since the isometries are determined by the first jets at one point. But for example, the groups of morphisms of symplectic manifolds are infinite dimensional and the symplectic structures admit prolongations of any order.

6.1. Let us remind the construction from 2.11, 2.12. For each closed Lie subgroup in the jet group $B \subset G^r_m$ we obtain a category of manifolds with B-structures, $\mathcal{M}f_m(B)$. In particular there are distinguished natural principal bundles $P\colon \mathcal{M}f_m(B) \to \mathcal{PB}_m(B)$ with structure group B over the m-dimensional objects. For each linear representation $\lambda\colon B \to GL(V)$ we obtain the corresponding natural bundle F_λ. The Lie derivative of sections of natural bundles is defined for all vector fields with flows formed by morphisms of the category $\mathcal{M}f_m(B)$, the so called $\mathcal{M}f_m(B)$-fields. But the values are in the vertical bundles. If the bundles themselves are vector bundles, we recover the usual Lie derivative and it is easy to see that the linear natural operators have to commute with the Lie derivative and vice versa. For the proofs see [Kolář, Michor, Slovák, 93, Section 48] or [Cap, Slovák, 92] where the result is proved in the non-linear setting. Each natural bundle F_λ admits the so called flow operator $\mathcal{F}_\lambda$, a natural operator which transforms $\mathcal{M}f_m(B)$-fields on M into vector fields on FM. The flow of its value $\mathcal{F}_\lambda X$ is defined by the application of the functor F_λ to the flow of the $\mathcal{M}f_m(B)$-field X. If $P\colon \mathcal{M}f_m(B) \to \mathcal{PB}_m(B)$ is a natural principal bundle, then $\mathcal{P}X$ is right invariant for all $\mathcal{M}f_m(B)$-fields X.

Let us consider a linear representation λ of the Lie group B in a vector space V and the associated bundle $F_\lambda M$ to the principal bundle $p\colon PM \to M$. Let us write $\{u, v\}$ for the class in $F_\lambda M$ determined by $(u, v) \in PM \times V$. The Lie derivative of the V-valued functions on PM is defined as usual.

Lemma. *The set of all smooth section $C^\infty(F_\lambda M)$ is identified with the set of B-equivariant mappings in $C^\infty(PM, V)^B$, $s \mapsto \tilde{s}$, $s(p(u)) = \{u, \tilde{s}(u)\}$, and for all $\mathcal{M}f_m(B)$-fields $X \in \mathcal{X}(M)$ and sections $s \in C^\infty(F_\lambda M)$. The Lie derivative $\mathcal{L}_X s$ corresponds to $\mathcal{L}_{\mathcal{P}X}\tilde{s}$.*

Proof. We have only to write down explicitely the definition of the Lie derivative and to compare it with the identification from the lemma. □

6.2. The natural operators. In view of the above discussion, we can define the natural linear operators D as those systems of operators for which $D_M(\mathcal{L}_{\mathcal{P}X}\tilde{s}) = \mathcal{L}_{\mathcal{P}X}(D_M\tilde{s})$ for all sections and $\mathcal{M}f_m(B)$-fields. We get exactly the linear natural operators acting on the natural bundles on the categories over manifolds with B-structures (defined separately for each manifold), but with this formulation we are able to involve also some covering fenomena. Let us consider P and B as in 6.1, a covering $\bar{B}$ of B and two representations λ_1, λ_2 of $\bar{B}$ in V and W. Then some of the natural bundles PM can be covered by principal $\bar{B}$-bundles $\bar{P}M$. Let us consider the manifolds M together with such coverings $\bar{P}M$ as distinguished objects. Now, each $\bar{P}M$ yields the bundles $F_{\lambda_i}M$ and each $\mathcal{M}f_m(B)$-field X determines a unique right invariant lift, denoted by the same symbol $\mathcal{P}X$, on $\bar{P}M$. Hence in this setting we can define natural operators between bundles corresponding to representations of the finite dimensional coverings of B. Of course, such operators need not to be defined on all $\mathcal{M}f(B)$-objects M, they are well defined only on those ones where the coverings $\bar{P}M$ do exist, cf. 2.14.

Definition. Let $\lambda_1 : \bar{B} \to GL(V)$, $\lambda_2 : \bar{B} \to GL(W)$ be finite dimensional linear representations. A system of local operators $D_M : C^\infty(F_{\lambda_1}M) \to C^\infty(F_{\lambda_2}M)$, $M \in \mathrm{Ob}\mathcal{M}f_m(B)$, is called an *infinitesimally natural operator* if and only if $D(\mathcal{L}_{\mathcal{P}X}\tilde{s}) =$

$\mathcal{L}_{\mathcal{P}X}(D\tilde{s})$ for all $\mathcal{M}f_m(B)$-fields X on M, and $D_U(s|U) = D_M(s)|U$ for all section $s \in C^\infty(F_{\lambda_1}M)$ and open submanifolds $U \subset M$.

In the sequel, we shall write B for the connected component of the unit in the Poincaré conformal group and G for the connected component of the unit in $O(m'+1, n+1)$ or their double coverings. We have described in detail the Lie algebra $\mathfrak{g}$ of G, $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1$ in the last section. We have seen that the subgroup of conformal transformations of the sphere $S_{(m',n)}$ fixing a point can be identified with a subgroup in the second jet group G^2_m. As mentioned above, there is the natural principal bundle functor $P \subset P^2$ on the conformal manifolds and each representation $\lambda\colon B \to GL(V)$ gives rise to a vector bundle functor F_λ on $\mathcal{M}f_m(B)$, $FM = PM \times_\lambda V$. The representations of its double-covering will be referred to as two-valued representations of B, the classical terminology which is useful since we shall work on the level of Lie algebras.

In order to get general information on the invariant operators, we have to restrict our class of natural vector bundles to those coming from (finite dimensional) irreducible representations of B. Unfortunately, we exclude a lot of representations of B which are not completely reducible, but we still cover all first order bundles. The normal subgroup B_1 corresponding to $\mathfrak{b}_1$ is commutative and B/B_1 is isomorphic to $SO_0(m', n, \mathbb{R}) \times \mathbb{R}^\bullet$ (or its double-covering), where $\mathbb{R}^\bullet$ means the commutative multiplicative group in $\mathbb{R}$ (remember, $SO_0(m', n, \mathbb{R})$ denotes the connected component of the unit for all signatures of the metrics). On the Lie algebra level, we get the induced representation $\lambda' = T_e\lambda$ and the ideal $\mathfrak{b}_1$ acts by nilpotent endomorphisms by the Engel's theorem ($\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$ is the Levi decomposition). By the irreducibility, the action of $\mathfrak{b}_1$ must be trivial. Thus, λ is a trivial extension of an irreducible representation λ_1 of $SO_0(m', n, \mathbb{R}) \times \mathbb{R}^\bullet$ and the vector bundles in question are associated bundles $P_1 \times_{\lambda_1} V$, where $P_1 = P/B_1 \subset P^1$ is a sub bundle in the linear frame bundle with structure group $SO_0(m', n, \mathbb{R}) \times \mathbb{R}^\bullet$. On the spin manifolds M, the principal bundle P_1M lifts to $\tilde{P}_1M$ with structure group $Spin(m', n, \mathbb{R}) \times \mathbb{R}^\bullet$ and there is the associated vector bundle $\tilde{P}_1M \times_\lambda V$ for each two-valued representation λ of $SO_0(m', n, \mathbb{R}) \times \mathbb{R}^\bullet$.

6.3. The conformal weight. The reductive part $\mathfrak{b}_0$ in the Levi decomposition $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$ decomposes further to the center and the semisimple part, $\mathfrak{b}_0 = \mathbb{R} \oplus \mathfrak{o}(m', n, \mathbb{R})$. So an irreducible representation $\bar{\lambda}$ of $\mathfrak{b}_0$ (i.e. also of $\mathfrak{b}$) is determined by an element α from the dual of the center $\mathbb{R}^*$ and a dominant integral weight λ for $\mathfrak{o}(m', n, \mathbb{R})$. The element $-\alpha$ is a real number called *the conformal weight* of the irreducible representation $\bar{\lambda}$. We shall write V_λ for the irreducible $\mathfrak{o}(m', n, \mathbb{R})$-module corresponding to the given dominant weight λ and $V_\lambda(\alpha)$ will denote the irreducible representation with the conformal weight α. The action of $t + A \in \mathbb{R} \oplus \mathfrak{o}(m', n, \mathbb{R})$ on $v \in V_\lambda(\alpha)$ is $v \mapsto -\alpha t.v + (A.v)$ where the dot denotes the action of $\mathfrak{o}(m', n, \mathbb{R})$ and the multiplication by a scalar is without notation. On the Lie group level we get $(tA).v = t^{-\alpha}(A.v)$. The sign convention is used so that the conformal weight of the metrics is two. This enables the usual identification of sections of the bundles with a conformal weight α with the sections of the corresponding bundles on the underlying Riemannian manifolds (without the conformal weight) which depend on the chosen metric and 'rescale' by multiplication by the function f^α if the metric is rescaled by f^2.

For example, let us consider the standard $GL(m,\mathbb{R})$-representations $\mathbb{R}^m$, $\mathbb{R}^{m*}$, $\Lambda^{m-1}\mathbb{R}^*$. With the restrictions of the representations to the pseudo orthogonal groups, all these $O(m',n,\mathbb{R})$-representation are equivalent. However, the restrictions to the conformal groups yield representations with the conformal weights -1, 1 and $m-1$.

6.4. Representations of the conformal groups. In 10.10 and 10.11, we find the description of the irreducible representations of the complex orthogonal algebras in the terms of the dominant weights. There is a general theorem, [Zhelobenko, 70, p. 526] which enables to use this description also in the real case.

Let G be a semisimple real connected Lie Group and $G_{\mathbb{C}}$ be its connected complex form. Then each irreducible finite dimensional representation of G is uniquely determined (up to equivalence) by one of the dominant weights of a covering of $G_{\mathbb{C}}$.

If we start with a concrete dominant weight, we take the corresponding complex representation space, we view this space as a complexification of a real one and restrict the action of the complex group to the real subgroup. It is even possible to verify directly that we get irreducible representations in this way using the method mentioned in the footnote in 3.13 and Lemma 3.16 where we proved that $SO_0(m',n,\mathbb{R})$ is birationally isomorphic to an affine space.

Of course, there is a difference concerning the possible conformal weights. If dealing with representations of the Lie algebras, they are quite arbitrary elements in the center of $\mathfrak{co}_0(m',n,\mathbb{K})$, hence arbitrary real or complex numbers. However only in the real case all of them also exponentiate to representations of the connected components of the unit in $CSO_0(m',n,\mathbb{R})$.

6.5. Remark. For many Lie subgroups $B \subset G^r_m$, the category $\mathcal{M}f_m(B)$ of manifolds with B-structures involves enough local isomorphisms to be locally homogeneous (i.e. there is a local model for all objects and morphisms) and all local isomorphisms belong to flows of $\mathcal{M}f_m(B)$-fields. In such a situation, the infinitesimally natural operators are systems of operators commuting with the actions of the morphisms, hence the usual natural operators. For detailed discussion see [Cap, Slovák, 92].

Unfortunately, dealing with the category of conformal manifolds, we are very far from the latter situation. On the contrary, the manifolds (generically) admit no conformal vector fields, and the objects are highly non-homogeneous. Thus, our definition of infinitesimally natural operators yields systems of operators which commute with the actions of the morphisms on subcategories which are homogeneous enough, e.g. on the locally conformally flat manifolds.

6.6. The first order operators on conformal manifolds. In the rest of this section, we shall solve the following problem: *For a given dimension m find all non-zero first order natural operators $D\colon F_{\lambda,\alpha} \to F_{\rho,\beta}$ between the vector bundles corresponding to dominant weights λ, ρ of $\mathfrak{o}(m',n,\mathbb{R})$ and conformal weights α, β.*

So let us fix the weights λ, ρ, α, β and write $E \to S_{(m',n)}$, $F \to S_{(m',n)}$ for values of the corresponding natural bundles $E_{\lambda,\alpha}$, $F_{\rho,\beta}$ on the pseudo-spheres. Let us notice that the pseudo-spheres are always spin manifolds, so that this is possible for all dominant weights. In view of the general theory of natural bundles, the description of all infinitessimally natural operators on the pseudo-spheres and their open submanifolds yields the description of all natural operators on the conformally flat Riemannian manifolds.

In general, it is a difficult problem to find all possible extensions of a given operator to the whole category of conformal Riemannian manifolds, we shall touch it in Section 9. However, dealing with first order operators only, the situation is very simple and we can give a complete answer just now.

Our first observation will be that each first order operator $D\colon C^\infty E \to C^\infty F$ on the pseudo-sphere with the flat conformal structure which intertwines the action of the conformal isomorphisms determines a natural operator defined on the whole category of conformal manifolds. In view of this fact, we shall often refer to D as to a natural operator on the conformal manifolds in the sequel.

6.7. Proposition. *Every infinitesimally natural first order operator $D\colon C^\infty E \to C^\infty F$ is invariant with respect to the whole group $SO_0(m'+1, n+1)$ of conformal transformations and extends to a natural operator $\tilde{D}$ on the whole category of oriented conformal manifolds and their morphisms.*[19]

Proof. Let us write briefly G for the connected component of the unit of the pseudo-orthogonal group or the spin group. The pseudo-spheres are then homogeneous spaces G/B. As discussed in 2.10, the left action of $h \in G$ on the sections (viewed as mappings in $C^\infty(G, V_\lambda(\alpha))^B$) is given by the left multiplication by the inverse h^{-1}. This action coincides with the induced action of the principal bundle morphism h (acting by left multiplication) on the sections viewed as elements in $C^\infty(E)$, see 2.10. Thus, given a flow of a conformal vector field (i.e. a one-parameter subgroup in G) its action on the sections is just the left multiplication by $\exp tX$ for some element X in the Lie algebra of G. If we differentiate this action, we get just the Lie derivative with respect to $-X$, where X stands for the right invariant vector field now. So the flow lifts to a one parametric subgroup of principal bundle morphisms which are just the flow of the above right invariant vector field. Hence the infinitesimal invariance is equivalent to the usual invariance with respect to the whole group G, for the image of the exponential mapping generates the whole connected component of the unit.[20]

Now, assume we have found an infinitesimally invariant operator $D\colon C^\infty E \to C^\infty F$. We have to prove that D extends uniquely to the whole category of conformal manifolds, i.e. D determines the linear first order operators $D_M : C^\infty F_\lambda(\alpha)M \to F_\rho(\beta)M$ for all conformal manifolds M and, moreover, if we deal with representa-

[19]The description of the morphisms is a little unpleasant, in general. In the positive definite case, the latter are just the local conformal isomorphisms keeping the orientations, so there are no problems. However, in the case of a general signature, there are four components of the unit and two of them are described by the value of the determinant (i.e. they form $SL(m,\mathbb{R}) \cap O(m', n, \mathbb{R})$) and they are further distinguished by certain subdeterminants.

[20]The equivalence of infinitesimal invariance and the usual invariance remains valid also for the B-structures with infinite dimensional groups of automorphisms, see [Cap, Slovak, 92]. In fact, the above arguments involve a lot of identifications. A geometric definition of the Lie differentiation leads to an operation with values in the vertical bundles (since the Lie derivative should have values in the 'tangent space to the space of sections' being itself a derivative of curves) and using this definition, the whole problem becomes very clear, provided the dimension of the group of transformations in question is finite. In the cited paper, the main point is to apply suitable analytical tools in order to reduce the problem to a finite dimensional one.

tions of the pseudo-orthogonal groups then the latter operators have to intertwine the actions of the local conformal isomorphisms while in the spin case they have to intertwine with the acton of the coverings, see 2.14.

Each first order operator D_M factorizes through a mapping D^1_M defined on the first jets of sections, see 2.5. Let us fix a point $x \in M$ and consider the normal coordinates with respect to one of the metrics in the conformal class. If we compose the latter mapping with the inverse to the stereographic projection, we get a locally defined mapping $\varphi\colon S_{(m',n)} \to M$ with $\varphi(0) = x$, where $0 \in G/B = S_{(m',n)}$ is the point represented by the unit $e \in G$, and the second jet $j^2_0\varphi$ transforms the flat conformal metric on the sphere into the first jet of the conformal metric on M at x. Having φ we also have the principal fiber bundle morphism $P^1\varphi\colon P^1S_{(m',n)} \to P^1M$ and we can choose its covering $\tilde{P}^1\varphi$, if necessary. The restrictions of the first jet prolongations of the induced mappings on the associated bundles to the fibers over 0 depend only on $j^2_0\varphi$. Hence we can transform the first jets of sections of the homogeneous bundles on the pseudo-sphere at 0 into first jets of sections of the corresponding bundles on M at x using the second jet $j^2_0\varphi$ only, see the diagram below. The induced mappings Φ_i define the restriction of the mapping D^1_M to the fiber over x which also depends only on the second jet of φ at 0 (and our choice of the covering if any). If we choose another $\bar\varphi$ instead of φ with $\bar\varphi(x) = 0$ and $\bar\varphi$ transforming the first jets of the conformal metric on M into the flat conformal metric on the sphere, then their second jets differ by a jet of a conformal transformation on $S_{(m',n)}$ (more explicitly, by a left action of an element from G). Since the operator $D_{S_{(m',n)}}$ is a first order operator which is invariant with respect to the action of G by the first part of the proof, the whole mapping $D_{S_{m',n}}$ is completely determined by the restriction of the induced mapping on the first jets of sections to the fiber over zero. Thus every choice of φ leads to the same mapping D^1_M on the fiber over x and we have got a well defined first order operator D_M on all confomal Riemannian (spin) manifolds M.

On the other hand, the action of an arbitrary local transformation $f\colon M \to N$ on the first jets of sections of $F_\lambda(\alpha)M$ depends on the second jets of f in the underlying points and so the action of each local conformal transformation $f\colon M \to N$ (or the appropriate covering in the spin case) is reflected pointwise as an action of a conformal mapping on the sphere in a similar way, see the diagram below. Consequently, $D_{S_{(m',n)}}$ extends canonically to a system of first order operators D_M invariant with respect to all local conformal isomorphisms.

$$\begin{tikzcd}
J^1_0E \arrow[rrr, dashed, "\Phi_1"] \arrow[dr] \arrow[dd, "D_{S_{(m'n)}}"'] & & & J^1_x(F_{\lambda,\alpha}M) \arrow[rr, "J^1F_{\lambda,\alpha}f"] \arrow[dl] \arrow[dd, dashed, "D_M"] & & J^1_{f(x)}(F_{\lambda,\alpha}N) \arrow[dl] \arrow[dd, dashed, "D_N"] \\
& S_{(m',n)} \arrow[r, "\varphi"] & M \arrow[rr, "f"] & & N & \\
F_0 \arrow[ur] \arrow[rrr, dashed, "\Phi_2"] & & & (F_{\rho,\beta}M)_x \arrow[ul] \arrow[rr, "F_{\rho,\beta}f"] & & (F_{\rho,\beta}N)_{f(x)} \arrow[ul]
\end{tikzcd}$$

□

6.8. Remarks. We can formulate the first part of the above proposition for the invariant operators on homogeneous bundles, cf. 2.10. Then the first part of the proof shows that the infinitesimal naturality is equivalent to the invariance of the operators with respect to the action of the group elements by the left multiplication and the proof goes through for every connected finite dimensional Lie group G and its closed Lie subgroup B. Such operators are usually called the *translation invariant operators on homogeneous vector bundles.*

In the other part of the proof, we found certain canonical extension of a given translation invariant operator to the whole category of conformal manifolds. But we have not mentioned any uniqueness. If we forget about the spin cases, we can formulate the whole naturality problem for operators on natural bundles over the whole category of m-dimensional manifolds, we add the metrics as additional arguments and the conformal invariance is then reflected as a special kind of homogeneity in the metric argument (cf. Section 4). From this point of view, the above uniqueness problem reads: How far is the natural operator determined by its restriction to the conformally flat metrics on $\mathbb{R}^m$? For higher order operators, even the existence problem of such an extension has not been solved yet in general.

6.9. The symbols. By the definition, the infinitesimally conformally invariant first order operators on the pseudo-spheres $D_{S_{(m',n)}}$ are in bijection with $\mathfrak{g}$-equivariant mappings $D\colon (J^1E)_0 \to F$ on the fiber over $0 \in S_{(m',n)}$, see 2.6. The latter vector space splits as a sum of the representation space $V = V_\lambda(\alpha)$ and $V_1 = V \otimes \mathbb{R}^{m*} = V \otimes (\mathfrak{g}/\mathfrak{b})^*$, we shall write $J^1_0 E = V \oplus V_1$.

Recall from 2.9 that there is the exact sequence

$$0 \to V \otimes (\mathfrak{g}/\mathfrak{b})^* \xrightarrow{i} V \oplus V_1 \xrightarrow{\pi^1_0} V \to 0 \tag{1}$$

and the composition $D \circ i$ defines the symbol of D which is equivariant too. We have seen in 2.16 that in the Riemannian case each equivariant symbol is a symbol of a natural operator and it follows from the results of Section 4 that all first order natural operators on Riemannian manifolds are obtained as composition of the first covariant derivative and an operator of order zero in the covariant derivative but of an arbitrary order in the metric itself.

The conformal situation is quite similar in the first order case, however there are much more bundles but less operators. Each conformally invariant linear operator is clearly invariant with respect to all isometries of any metric in the class. We shall distinguish some of these Riemannian invariant operators, we shall show that there are uniquely defined conformal weights of the bundles for which we get conformally invariant operators and we shall prove that there are no other invariant operators on the pseudo-spheres.

Let us write $E_\lambda(\alpha)$ for the homogeneous vector bundle over pseudo-sphere corresponding to the dominant weight λ of $\mathfrak{o}(m',n)$ and conformal weight α. The symbol V_λ will denote the corresponding $\mathfrak{o}(m',n)$-module. For further notation concerning the weights see the Appendix.

6.10. Theorem [Fegan, 76]. *Let λ be a dominant weight of $\mathfrak{o}(m',n)$, $m'+n=m$, and let $\mathbb{R}^{m*} \otimes V_\lambda = \sum_\rho V_\rho$ be the decomposition into irreducible representations* *with dominant weights ρ of $o(m', n)$. Let us define*

$$\alpha(\lambda, \rho) = \frac{1}{2}(m-1) + \langle \rho, 2\delta + \rho \rangle - \langle \lambda, 2\delta + \lambda \rangle$$

where δ is half the sum of all positive roots of $\mathfrak{o}(m', n)$ (equivalently δ is the lowest form, i.e. the sum of all fundamental forms) and $\langle\ ,\ \rangle$ is the Killing form.

Then, beside the zero operators and the constant multiples of the identities, all linear infinitesimally natural first order operators which are defined on the sections of the vector bundles $E_{\lambda,\alpha}$ with some conformal weight α are given by the projections

$$\sigma_\rho \circ \nabla \colon C^\infty(E_\lambda(\alpha)) \xrightarrow{\nabla} C^\infty(\mathbb{R}^{m*} \otimes E_\lambda(\alpha)) \xrightarrow{\sigma_\rho} C^\infty(E_\rho(\beta))$$

of the first covariant derivatives with respect to an arbitrary metric from the conformal class onto the irreducible components V_ρ, and the conformal weight of $E_{\lambda,\alpha}$ is then $\alpha = \alpha(\lambda, \rho)$, while the conformal weight of $E_\rho(\beta)$ equals to $\alpha(\lambda, \rho) + 1$.

Furthermore, each irreducible component V_ρ has multiplicity one and all the dominant weights ρ are listed below:

(i) *If $m = 2l$, then $\rho = \lambda \pm e^i$, $1 \le i \le l$.*
(ii) *If $m = 2l+1$ and e^l appears in λ with a non-zero coefficient, then $\rho = \lambda \pm e^i$, $1 \le i \le l$, or $\rho = \lambda$.*
(iii) *If $m = 2l+1$ and e^l does not appear in λ, then $\rho = \lambda \pm e^i$, $1 \le i \le l-1$, or $\rho = \lambda + e^l$.*

For the notation concerning the weights see 10.10. Let us notice that the weights $\pm e^i$ are (possibly) not dominant, while the resulting ρ must be dominant and so only some values of i are allowed for each given λ.

6.11. Remarks. We shall present concrete examples in 6.21 and 6.22, in fact we will specify the latter theorem for all fundamental weights λ.

As we have seen, the operators $D_{S_{(m',n)}}$ extend canonically to natural operators defined for all conformal manifolds. Of course, there are operators like the tensor product with conformal curvature which cannot appear in our list since they are zero on the conformally flat manifolds.

Even for the integral weights, we cannot treat the problem in the same manner as for Riemannian manifolds in Section 4, since only very specific conformal weights allow the existence of the operators. A possibility to overcome this difficulty is to incorporate the general conformal weights as certain homogeneity condition (linking the argument of our linear operation and the metric) into the concept of the natural operators. This is the point of view adopted by many authors, see e.g. [Ørsted, 81], [Branson, 85], [Wünsch, 86].

6.12. The complex case. We shall see that the description of all infinitesimally natural operators on the homogeneous complex vector bundles on the complex spheres coincides with the real situation. In fact we shall prove both theorems together. We shall keep the same notation as in the real case for the complex bundles. Theorem 6.10 remains true without any change, i.e. all operators result from decompositions of the target space of the complex Riemannian covariant derivative, $\mathbb{C}^{m*} \otimes V_\lambda = \sum_\rho V_\rho$, into complex irreducible representations with dominant weights ρ of $o(m', n)$ (which is in fact the same as in the real case) and the conformal weights are prescribed by the same formula.

6.13. Idea of the proof. The whole proof will need several lemmas, but the main idea is quite simple. Let us consider some infinitesimally natural operator D and let us come back to the brief notation from 6.6 and 6.9, so that $V_\lambda(\alpha) = V$, $E_\lambda(\alpha) = E$, $J^1_0 E = V \oplus V_1$ and write $V_\rho(\beta) = W$.

Consider an equivariant symbol mapping $\sigma\colon V_1 = (\mathfrak{g}/\mathfrak{b})^* \otimes V \to W$. If we dualize the sequence 6.9.(1) and σ, we get

$$\begin{tikzcd}
0 & \mathfrak{g}/\mathfrak{b} \otimes V^* \arrow[l] & (J^1_0 E)^* \arrow[l, "i^*"'] & V^* \arrow[l, "(\pi^1_0)^*"'] & 0 \arrow[l] \\
& W^* \arrow[u, "\sigma^*"] \arrow[ur, dashed, "D^*"] & & &
\end{tikzcd}$$

We have a non-trivial action of $\mathfrak{b}_1$ on the term in the middle of the row, but $\mathfrak{b}_1$ acts trivially on the three remaining non-zero terms. Since W, and so also W^*, is irreducible, σ^* must be a linear combination of embeddings of $\mathfrak{co}(m', n)$-invariant linear subspaces. Since D^* is $\mathfrak{g}$-equivariant, the image of D^* must be an invariant subspace with a trivial action of $\mathfrak{b}_1$. Now it is easy to read from the diagram the conditions for σ being a symbol of an invariant operator. However, we shall proceed in a more direct way:

If there is a non-zero element $y \in (J^1_0 E)^*$ with the trivial action of the whole $\mathfrak{b}_1$, then the whole linear subspace generated by the orbit $\mathfrak{b}_0.y$ consists of points with the trivial action of $\mathfrak{b}_1$. We assume that W is irreducible and so D^* must be a linear combination of embeddings of irreducible components. Since the conformal weight of the action on $(V_1)^*$ is by one less then that on V^*, we have either $y \in V^*$ or $y \in (V_1)^*$. The first possibility yields the constant multiples of the identity operator, for both V and W are irreducible. Hence we have got: *the existence of a non-trivial linear infinitesimally conformally invariant operator D is equivalent to the existence of a vector $y \in (V_1)^*$ with a trivial action of $\mathfrak{b}_1$.*

If we deal with a concrete bundle $E = E_{\lambda,\alpha}$, then this is a very good starting point to find all operators defined on E. Indeed, it is enough to find all y's with trivial actions of $\mathfrak{b}_1$ which are at the same time highest weight vectors for $\mathfrak{o}(m', n, \mathbb{R})$. The latter means, we have a rather explicit system of equations for such y's. Moreover, this point of view restricts the whole proof to certain discussion on the highest weight vectors. It is convenient to prove the theorem in the complex case and specialize at the very end to the real case. In particular we sahll not need to take care of the signature of our metric.

For given concrete bundles we even do not have to insist on the irreducibility of the representations. However in our general setting we have to proceed more intricate than to use directly the above idea.

6.14. The action on the first jets. For every $\mathfrak{b}/\mathfrak{b}_1$-module E there is a very simple formula for the action of $\mathfrak{g}$ on the dual of the first jet space. Notice, we do not require that the representation is irreducible here. In fact, this is a very special case of the identification of $(J^\infty_0 E)^*$ with the universal enveloping algebra of $\mathfrak{g}$ which we shall use heavily in Section 8.

Lemma. *We have $(J^1_0 E)^* = V^* \oplus (\mathfrak{b}_{-1} \otimes V^*)$ with the action of $\mathfrak{b}_1$ on the second summand given by $X.(a\otimes y) = -[X,a].y \in V^*$, while the action of $\mathfrak{b}_0$ is $X.(a\otimes y) = -[X,a]\otimes y + a\otimes X.y$.*

Proof. Clearly $J^1_0 E = V \oplus (\mathfrak{b}_1^* \otimes V)$ as a vector space. The sections can be considered as B-equivariant mappings $s\colon G \to V$, hence also $\mathfrak{b}$-equivariant. The jets from the fiber in question are then identified with the expressions $j^1_e s$, $e \in G$ being the unit. The induced action of $\mathfrak{b}$ must be also respected. As derived in 6.7, the action of $X \in \mathfrak{g}$ is given by the Lie derivative $\mathcal{L}_{-X}$ with respect to the right invariant vector field X on G.

We shall identify an element $a\otimes y \in (V_1)^*$ with the linear functional defined by $\langle a\otimes y, j^1_e s\rangle = \langle \mathcal{L}_a s(e), y\rangle$. Now it is easy to express the action of $\mathfrak{b}$:

$$\begin{aligned}\langle X.(a\otimes y), j^1_e s\rangle &= -\langle a\otimes y, X.j^1_e s\rangle = -\langle a\otimes y, j^1_e(\mathcal{L}_{-X} s)\rangle\\ &= \langle \mathcal{L}_a\mathcal{L}_X s(e), y\rangle = \langle \mathcal{L}_X\mathcal{L}_a s(e), y\rangle + \langle \mathcal{L}_{[a,X]} s(e), y\rangle\\ &= \langle \mathcal{L}_a s(e), X.y\rangle - \langle \mathcal{L}_{[X,a]} s(e), y\rangle.\end{aligned}$$

If $X \in \mathfrak{b}_1$, then its action on y is zero and $[X,a] \in \mathfrak{b}_0$ and we get the first formula. Similarly we get the other expression for $X \in \mathfrak{b}_0$. $\square$

6.15. Let us pass to the complex setting now. The action of the kernel $\mathfrak{b}_1$ can be written as a linear mapping $\varphi\colon \mathfrak{b}_1 \otimes \mathbb{C}^m \otimes V^* \to V^*$. Since $\mathfrak{b}_1 = \mathbb{C}^{m*}$, this gives rise to the induced linear mapping $\psi\colon \mathbb{C}^m \otimes V^* \to \mathbb{C}^m \otimes V^*$.

Lemma. *There is a non-zero element in $(V_1)^*$ with trivial action of $\mathfrak{b}_1$ if and only if ψ is singular.*

Proof. Notice $\psi(Y)(X) = \varphi(X\otimes Y)$. $\square$

6.16. Lemma. *It holds $\psi = -\alpha\mathbb{I}_m - B$, where α is the conformal weight of E and B is defined by $B(a\otimes y) = \sum_k e_k \otimes (ae^k - \mathbb{J}(ae^k)\mathbb{J}).y$.*

Proof. We have first to work out the formula for φ. This is easy using Lemma 6.14 and the description of $\mathfrak{g}$ from 5.9. For all $X \in \mathbb{C}^{m*}$, $a \in \mathbb{C}^m$, $y \in V^*$

$$\varphi(X\otimes a\otimes y) = -[X,a].y = -(Xa\mathbb{I}_m + (aX - \mathbb{J}(aX)^T\mathbb{J})).y$$

This yields

$$\begin{aligned}\psi(e_i\otimes y)(e^k) &= -\delta^k_i \mathbb{I}_m.y - (e_ie^k - \mathbb{J}(e_ie^k)^T\mathbb{J}).y\\ \psi(e_i\otimes y) &= -\alpha(e_i\otimes y) - \sum_k e_k\otimes(e_ie^k - \mathbb{J}(e_ie^k)\mathbb{J}).y \quad \square\end{aligned}$$

6.17. Lemma. $B = C_{\mathbb{C}^m} \otimes 1 + 1 \otimes C_{V^*} - C_{\mathbb{C}^m \otimes V^*}$. *Here C_Y stands for the Casimir operator of an $\mathfrak{o}(m)$-module Y.*

Proof. We shall verify the formula for the even dimension $m = 2p$, the odd case is analogous. Let us express $B(e_j \otimes y)$ in terms of the matrices E_{ij} and try to get an expression in the action of the root elements $h_{\omega^k \pm \omega^l}$, cf. 10.10. We write $h_k = E_{2k-1,2k-1} - E_{2k,2k}$ for the orthogonal basis of the Cartan algebra. We have $h_{\omega^k \pm \omega^l}.e_i = 0$ for nearly all j and an elementary (but long) computation leads to ($\mathbb{J}$ is now the symmetric matrix used in 10.10)

$$B(e_j \otimes y) = \sum_{k=1}^{m} e_k \otimes (E_{jk} - \mathbb{J} E_{kj} \mathbb{J}).y = \sum_{\substack{\omega^l \pm \omega^k \\ l>k}} h_{\omega^l \pm \omega^k}.e_j \otimes h_{\omega^k \pm \omega^l}.y +$$
$$+ \sum_{\substack{\omega^l \pm \omega^k \\ l<k}} h_{\omega^l \pm \omega^k}.e_j \otimes h_{\omega^k \pm \omega^l}.y + \sum_{k=1}^{p} h_k.e_j \otimes h_k.y$$

The Killing form $\langle\ ,\ \rangle$ on the dual to the real part of the Cartan subalgebra $\mathfrak{h}_0 = \mathbb{R}^p$ is the standard Euclidean scalar product with factor $-\frac{1}{2}$. Hence by the definition, our root elements satisfy $\langle h_{\omega^k \pm \omega^l}, h_{\omega^l \pm \omega^k} \rangle = 1$ and $\langle h_k, h_k \rangle = -2$. Thus the root elements $h_{\omega^k \pm \omega^l}$ together with the multiples $\frac{1}{\sqrt{2}} h_k$ form two dual bases of the Lie algebra with respect to the Killing form. In view of this choice of dual bases A_i, B_i, the above formula for B reads $B(a \otimes y) = -\sum_i (A_i.a \otimes B_i.y + B_i.a \otimes A_i.y)$.

The Casimir operator of a representation φ (one of the possible definitions) is given by the action of an arbitrary pair of dual basis A_i, B_i through $\sum \varphi(A_i) \circ \varphi(B_i)$. This is independent of our choice of the basis, see e.g. [Samelson, 89, p. 120]. By the definition of the tensor product of representations we get

$$\begin{aligned} C_{\mathbb{C}^m \otimes V^*}(a \otimes y) &= \sum_i A_i B_i (a \otimes y) \\ &= \big(\textstyle\sum_i A_i B_i.a\big) \otimes y + a \otimes \big(A_i B_i\big).y + \textstyle\sum_i (A_i.a \otimes B_i.y + B_i.a \otimes A_i.y) \\ &= (C_{\mathbb{C}^m} \otimes 1)(a \otimes y) + (1 \otimes C_{V^*}) - B(a \otimes y) \end{aligned}$$

and the lemma is proved. □

A classical result states, [Samelson, 89, p. 121]

6.18. Proposition. *The Casimir operator of the irreducible representation corresponding to a dominant weight λ is $C_\lambda = \langle \lambda, \lambda + 2\delta \rangle$ where 2δ is the sum of all positive roots and $\langle\ ,\ \rangle$ is the Killing metric.*

6.19. Corollary. $C_{\mathbb{C}^m} = -\frac{1}{2}(m-1)$.

Proof. As mentioned in the formulation of Theorem 6.10, half the sum of all positive roots equals to the sum of all fundamental forms (the so called lowest form), see [Samelson, p. 91]. Hence we can compute: If $m = 2l+1$, then twice the lowest form equals $2\delta = (2l-1)e^1 + (2l-3)e^2 \cdots + e^l = (m-2)e^1 + \cdots + (m-2l)e^l$ and for $m = 2l$ we get (surprisingly) the same $2\delta = (2l-2)e^1 + (2l-4)e^1 + \cdots + 2e^{l-1} =$

$(m-2)e^1 + \cdots + (m-2l)e^l$. The dominant form corresponding to $\mathbb{C}^m$ and $\mathbb{R}^m$ is e^1 and the Killing form differs by the factor $-\frac{1}{2}$ from the Euclidean one. Hence $C_{\mathbb{C}} = \langle e^1, e^1 + 2\delta \rangle = \langle e^1, e^1 + (m-2)e^1 \rangle = -\frac{1}{2}(m-1)$. $\square$

Now we are able to prove the most of Theorem 6.10. The operator $D\colon C^\infty E \to C^\infty F$ must be invariant with respect to the isometries of each of the metrics from the conformal class. In view of the discussion from Section 4, D must be expressed by means of the first covariant derivative only (every curvature term would kill the operator on the Euclidean space). Hence it must be determined by some projection of V_1 onto an irreducible component corresponding to a dominant weight ρ. Such a projection gives rise to an invariant operator (i.e. we are able to find suitable conformal weights for the bundles) if and only if the restriction of ψ to this component is singular. The Casimir operator $C_{\mathbb{C}^m \otimes V^*}$ is constant on the irreducible components and the mapping ψ is singular if and only if $-\alpha$ is an eigen value of B by Lemma 6.16. The latter means $-\alpha = C_{\mathbb{R}^m} \otimes 1 + 1 \otimes C_\lambda - C_\rho = -\frac{1}{2}(m-1) + \langle \lambda, 2\delta + \lambda \rangle - \langle \rho, 2\delta + \rho \rangle$ by Lemma 6.17. This is the formula for the conformal weights in Theorem 6.10.

6.20. The last claim we need for the proof of Theorem 6.10 is that each dominant weight ρ which appears in $\mathbb{R}^m \otimes V$ has multiplicity one and we have to find all of them. We shall use the Klimyk's formula, see [Samelson, 89, p. 128], and since we know all weights of $\mathbb{R}^m$ this happens to be rather easy.

Our notation will slightly differ from that in Samelson. Let us denote A_ν the operator on the weights given by $A_\nu(\mu) = \sum_{s \in W} (\operatorname{sgn} s) \delta^\mu_{s(\nu)}$ where the sum goes over the Weyl group and the Kronecker δ symbol is zero or one as usual. By the definition, $A_\nu(\mu) = \operatorname{sgn} s A_{s(\nu)}(\mu) = \operatorname{sgn} s A_\nu(s(\mu))$ and $A_\nu \neq 0$ if and only if ν is regular, i.e. it cannot belong to one of the walls of the Weyl chambers (if ν is regular, then all elements $s(\nu)$, $s \in W$, are distinct, but if ν is on a wall, then there is s with $\operatorname{sgn} s = -1$ and $s(\nu) = \nu$). The functionals A_ν with ν dominant are called the elementary alternating functionals.

Consider now two dominant weights λ_1, λ_2 and the decomposition of the tensor product $V_{\lambda_1} \otimes V_{\lambda_2} = \sum_\rho n_\rho V_\rho$ where n_ρ are the multiplicities and we sum over all dominant weights ρ. The Klimyk's formula reads:

Proposition. *For each dominant weight ρ the multiplicity n_ρ is given by*

$$n_\rho = \sum_\sigma m_\sigma A_{\sigma + \lambda_2 + \delta}(\rho + \delta)$$

where the sum goes over all (not only dominant) weights σ of V_{λ_1}, m_σ is the multiplicity of the weight σ in V_{λ_1} and δ is the lowest form.

We shall apply the proposition to $V_{\lambda_1} = \mathbb{C}^m$ and $\lambda_2 = \lambda$. In order to find all weights σ appearing in $\mathbb{C}^m$, we have to apply the Weyl group to the dominant weight e^1. According to the descriptions in 10.10, we can get all $\pm e^i$, $1 \le i \le l$, where l is the rank of the algebra as usual, and additionally the weight 0 in the odd dimensional case. Since the corresponding weight spaces yield the full dimension of $\mathbb{C}^m$, we have found all weights.

Now, let us notice that for a dominant weight μ and strongly dominant weight ν (i.e. ν is not on a wall of the fundamental Weyl chamber), we always have

$$A_\nu(\mu) = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{otherwise.} \end{cases}$$

In our case, $\delta = \frac{1}{2}((m-2)e^1 + (m-4)e^2 + \dots + (m-2l)e^l)$, $\rho + \delta$ must be strongly dominant by the definition, but we also have

Sublemma. *$\lambda + \sigma + \delta$ is dominant for all weights σ appearing in $\mathbb{C}^m$ with the only exception when $\sigma = -e^l$ and λ does not involve e^l.*

Proof. A weight is dominant if it is a linear combination of the fundamental weights with non-negative coefficients, i.e. they are of the form $\sum_1^l \alpha_i e^i$ with all α_i integral or half-integral and $\alpha_1 \geq \dots \geq \alpha_l \geq 0$ in the odd dimensional case and $\alpha_1 \geq \dots \geq |\alpha_l|$ in the even dimensions. But for the weight $\lambda + \delta$ we have $\alpha_1 > \alpha_2 > \dots > \alpha_{l-1}$ and $\alpha_{l-1} > \alpha_l \geq \frac{1}{2}$ or $\alpha_{l-1} > |\alpha_l|$ in the even or odd dimensional cases, respectively. So we can always subtract e^i in the even dimensional case without running away from the class of dominant weights. The same holds in the odd dimensions m except λ does not involve e^l, for then we get $\alpha_l = -1$ after the subtraction. Adding of e^i or the zero weight cannot cause any difficulty. $\square$

Now, everything is prepared to finish the proof of Theorem 6.10. By the Klimyk's formula, the multiplicity equals either zero or one in the cases 6.10.(i) and (ii), for two different weights σ cannot contribute to the same multiplicity. The multiplicity one is obtained if and only if $\rho = \lambda \pm e_i$ is dominant. In the case 6.10.(iii) we can apply the same argument, except $i = l$ and this possibility remains for check. Let us choose the element $s \in W$ with $s(e^i) = e^i$, $1 \leq i \leq l-1$, and $s(e^l) = -e^l$, cf. 10.10. Hence $s(\delta + \lambda - e^l) = \delta + \lambda$ and so if we choose ρ with $\rho = \lambda - e^l$ then the contribution of the weight $-e^l$ cancels with the contribution of the weight zero. This proves the case 6.10.(iii) and Theorem 6.10 is proved in the complex case. But its real version follows immediately since we can complexify the space $V^* \oplus V_1^*$ and seek for the heighest weight vectors with trivial actions of $\mathfrak{b}_1$ there. During the complexification, the highest weight vectors either remain the same ones or they are doubled. Thus each real morphism must be reflected also in the complex case and each complex highest weight vector gives rise to a morphism in the real case. This completes the proof of Theorem 6.10.

6.21. Examples. Let us discuss the operators defined on the fundamental representations of $SO_0(m', n, \mathbb{R})$.

Take first $\lambda = e^1 + \dots + e^i$, $1 \leq i < l$ if $m = 2l+1$, $1 \leq i < l-1$ if $m = 2l$. As we know, this dominant weights correspond to the exterior forms of degree i. It is easy to find all irreducible components in $\mathbb{R}^{m*} \otimes V_\lambda$ (It is the same as for $\mathbb{R}^m \otimes V_\lambda$):

$$\begin{aligned} \rho_1 &= e^1 + \dots + e^{i-1} + e^i + e^{i+1} \\ \rho_2 &= e^1 + \dots + e^{i-1} \\ \rho_3 &= 2e^1 + e^2 + \dots + e^i. \end{aligned}$$

The ρ_1 and ρ_3 result from adding one e^j, the ρ_2 is the only possibility obtained through subtracting an e^j. We should notice that the dominant weight $\rho_1 = e^1 +$

$\dots + e^l$ does not describe the l-th degree exterior forms but only the self-dual part Ω^l_+, i.e. the $+1$-eigen space of the Hodge operator.

We have to work out the conformal weights. The formula from Theorem 6.10 yields

$$\alpha_j = \frac{1}{2}(m-1) - \langle\lambda, \lambda + 2\delta\rangle + \langle\rho_j, \rho_j + 2\delta\rangle \qquad j = 1, 2, 3$$

and, as used several times above, the Killing form differs form the standard Euclidean product by the factor $-\frac{1}{2}$. Hence we get

$$\begin{aligned}
\alpha_1 = {} & \frac{1}{2}(m-1) - \\
& \langle e^1 + \dots + e^i, (m-1)e^1 + \dots + (m-2i+1)e^i + (m-2i-2)e^{i+1} + \dots (m-2l)e^l\rangle + \\
& \langle e^1 + \dots + e^{i+1}, (m-1)e^1 + \dots + (m-2i-1)e^{i+1} + \dots (m-2l)e^l\rangle = \\
& = \frac{1}{2}(m-1) + \langle e^{i+1}, (m-2i-1)e^{i+1}\rangle = \frac{1}{2}(m-1-m+2i+1) = i
\end{aligned}$$

This computation was a good test for the formula since the operator corresponding to ρ_1 must be of course the exterior derivative which is invariant with respect to all diffeomorphisms. Therefore, we have known from the beginning that the weight must correspond to the restriction of the canonical tensor representation of $GL(m)$. (In the case $i = l-1$, the operator is the composition of d with the projection onto the irreducible component Ω^l_+ or Ω^l_-.)

A similar computation for ρ_2 leads to $\alpha_2 = \frac{1}{2}(m-1) - \langle e^i, (m-2i+1)e^i\rangle = \frac{1}{2}(m-1+m-2i+1) = m-i$. This yields the codifferential δ acting on the bundle of exterior forms of degree i with conformal weight $m-i$ and valued in exterior forms of degree $i-1$ with conformal weight $m-i+1$.

For ρ_3 we get $\alpha_3 = \frac{1}{2}(m-1) - \langle e^1, (m-1)e^1\rangle - \langle 2e^1, me^1\rangle = \frac{1}{2}(m-1+m-1-2m) = -1$. As an operator invariant with respect to the isometries, this is the trace-free part of the covariant derivative symmetrized in the last two indices.

6.22. Examples. Let us consider the remaining fundamental representation $\lambda = \frac{1}{2}(e^1 + \dots + e^l)$ in the odd dimensional case $m = 2l+1$. We get only two possibilities for the weights

$$\begin{aligned}
\rho_1 &= \lambda \\
\rho_2 &= \frac{1}{2}(3e^1 + e^2 + \dots + e^l).
\end{aligned}$$

The conformal weight α_1 equals $\frac{1}{2}(m-1)$ and we evaluate $\alpha_2 = \frac{1}{2}(m-1) - \langle\frac{1}{2}e^1, (m-\frac{3}{2})e^1\rangle + \langle\frac{3}{2}e^1, (m-\frac{1}{2})e^1\rangle = \frac{1}{2}(m-1+\frac{1}{2}m-\frac{3}{4}-\frac{3}{2}m+\frac{3}{4}) = -\frac{1}{2}$. We shall see in the next section that ρ_1 corresponds to the Dirac operator while the other one yields the twistor operator.

If the dimension is $m = 2l$, we have still to discuss two fundamental representations $\lambda^+ = \frac{1}{2}(e^1 + \dots + e^l)$ and $\lambda^- = \frac{1}{2}(e^1 + \dots + e^{l-1} - e^l)$. We get the components

$$\begin{aligned}
\rho_1^+ &= \frac{1}{2}(e^1 + \dots + e^{l-1} - e^l) \\
\rho_2^+ &= \frac{1}{2}(3e^1 + e^2 + \dots + e^l) \\
\rho_1^- &= \frac{1}{2}(e^1 + \dots + e^l) \\
\rho_2^- &= \frac{1}{2}(3e^1 + e^2 + \dots + e^{l-1} - e^l)
\end{aligned}$$

The weights $\rho_1^\pm$ correspond to the Dirac operators, the other ones to the twistor operators. The conformal weights are: $\alpha_1^\pm = \frac{1}{2}(m-1)$ and $\alpha_2^\pm = -\frac{1}{2}$.

7. The spinors and the Dirac operators

We want to work out a geometric description of the bundles corresponding to the half integral dominant forms from the proceeding section and, of course, also of the operators between them, at least for those discussed in the Examples 6.22. First of all we need to understand the double coverings of the orthogonal groups. The most efficient way is to view them as subgroups in the so called Clifford algebras. Hence we start with the necessary algebraic considerations. The topic is standard and can be found in several nice books, see e.g. [Budinich, Trautman, 88], [Lawson, Michelsohn, 89], [Gilkey, 84].

7.1. Clifford algebras. Let $\mathbb{K}$ be any commutative field, V be a finite dimensional vector space over $\mathbb{K}$ and let Q be a quadratic form on V. We write $T(V) = \sum_{k=0}^{\infty} \otimes^k V$ for the tensor algebra of V and $\mathcal{C}\ell(V) = T(V)/I_Q$ is the quotient algebra with respect to the two-sided ideal $I_Q \subset T(V)$ generated by the expressions $x \otimes x - Q(x)$, $x \in V$. The $\mathbb{K}$-algebra $\mathcal{C}\ell(V)$ is called the *Clifford algebra*. The composition $V \to T(V) \to T(V)/I_Q$ defines the injection $i_Q \colon V \to \mathcal{C}\ell(V)$, for if $v - w \in I_Q$ then it cannot be an element in $V \subset T(V)$ for homogeneity reasons. We shall often identify V with $i_Q(V) \subset \mathcal{C}\ell(V)$ in the sequel. The tensor multiplication on $T(V)$ induces a multiplication on $\mathcal{C}\ell(V)$ which we shall denote by $*$. The canonical filtration $F^q \subset T(V)$, $F^q = \sum_{k=0}^{q} \otimes^k V$, induces a filtration on $\mathcal{C}\ell(V)$ denoted by $F^i_{I_Q}$. In this way we get a canonical grading on $\mathcal{C}\ell(V)$. As a vector space, $\mathcal{C}\ell(V) = F^0_{I_Q} + F^1_{I_Q}/F^0_{I_Q} + F^2_{I_Q}/F^1_{I_Q} + \dots$. The exterior forms are also a quotient of the tensor algebra, $T(V)/J$ with $J = \langle x \otimes y + y \otimes x \rangle$. Since $Q(x+y) = (x+y)*(x+y) = Q(x) + Q(y) + x*y + y*x$ on V, the identity mapping on the tensor algebra $T(V)$ induces the isomorphisms

$$\Lambda^i(V) = F^i/(F^{i-1} + J \cap F^i) \simeq F^i/(F^{i-1} + I_Q \cap F^i) = F^i_{I_Q}/F^{i-1}_{I_Q}.$$

Thus, the Clifford algebra $\mathcal{C}\ell(V)$ is as a vector space isomorphic to the exterior algebra $\Lambda(V)$. In particular, its dimension is $2^{\dim V}$ and if e_i, $i = 1, \dots, m$, is a basis of $i_Q(V)$, then the unit $1 \in \mathbb{K}$ together with the products $e_{i_1} * \dots * e_{i_p}$, $i_1 < \dots < i_p$, form the basis for $\mathcal{C}\ell(V)$ (as a vector space). The multiplication on

$\mathcal{C}\ell(V)$ does not respect the grading, so the Clifford algebra $\mathcal{C}\ell(V)$ is not a $\mathbb{Z}$-graded algebra, however the induced $\mathbb{Z}_2$-grading $\mathcal{C}\ell(V) = \mathcal{C}\ell^0(V) + \mathcal{C}\ell^1(V)$ is respected (with the proper signs). As vector spaces, the homogeneous components $\mathcal{C}\ell^i(V)$, $i = 0, 1$, are isomorphic to the even and odd degree exterior forms in $\Lambda(V)$, and so they have the same dimension $2^{\dim V - 1}$. The Clifford algebra $\mathcal{C}\ell(V)$ is universal with respect to the linear maps $\varphi\colon V \to A$ with the property $\varphi(x)^2 = Q(x)1$, where A is an arbitrary $\mathbb{K}$-algebra: Indeed, each linear φ extends to an algebra homomorphism on the tensor algebra $T(V)$, but this extension is trivial on the ideal I_Q, hence it factors to an algebra homomorphism on $\mathcal{C}\ell(V)$.

7.2. There are several canonical automorphisms or anti-automorphisms of $\mathcal{C}\ell(V)$:

First of all, the map defined by $x = x_1 \otimes \cdots \otimes x_p \mapsto x^t = x_p \otimes \cdots \otimes x_1$ on $T(V)$ leaves the ideal I_Q invariant and so we get the induced mapping $y \mapsto y^t$ on $\mathcal{C}\ell(V)$, a well defined anti-automorphism.

Further, there is the algebra automorphism α generated by $-i_Q$: we have $\alpha(x)^2 = Q(x)1$, $x \in V$, and so α extends by the universal property. This homomorphism acts by multiplication by ± 1 and the $\mathbb{Z}_2$-grading of $\mathcal{C}\ell(V)$ consists just of the $+1$ and -1-eigen spaces of α.

Finally, we have the 'bar' anti-automorphism $x \mapsto \bar{x} = \alpha(x^t)$.

7.3. We shall consider only $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and Q will be always the canonical quadratic form with signature (p, q), $p + q = m$, where m is the dimension of $V = \mathbb{K}^m$. The corresponding Clifford algebras will be denoted by $\mathcal{C}\ell_m(\mathbb{R})$ or $\mathcal{C}\ell_m(p, q)$ and $\mathcal{C}\ell_m(\mathbb{C})$ (in the complex case there is no reason to point out the signature of Q). Since $(V \otimes_{\mathbb{R}} V) \otimes_{\mathbb{R}} \mathbb{C} = (V \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (V \otimes_{\mathbb{R}} \mathbb{C})$, we have $\mathcal{C}\ell_m(\mathbb{C}) = \mathcal{C}\ell_m(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$. Thus, we can often discuss both cases together.

Let us fix $e_i \in \mathbb{R}^m$, the canonical base. Hence $Q(e_i, e_j) = 0$ for all $i \neq j$, while $Q(e_i, e_i) = \pm 1$. Thus, $\mathcal{C}\ell_m(p, q)$ is an algebra generated by $1 \in \mathbb{R}$ and e_i subject to the relations $e_i * e_j = -e_j * e_i$ if $i \neq j$, $e_i * e_i = 1$, $1 \leq i \leq p$, and $e_i * e_i = -1$, $p < i \leq m$. Once we have fixed the orthonormal basis, there is the distinguished element $\nu = e_1 * \cdots * e_m$, the so called *volume element in* $\mathcal{C}\ell_m(\mathbb{K})$.

Now, the idea is to find a suitable subgroup of invertible elements in $\mathcal{C}\ell_m(p, q)$ acting on $\mathbb{R}^m$ by isometries.

We consider $\mathbb{R}^m$ as the subspace $\mathbb{R}^m \subset \mathcal{C}\ell_m(\mathbb{R})$ and so we can always act by conjugation (accomplished with suitable sign changes): $\mathbb{R}^m \ni y \mapsto \alpha(x) * y * x^{-1} \in \mathcal{C}\ell_m(\mathbb{R})$, with $x \in \mathcal{C}\ell_m(\mathbb{R})^*$, the multiplicative group of invertible elements in $\mathcal{C}\ell(V)$. The subgroup $\Gamma \subset \mathcal{C}\ell_m(\mathbb{R})^*$ of elements with $\alpha(x) * y * x^{-1} \in \mathbb{R}^m$ for all $y \in \mathbb{R}^m$ is called the *Clifford group*. Let us denote by $\rho\colon \Gamma \to GL(m)$ the induced group homomorphism. In [Atiyah, Bott, Shapiro, 64], ρ is called the *twisted adjoint representation* of Γ on $\mathbb{R}^m$. It is easy to see that all three canonical maps from 7.2 preserve Γ. Let us further define another mapping $N\colon \Gamma \to \Gamma$, $N(x) = x * \bar{x}$, i.e. $\rho(N(x)) = \rho(x) \circ \rho(\bar{x})$.

We define the $Pin(p, q)$ as the subset $\{y \in \Gamma \subset \mathcal{C}\ell_m(p, q); N(y) = 1 \text{ or } N(y) = -1\}$. Let us notice that the multiplicative subgroup $\mathcal{C}\ell_m(p, q)^*$ in $\mathcal{C}\ell_m(p, q)$ is a Lie group (a closed subgroup of a matrix group) and $Pin(p, q)$ is a closed subset, by the definition.

7.4. Theorem. *$Pin(p, q)$ is a Lie subgroup in the Clifford group. The restriction $\rho|Pin(p, q)$ is a surjection of $Pin(p, q)$ onto $O(p, q)$ with kernel $\mathbb{Z}_2 = \{\pm 1\} \subset \Gamma$. Let*

$Spin(p,q)$ be the inverse image of the connected component of the unit $SO_0(p,q) \subset O(p,q)$. If $m \geq 3$, then the restriction $\rho\colon Spin(p,q) \to SO_0(p,q)$ is the non-trivial connected and simply connected double covering of $SO_0(p,q)$.

Proof. First of all we have to show that ρ has values in the subgroup of all isometries and this requires to study the kernels of ρ and N. Then it will be easy to see that our choice of the subset $Pin(p,q)$ yields a subgroup, i.e. a Lie group by the remark at the end of 7.3, and that $\rho|Pin(p,q)$ is surjective.

Sublemma. *The kernel of $\rho\colon \Gamma \to GL(m)$ is precisely the multiplicative subgroup $\mathbb{R}^* \subset \Gamma$ generated by 1. For each $x \in \Gamma$ the value $N(x)$ belongs also to $\mathbb{R}^*$.*

Proof. Let $x \in \ker\rho$, so that $\alpha(x) * y = y * x$ for all $y \in \mathbb{R}^m$. As an element in $\mathcal{C}\ell_m(\mathbb{R})$, x decomposes into the homogeneous parts $x = x^0 + x^1$ and the condition for x being in the kernel splits into two conditions

$$x^0 * y = y * x^0 \quad \text{and} \quad x^1 * y = -y * x^1. \tag{1}$$

As usual e_i are the elements of the canonical basis in $\mathbb{R}^m$. Let us fix some e_i and write $x^0 = a^0 + e_i * a^1$, $x^1 = b^1 + e_i * b^0$, where the elements a^0, a^1, b^1, b^0 do not involve e_i. The first condition in (1) with $y = e_i$ now implies $a^0 * e_i + e_i * a^1 * e_i = e_i * a^0 + e_i * e_i * a^1$. Since a_0 is an even element without e_i while a^1 is an odd one, we have $a^0 * e_i = e_i * a^0$ and $a^1 * e_i = -e_i * a^1$. Hence we get $e_i * a^0 - Q(e_i)a^1 = e_i * a^0 + Q(e_i)a^1$, so that $a^1 = 0$ and therefore the even part x^0 does not involve e_i. Since i was arbitrary, x_0 is a multiple of 1. Similarly, the second condition in (1) yields $b^1 * e_i + b^0 * e_i * e_i = -e_i * b^1 - e_i * b^0 * e_i$ where b^1 is odd and b^0 even. Thus, $-e_i * b^1 + b^0 Q(e_i) = -e_i * b^1 - b^0 Q(e_i)$ and so b^0 is zero, x^1 does not involve any e_i, i.e. x^1 is a multiple of 1. On the other hand, x^1 is odd, hence zero, and the first claim is proved.

Since $y^t = y$ for all $y \in \mathbb{R}^m$, we have $\alpha(x) * y * x^{-1} = (x^t)^{-1} * y * \alpha(x^t)$ and so $(\rho(\alpha(x^t)) \circ \rho(x))(y) = y$, since α^2 is the identity on $\mathcal{C}\ell_m(\mathbb{R})$. Thus, we have shown that $N(x) \subset \ker\rho$ for all $x \in \Gamma$. $\square$

Now, $N(x * y) = x * y * \bar{y} * \bar{x} = x * N(y) * \bar{x} = x * \bar{x} * N(y) = N(x)N(y)$ and so $N\colon \Gamma \to \mathbb{R}^*$ is a group homomorphism. Moreover $N(\alpha(x)) = \alpha(x) * x^t = \alpha(N(x)) = N(x)$, for $N(x)$ is an element of degree zero. But this implies $N(\rho(x)(y)) = N(\alpha(x))N(y)N(x^{-1}) = N(y)$ for all $x \in \Gamma$ and $y \in \mathbb{R}^m$. For each element $x \in \mathbb{R}^m$, $N(x) = x*(-x) = -x*x = -Q(x)$, i.e. $N|\mathbb{R}^m$ is the negative of the standard scalar product with signature (p,q). But then the formula for $N(\rho(x)(y))$ claims precisely $\rho\colon \Gamma \to O(p,q)$.

Let us write $\langle x, y\rangle$ for the scalar product of x, $y \in \mathbb{R}^m$ induced by Q. For all elements $y \in \mathbb{R}^m \subset \mathcal{C}\ell_m(p,q)$ with $N(y) = -Q(y) = 1$ and $x \in \mathbb{R}^m$ we have $y^{-1} = -y$ and

$$\rho(y)(x) = \alpha(y) * x * y^{-1} = y * x * y = x + 2\langle x, y\rangle y$$

where the last equality follows from $2\langle x, y\rangle + Q(x) + Q(y) = (x+y)*(x+y) = Q(x) + Q(y) + x * y + y * x$. Similarly, if $Q(y) = 1$, then $y = y^{-1}$ and

$$\rho(y)(x) = -y * x * y = x - \frac{2\langle x, y\rangle}{\langle y, y\rangle} y$$

and this formula holds for both cases. It is well known in the definite case that the latter transformations are precisely all reflections in hyperplanes in $\mathbb{R}^m$ which generate the whole orthogonal group $O(m,\mathbb{R})$. Since the real groups of pseudo-orthogonal transformations all admit the same complexification, they must be also generated by these transformations (this argument applies immediately for the connected components of the unit, the whole groups need more detailed consideration). Thus $\rho\colon Pin(p,q) \to O(p,q)$ is onto. The kernel of this map is the intersection $\ker\rho \cap \{N(x)^2 = 1\}$. Since the kernel of ρ coincides with the multiplicative group $\mathbb{R}^*$ and $N(\lambda.1) = \lambda^2$, the kernel of $\rho|Pin(p,q)$ must be $\mathbb{Z}_2$ (as a multiplicative group).

We already know that $Spin(p,q)$ is a double covering of $SO_0(p,q)$. In order to show that this is a non-trivial covering, it suffices to connect $+1$ and -1, i.e. the elements of the kernel of $\rho|Spin(p,q)$, by a curve in $Spin(p,q)$. Let us consider $t \mapsto c(t) = a(t) + b(t)e_1 * e_2$. We have $N(a(t) + b(t)e_1 * e_2) = (a(t) + b(t)e_1 * e_2) * (a(t) + b(t)e_2 * e_1) = a(t)^2 + b(t)^2 Q(e_2)Q(e_1)$. If $Q(e_1)Q(e_2) = 1$, then we choose $a(t) = \cos t$, $b(t) = \sin t$. Then $N(c(t)) = 1$ and $c(0) = 1$, $c(\pi) = -1$ so that it yields a suitable curve. If $m > 2$, we can always find two generators e_i, e_j with $Q(e_i)Q(e_j) = 1$. Since $SO_0(p,q)$ is connected by our definition and its fundamental group is $\mathbb{Z}_2$ if $m \geq 3$, $Spin(p,q)$ must be simply connected in dimensions $m \geq 3$. $\square$

7.5. Remark. Let us consider the positive definite case $O(m,\mathbb{R})$. Each element $\rho(y) \in O(m,\mathbb{R})$ equals to a composition $x_p \circ \ldots \circ x_1$ of reflections in hyperplanes and we have seen, there are always elements $y_i \in \mathbb{R}^m \subset \Gamma$ with $\rho(y_i) = x_i$. By Theorem 7.4, there is $y \in Pin(m)$, $y = \pm y_p * \cdots * y_1$. Let us write $Pin^j(m) = Pin(m) \cap \mathcal{C}\ell^j_m(\mathbb{R})$, $j = 0, 1$. The element y must be either in $Pin^0(m)$ or in $Pin^1(m)$. But we know that $y \in Spin(m)$ if and only if the number of the reflections involved is even. Thus $Spin(m) = Pin^0(m)$ and we see that the elements in $Spin(m)$ are just the products $y = y_1 * \cdots * y_{2j}$ with $y_i \in \mathbb{R}^m$, $Q(y) = -1$. Then $y^{-1} = y^t$ and $\rho(y)(x) = y * x * y^t$. Let us remark that $Spin(2,0)$ and $Spin(0,2)$ are also non-trivial coverings by the argument from the proof. Since they are generated by $e_1 * e_2$, they are one-dimensional ($N(a.1 + be_1 * e_2) = 1$). As a double-covering of the circle $S^1 \subset \mathbb{R}^2$ it must also be S^1.

In the case of a general signature, we still get $Spin(m) \subset Pin^0(m)$ but the whole $Pin^0(m)$ is not involved. The group $SO_0(1,1)$ equals $\mathbb{R}$ so that it does not admit a non-trivial covering.

7.6. The complex spin groups. We have noticed that the complex Clifford algebras are $\mathcal{C}\ell_m(\mathbb{C}) = \mathcal{C}\ell_m(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, i.e. the complexified real Clifford algebras. All the previous definitions and considerations have their complex analogies (working best with the negative definite bilinear form Q) and we get the complex groups $Pin(m,\mathbb{C})$ and $Spin(m,\mathbb{C})$ which are non-trivial double coverings of the complex orthogonal group if $m \geq 2$. We shall see below that all the real spin groups are matrix groups and their complexifications are just the complex spin groups.

Let us remark that there are other complex Lie groups sitting in the complex Clifford algebras, the groups $Pin^{\mathbb{C}}(m)$ which are important in the K-theory. The latter groups are quite different from $Pin(m,\mathbb{C})$ defined above and should be carefully distinguished. Namely, we can change our definition of the basic operation by setting $\alpha_{\mathbb{C}}(x \otimes z) = \alpha(x) \otimes z$, $(x \otimes z)^T = x^t \otimes \bar{z}$ and the 'bar' operation and $N^{\mathbb{C}}$ is defined in terms of $\alpha_{\mathbb{C}}$ and $(\ )^T$ as before. The (other) complex Clifford group $\Gamma^{\mathbb{C}}$ contains the elements $x \in \mathcal{C}\ell_m(\mathbb{R}) \otimes \mathbb{C}$ with $\alpha_{\mathbb{C}}(x) * y * x^{-1} \in \mathbb{R}^m$ for all $y \in \mathbb{R}^m$ (so it might be bigger).

Going through the above proof, nearly everything goes through with $\mathbb{R}^*$ replaced by $\mathbb{C}^*$ (notice that the generators e_i which are used in the proof remain the same, i.e. real) and the mapping ρ takes values in the real orthogonal group $O(m, \mathbb{R})$. We define the complex group $Pin^{\mathbb{C}}(m)$ as the kernel of $N^{\mathbb{C}}\colon \Gamma^{\mathbb{C}} \to \mathbb{C}^*$. At the end we get as before that the kernel of ρ consists of non-zero complex numbers $1 \otimes z \in \mathcal{C}\ell_m(\mathbb{C})$ with $N^{\mathbb{C}}(1 \otimes z) = z\bar{z} = 1$. Thus, we get for all $m \geq 1$ the exact sequences of Lie groups

$$1 \to U(1) \to Pin^{\mathbb{C}}(m) \to O(m, \mathbb{R}) \to 1$$
$$1 \to U(1) \to Pin(m, \mathbb{R}) \times_{\mathbb{Z}_2} U(1) \to Pin(m, \mathbb{R})/\mathbb{Z}_2 \to 1.$$

This induces an isomorphism $Pin^{\mathbb{C}}(m) \simeq Pin(m, \mathbb{R}) \times_{\mathbb{Z}_2} U(1)$.

7.7. It is possible to view the Clifford algebras as matrix algebras, the concrete identifications will require some effort. Let us write $\mathrm{Mat}_m(\mathbb{K})$ for the algebra of $(m \times m)$-matrices over $\mathbb{K}$, i.e. $\mathrm{End}(\mathbb{K})$. Beside the real and complex numbers, we shall also meet $\mathbb{K} = \mathbb{H}$, the quaternions.

Proposition. *There are the following identities of algebras:*

$$\begin{aligned}
\mathrm{Mat}_m(\mathbb{K}) &\simeq \mathrm{Mat}_m(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{K} && \text{over } \mathbb{K}\\
\mathrm{Mat}_m(\mathbb{R}) \otimes \mathrm{Mat}_n(\mathbb{R}) &\simeq \mathrm{Mat}_{mn}(\mathbb{R}) && \text{over } \mathbb{R}\\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\simeq \mathbb{C} \oplus \mathbb{C} && \text{over } \mathbb{C}\\
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} &\simeq \mathrm{Mat}_2(\mathbb{C}) && \text{over } \mathbb{C}\\
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\simeq \mathrm{Mat}_4(\mathbb{R}) && \text{over } \mathbb{R}.
\end{aligned}$$

Proof. The first identity is clear. In the second one, we define the tensor product of the generating matrices $E_{ij} \otimes E_{pq}$ as the block matrix $A = (A_{ab})$ with $A_{ij} = E_{pq}$ and $A_{kl} = 0$ for all other indices. This generates the required isomorphism. The third one is defined on generators as follows: $\sqrt{-1} \otimes 1 \mapsto \sqrt{-1} \oplus \sqrt{-1}$ and $1 \otimes \sqrt{-1} \mapsto \sqrt{-1} \oplus -\sqrt{-1}$. Hence we get $a \otimes b \mapsto (ab, a\bar{b})$.

The algebras $\mathrm{Mat}_2(\mathbb{R})$ and $\mathrm{Mat}_2(\mathbb{C})$ are generated by two matrices

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which form a linear bases together with the identity matrix $\mathbb{I}_2$ and the matrix

$$\nu = \sigma\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These matrices satisfy $\sigma\tau = -\tau\sigma$, $\sigma^2 = \tau^2 = -\nu^2 = \mathbb{I}_2$.

The next isomorphism is obtained through $\sqrt{-1} \otimes 1 \mapsto \sqrt{-1}\mathbb{I}_2$, $1 \otimes i \mapsto -\sqrt{-1}\sigma$, $1 \otimes j \mapsto \nu$ on generators.

The last isomorphism is defined on the generators by $1 \otimes i \mapsto \nu \otimes \sigma$, $1 \otimes k \mapsto \nu \otimes \tau$, $i \otimes 1 \mapsto \sigma \otimes \nu$, $k \otimes 1 \mapsto \tau \otimes \nu$. $\square$

7.8 Proposition. *There are isomorphisms*

$$\mathcal{C}\ell_2(2,0) \otimes_{\mathbb{R}} \mathcal{C}\ell_m(p,q) \simeq \mathcal{C}\ell_{m+2}(q+2,p)$$
$$\mathcal{C}\ell_2(1,1) \otimes_{\mathbb{R}} \mathcal{C}\ell_m(p,q) \simeq \mathcal{C}\ell_{m+2}(p+1,q+1)$$
$$\mathcal{C}\ell_2(0,2) \otimes_{\mathbb{R}} \mathcal{C}\ell_m(p,q) \simeq \mathcal{C}\ell_{m+2}(q,p+2).$$

The low-dimensional algebras and their even parts are

$$\begin{array}{ll} \mathcal{C}\ell_1(1,0) \simeq \mathbb{R} \oplus \mathbb{R} & \mathcal{C}\ell_1^0(1,0) \simeq \mathbb{R} \\ \mathcal{C}\ell_1(0,1) \simeq \mathbb{C} & \mathcal{C}\ell_1^0(0,1) \simeq \mathbb{R} \\ \mathcal{C}\ell_2(2,0) \simeq \mathrm{Mat}_2(\mathbb{R}) & \mathcal{C}\ell_2^0(2,0) \simeq \mathbb{C} \\ \mathcal{C}\ell_2(1,1) \simeq \mathrm{Mat}_2(\mathbb{R}) & \mathcal{C}\ell_2^0(1,1) \simeq \mathbb{R} \oplus \mathbb{R} \\ \mathcal{C}\ell(0,2) \simeq \mathbb{H} & \mathcal{C}\ell_2^0(0,2) \simeq \mathbb{C}. \end{array}$$

Proof. We shall give explicit formulas for these isomorphisms on the generators.

Let us consider the canonical basis e_i in $\mathbb{R}^m$, f_1, f_2 in $\mathbb{R}^2$, and e'_j in $\mathbb{R}^{m+2}$. Let us define the elements $g_1 = f_1 \otimes 1$, $g_2 = f_2 \otimes 1$, $g_{i+2} = f_1 * f_2 \otimes e_i$ in the tensor products of the Clifford algebras. We shall show that the linear map ψ defined by $\psi(e'_j) = g_j$ satisfies in all three cases the universal property from 7.1 and so extends to an algebra homomorphism. If $j = 1$ or $j = 2$, we get $\psi(e'_j)^2 = f_j \otimes 1 * f_j \otimes 1 = Q(f_j) 1 \otimes 1$ and for e'_{i+2} it holds $\psi(e'_{i+2})^2 = (f_1 * f_2 \otimes e_i) * (f_1 * f_2 \otimes e_i) = -Q(e_i)Q(f_1)Q(f_2)(1 \otimes 1)$. Hence the roles of p and q interchange and two positive or two negative dimensions are added, or we add one positive and one negative dimension and the p and q remain. Then ψ extends to an algebra homomorphism and since the spaces in question have equal dimensions and generators are transformed into generators, it must be an isomorphism.

In the algebra $\mathcal{C}\ell_1(1,0)$, there is the generator e_1 with $e_1^2 = 1$. Hence $\mathcal{C}\ell_1(1,0) = \mathbb{R} \oplus \mathbb{R}$ and the even part is $1\mathbb{R}$ with the isomorphism defined by $1 \mapsto (1,1)$, $e_1 \mapsto (1,-1)$. Similarly, e_1 with $e_1^2 = -1$ is the generator of $\mathcal{C}\ell_1(0,1)$ and so $e_1 \mapsto \sqrt{-1}$ defines the isomorphism with $\mathbb{C}$. The even part is then $1\mathbb{R}$.

The algebra $\mathcal{C}\ell_2(2,0)$ is generated by e_1, e_2 with $e_1^2 = e_2^2 = 1$, $e_1 * e_2 = -e_2 * e_1$. We define $e_1 \mapsto \sigma$, $e_2 \mapsto \tau$ and the matrices σ and τ are declared as odd elements. Then 1 and ν are even, $\nu^2 = -1$ and we get the required isomorphisms.

If the signature is $(1,1)$, we associate e_1 to σ (the positive dimension), while $e_2 \mapsto \nu$. The latter are the odd elements and so the even subalgebra is generated by 1 and τ, hence equals to $\mathbb{R} \oplus \mathbb{R}$.

Finally, in the negative definite case $e_1^2 = e_2^2 = -1$ and we can identify e_1 and e_2 with the generators i, j of the quaternions $\mathbb{H}$. □

7.9. Proposition. *For each dimension m it holds*

$$\mathcal{C}\ell_{m+8}(p+8,q) \simeq \mathcal{C}\ell_{m+8}(p+4,q+4) \simeq \mathcal{C}\ell_{m+8}(p,q+8) \simeq \mathcal{C}\ell_m(p,q) \otimes Mat_{16}(\mathbb{R})$$
$$\mathcal{C}\ell_{m+2}(\mathbb{C}) \simeq \mathcal{C}\ell_m(\mathbb{C}) \otimes Mat_2(\mathbb{C}).$$

The Clifford algebras in dimensions less then eight are listed below for all definite scalar products.

m	$\mathcal{C}\ell_m(0,m)$	$\mathcal{C}\ell_m(m,0)$	$\mathcal{C}\ell_m(\mathbb{C})$
0	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$
1	$\mathbb{C}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{C}\oplus\mathbb{C}$
2	$\mathbb{H}$	$Mat_2(\mathbb{R})$	$Mat_2(\mathbb{C})$
3	$\mathbb{H}\oplus\mathbb{H}$	$Mat_2(\mathbb{C})$	$Mat_2(\mathbb{C})\oplus Mat_2(\mathbb{C})$
4	$Mat_2(\mathbb{H})$	$Mat_2(\mathbb{H})$	$Mat_4(\mathbb{C})$
5	$Mat_4(\mathbb{C})$	$Mat_2(\mathbb{H})\oplus Mat_2(\mathbb{H})$	$Mat_4(\mathbb{C})\oplus Mat_4(\mathbb{C})$
6	$Mat_8(\mathbb{R})$	$Mat_4(\mathbb{H})$	$Mat_8(\mathbb{C})$
7	$Mat_8(\mathbb{R})\oplus Mat_8(\mathbb{R})$	$Mat_8(\mathbb{C})$	$Mat_8(\mathbb{C})\oplus Mat_8(\mathbb{C})$
8	$Mat_{16}(\mathbb{R})$	$Mat_{16}(\mathbb{R})$	$Mat_{16}(\mathbb{C})$

Proof. The whole statement follows from the two above propositions, see [Budinich, Trautman, 88] if more details are necessary. $\square$

7.10. Proposition. *Let e'_i, $i = 1,\dots,m+1$, be the canonical basis on $\mathbb{R}^{m+1}$, e_i, $i = 1,\dots,m$, be that on $\mathbb{R}^m$, and let $\varphi\colon \mathbb{R}^m \to \mathcal{C}\ell^0_{m+1}(p,q+1)$ be defined by $\varphi(e_i) = e'_{m+1} * e'_i$. Then φ extends uniquely to the algebra isomorphism $\mathcal{C}\ell_m(p,q) \simeq \mathcal{C}\ell^0_{m+1}(p,q+1)$. If we take e'_i, $i = 0,\dots,m$, as basis of $\mathbb{R}^{m+1}$ and define $\varphi(e_i) = e'_0 * e'_i$, we obtain the isomorphism $\mathcal{C}\ell_m(p,q) \simeq \mathcal{C}\ell^0_{m+1}(q+1,p)$. In the complex case we have $\mathcal{C}\ell_m(\mathbb{C}) \simeq \mathcal{C}\ell^0_{m+1}(\mathbb{C})$. Furthermore, $\mathcal{C}\ell^0_m(p,q) \simeq \mathcal{C}\ell^0_m(q,p)$.*

Proof. Since $\varphi(e_i)^2 = e'_j * e'_i * e'_j * e'_i = -Q(e'_j)Q(e'_i)$, φ extends uniquely by the universal property of the Clifford algebras if $Q(e'_j) = -1$ and $Q(e'_i) = Q(e_i)$ or if $Q(e'_j) = 1$ and $Q(e'_i) = -Q(e_i)$. This leads to homomorphisms between the indicated algebras. Since φ maps the generators of $\mathcal{C}\ell_m(p,q)$ to distinct elements, it must be injective. The dimensions of both spaces are equal and so φ is always an isomorphism. The complex case follows from the real considerations with negative definite scalar product.

The last isomorphism is obtained by composing the above isomorphisms but it can be also defined directly by $e_i * e_j \mapsto -e'_i * e'_j$. Indeed, the latter elements generate the even parts and the mapping is induced from $e_i \otimes e_j \mapsto -e'_i \otimes e'_j$ which leaves invariant the ideal $\langle x\otimes y + y\otimes x - 2Q(x,y)\rangle \subset \sum_k \otimes^{2k}\mathbb{R}^m$. The even parts are just the quotients by this ideal and so the homomorphism which is obvious on the tensor algebra descends to the even parts of the Clifford algebras. $\square$

7.11. Remark. Let us notice that the propositions above yield explicit identifications of the Clifford algebras with the (sums of) matrix algebras (as promised at the beginning of 7.7) and describe also explicitly the even parts of them. The whole situation is described by the 64 Clifford algebras $\mathcal{C}\ell_m(p,q)$ with $0 \le p,q \le 7$, the so called *spinorial chessboard*. We can find the main properties of Clifford algebras in this scheme.

It is possible to describe all real Clifford algebras by means of the so called *real* *clock*, [Budinich, Trautman, 88].

$$
\begin{tikzcd}
 & \mathbb{R}\oplus\mathbb{R} \arrow[r, "0"'] & \mathbb{R} \arrow[dr, "1"] & \\
\mathbb{R} \arrow[ur, "7"] & & & \mathbb{C} \arrow[d, "2"] \\
\mathbb{C} \arrow[u, "6"'] & & & \mathbb{H} \arrow[dl, "3"] \\
 & \mathbb{H} \arrow[ul, "5"] & \mathbb{H}\oplus\mathbb{H} \arrow[l, "4"'] &
\end{tikzcd}
$$

The usage: given p and q compute first the 'hour' μ such that $q-p = 8a+\mu$ where $a \in \mathbb{Z}$ and $0 \le \mu \le 7$. Then the algebras adjacent to the corresponding arrow determine the type of $\mathcal{C}\ell^0_{p+q}(p,q)$ (the source) and $\mathcal{C}\ell_{p+q}(p,q)$ (the target). The dimension of the full algebra is 2^{p+q} and we get the Clifford algebra by taking the proper matrix algebra.

For example: $p=2$ and $q=1$ yield $\mu=7$ and so $\mathcal{C}\ell_3(2,1) = \mathrm{Mat}_2(\mathbb{R})\oplus\mathrm{Mat}_2(\mathbb{R})$ while $\mathcal{C}\ell^0_3(2,1) = \mathrm{Mat}_2(\mathbb{R})$.

Similarly $\mathcal{C}\ell_8(3,5) = \mathrm{Mat}_8(\mathbb{H})$, for in this case $\mu = 2$ and $\mathrm{dim}\mathrm{Mat}_8(\mathbb{H}) = 2^8$, $\mathcal{C}\ell^0_8(3,5) = \mathrm{Mat}_8(\mathbb{C})$.

7.12. Clifford modules. A complex vector space V with an algebra homomorphism $\alpha\colon \mathcal{C}\ell_m(\mathbb{C}) \to \mathrm{End}(V)$ is called a (complex) *Clifford module*, α is called a representation of $\mathcal{C}\ell_m(\mathbb{C})$. Similarly a real vector space with a representation of a real Clifford algebra is called a (real) Clifford module. In fact, our aim is to understand the representations of the spin groups. However the study of the Clifford modules is a good way:

Proposition. *There are bijections between the representations of $Spin(m+1,\mathbb{C})$, the representations of $\mathcal{C}\ell_m(\mathbb{C})$ and the representations of $\mathcal{C}\ell^0_{m+1}(\mathbb{C})$. Furthermore, the decompositions into irreducible representations coincide.*

Proof. We consider the canonical negative-definite scalar product. The image of $Spin(m+1,\mathbb{C}) \subset \mathcal{C}\ell^0_{m+1}(\mathbb{C})$ in the inverse of the isomorphism φ from 7.10 generates the whole Clifford algebra $\mathcal{C}\ell_m(\mathbb{C})$ and the latter is isomorphic to $\mathcal{C}\ell^0_{m+1}(\mathbb{C})$. Since all the relations on the generators live in the image of the spin group as well, both the statements of the proposition are clear. □

7.13. Spinor bundles. Given any Clifford module V_γ with the representation γ of the real Clifford algebra $\mathcal{C}\ell_m(p,q)$, there is the corresponding bundle $F_\gamma M \to M$ over each oriented pseudo-Riemannian m-dimensional manifold M with a fixed spin structure (and the proper signature of the metric). This bundle is constructed as the associated bundle to the principal spin bundle $PM \to M$ with respect to the given representation γ. More generally, for each oriented pseudo-Riemannian vector bundle E with a spin structure there is the *spinor bundle* $\mathcal{C}\ell_\gamma(E) = P_{Spin}(E)\times_\gamma V_\gamma$ where the dimension of fibers in E is m and $P_{Spin}(E)$ is a covering of the $SO_0(p,q)$-frame bundle of E with structure group $Spin(p,q)$.

7.14. The Clifford multiplication. As we have seen in the proof of Theorem 7.4, the twisted adjoint representation of the Clifford group acts on $\mathbb{R}^m$ by the reflections and this is equivalent to the usual action of the spin group on $\mathbb{R}^m$ obtained from the identical representation of $GL(m, \mathbb{R})$. Thus the tangent functor T can be viewed as a very special example of a spinor bundle.

Proposition. *For each Clifford module V_γ there is the mapping $\bullet\colon \mathbb{R}^m \otimes V_\gamma \to V_\gamma$ defined by $y \otimes v \mapsto \gamma(y)(v)$, $y \in \mathbb{R}^m \subset \mathcal{C}\ell_m(p, q)$ which is $Spin(p, q)$-equivariant with respect to the twisted adjoint action on $\mathbb{R}^m$ and the action γ. An analogous mapping arises for complex Clifford modules.*

Proof. For all $x \in Spin(p, q)$, $y \in \mathbb{R}^m$ and $v \in V_\gamma$ we have

$$\alpha(x) * y * x^{-1} \otimes \gamma(x).v \mapsto \gamma(x)\gamma(y)\gamma(x^{-1})\gamma(x).v = \gamma(x)\gamma(y).v$$

since each element $x \in Spin(p, q)$ is even and so α disappears (the dot means the application of the endomorphisms). $\square$

This map is called the *Clifford multiplication* and since it is equivariant it extends to natural transformations defined on spinor bundles. Furthermore, there is the canonical natural equivalence $T^* \to T$ between the tangent and cotangent bundles on Riemannian manifolds. Hence there is also the natural bilinear transformation $\bullet_M\colon T^*M \otimes F_\gamma M \to F_\gamma M$ for each spinor bundle $F_\gamma M$. We shall call all these mappings Clifford multiplications.

7.15. The Dirac operators. For each spinor bundle $F_\gamma M$ on an oriented pseudo-Riemannian spin manifold M, there is the canonical Levi-Cività (or Riemannian) connection on the pseudo-orthogonal frame bundle. As an $\mathfrak{o}(p, q)$-valued right-invariant one-form, this connection lifts uniquely to the spin frame bundle on M. Let us write ∇ for the corresponding covariant derivative on the associated vector bundles. Then we have the following composition

$$D\colon C^\infty(F_\gamma M) \xrightarrow{\nabla} C^\infty(T^*M \otimes F_\gamma M) \xrightarrow{\bullet} C^\infty(F_\gamma M).$$

This operator is called the *Dirac operator.*

7.16. Our next aim is to describe the so called Dirac spinors and Weyl spinors and the Dirac operators on them. The *regular representation* of $\mathcal{C}\ell_m(\mathbb{C})$ is its representation on itself by left multiplication. This is a faithful representation.

Proposition. *The representations of the complex Clifford algebras are always completely reducible and each irreducible faithful representation of $\mathcal{C}\ell^0_{2n+1}(\mathbb{C})$ or $\mathcal{C}\ell_{2n}(\mathbb{C})$ is equivalent to a summand in the regular representation, i.e. to the identical representation of $\mathrm{Mat}_{2^n}(\mathbb{C})$ on $\mathbb{C}^{2^n}$. All irreducible faithful representations of $\mathcal{C}\ell_{2n}(\mathbb{C})$ are equivalent.*

Proof. According to 7.7, the complex Clifford algebras are always isomorphic to a sum of full matrix algebras over $\mathbb{C}$. Assume first $m = 2n$ so that $\mathcal{C}\ell_m(\mathbb{C}) = \mathrm{Mat}_{2^n}(\mathbb{C})$ and consider the regular representation of $\mathrm{Mat}_{2^n}(\mathbb{C})$) on itself. The matrix algebra decomposes under this representation into the sum of copies of

$\mathbb{C}^{2^n}$, each of them representing the matrices with one non-zero column allowed. All these representations are faithful (these are the identical representations of $\mathrm{Mat}_{2^n}(\mathbb{C}) = \mathrm{End}(\mathbb{C}^{2^n})$). Such matrices with one (fixed) non-zero column form minimal left ideals.

Let us consider a faithful irreducible representation φ of $\mathcal{C}\ell_{2n}(\mathbb{C})$ on some space S. Fix one of the above minimal ideals $\mathcal{B}$ and some elements $v \in S$, $x \in \mathcal{B}$ with $\varphi(x)(v) \neq 0$. We define $f\colon \mathcal{B} \to S$, $f(y) = \varphi(y)(v)$. The regular representation factors to a representation $\rho_{\mathcal{B}}\colon \mathcal{C}\ell_{2n}(\mathbb{C}) \to \mathrm{End}(\mathcal{B})$ (this is the above identical representation) and f intertwines $\rho_{\mathcal{B}}$ and φ by its definition. Since $f(\mathcal{B})$ contains non-zero elements, and since $\rho_{\mathcal{B}}$ is irreducible as $\mathcal{B}$ is minimal, f must be an isomorphism.

By a general theorem, each finite dimensional representation of a sum of full matrix algebras over an algebraically closed field is completely reducible, see [Boerner, 67, p. 68][21]. Hence the proposition is proved for even dimensions. But $\mathcal{C}\ell^0_{m+1}(\mathbb{C}) = \mathcal{C}\ell_m(\mathbb{C})$ by 7.10. $\square$

Let us notice that the complete reducibility of all representations of connected components of the identity in the complex pseudo-orthogonal groups also follows (cf. 7.12), for each representation of $SO(p,q,\mathbb{C})$ can be viewed as a representation of $Spin(p,q,\mathbb{C})$. The real case is then treated similarly to the discussion from 3.13.

7.17. Proposition. *The center $\mathcal{Z}$ of $\mathcal{C}\ell_m(\mathbb{K})$ is $1\mathbb{K} \oplus \nu\mathbb{K}$ if m is odd, and $1\mathbb{K}$ if m is even. The center of $\mathcal{C}\ell^0_m(\mathbb{K})$ equals to $1\mathbb{K} \oplus \nu\mathbb{K}$ for even dimensions and $1\mathbb{K}$ in odd dimensions.*

Proof. The proof goes similarly to the sublemma in 7.4. Consider an element $x \in \mathcal{C}\ell_m(\mathbb{K})$ which commutes or anti-commutes with each element $v \in \mathbb{K}^m$. We can decompose $x = x^0 + x^1$, the even and odd part of x, and the latter condition splits into

$$x^0 * e_i \mp e_i * x^0 = 0, \quad x^1 * e_i \mp e_i * x^1 = 0, \qquad i = 1, \dots, m.$$

Now, we fix e_i and express $x^0 = a^0 + e_i * a^1$ where a^j do not involve e_i. Hence we get $a^0 * e_i + e_i * a^1 * e_i \mp e_i * a^0 \mp e_i * e_i * a^1$ from the first condition and a^0 is an even element while a^1 is odd. Since they do not involve e_i we obtain $e_i * a^0 - Q(e_i)a^1 \mp e_i * a^0 \mp Q(e_i)a^1$. If x^0 commutes, this yields $2Q(e_i)a^1 = 0$, i.e. $a^1 = 0$. Since i was arbitrary this means x^0 does not involve any e_i, hence belongs to $\mathbb{K}$. If x^0 anti-commutes, then we get $2e_i * a^0 = 0$ and so $x^0 = e_i * a^1$ where a^1 does not involve e_i. Since this holds for all e_i, x^0 must be a multiple of $e_1 * \cdots * e_m$ which is possible only if m is even.

Similarly, write $x^1 = b^1 + e_i * b^0$ and apply the second condition. We get $b^1 * e_i + e_i * b^0 * e_i \mp e_i * b^1 \mp e_i * e_i * b^0 = 0$. Analogous considerations as above yield $b^1 = 0$ in the commuting case, so that $x^1 \in \nu\mathbb{K}$ and is non-zero only for odd m. If x^1 anti-commutes, then $b^0 = 0$, i.e. $x^1 \in \mathbb{K}$, hence zero.

So we have proved: if m is even, then $\mathbb{K}$ is the center and $\nu\mathbb{K}$ consists of all anti-commuting elements, while if m is odd, then the center is $\mathbb{K} \oplus \nu\mathbb{K}$ and there are no anti-commuting elements beside zero there.

[21] The proof involves only rather elementary manipulations with matrices, but it is not short.

Since the center of $\mathcal{C}\ell^0_m(\mathbb{K})$ consists of all elements which commute or anticommute with elements from $\mathbb{K} \subset \mathcal{C}\ell_m(\mathbb{K})$, the last statement of the proposition follows. $\square$

7.18. Dirac spinors and Weyl spinors. We have just proved that each of the algebras $\mathcal{C}\ell_{2n}(\mathbb{C})$ and $\mathcal{C}\ell^0_{2n+1}(\mathbb{C})$ admits precisely one faithful irreducible representation on the complex space $S = \mathbb{C}^{2^n}$, up to equivalence. The elements of this representation space S are called the *Dirac spinors.*

For example, starting with $V = \mathbb{C}^3$, we get the complex 2-component spinors, often also called the *Pauli spinors.* If $V = \mathbb{C}^4$, the Dirac spinors are complex 4-component. Let us remember, the Clifford algebras are explicitly identified with matrix algebras in even dimensions and so the generators e_i of $\mathcal{C}\ell_{2n}(\mathbb{C})$ act by the usual multiplication by the corresponding matrices γ_i, the so called *Dirac matrices.* For the explicit expressions of the Dirac matrices in low dimensions see 7.7 and 7.8.

Assume now, the dimension is even, $m = 2n$, and write $\gamma\colon \mathcal{C}\ell_m(\mathbb{C}) \to \mathrm{End}(S)$ for the faithful representation on the Dirac spinors. Fixing the canonical orthonormal base e_i in $V = \mathbb{C}^m$, the volume element $v = e_1 * \cdots * e_m$ satisfies $v * v = \pm 1$. We define $v' = v$ if $v^2 = 1$, while $v' = \sqrt{-1}v$ in the other case, so that $v'^2 = 1$. Hence $\gamma(v')\colon S \to S$ splits S into the ± 1-eigen spaces $S_\pm$. Since v' is in the center of $\mathcal{C}\ell^0_m(\mathbb{C})$, the restriction $\gamma_0 = \gamma|\mathcal{C}\ell^0_m(\mathbb{C})$ decomposes as $\gamma_0 = \gamma_+ \oplus \gamma_-$, $\gamma_\pm(y) = \frac{1}{2}(\mathrm{Id} \pm \gamma(v'))\gamma(y)$ for all $y \in \mathcal{C}\ell^0_m(\mathbb{C})$. Thus, we have got two irreducible inequivalent 2^{n-1}-dimensional (but not faithful) representations. The elements in S_+ and S_- are called the *Weyl spinors* of positive and negative *helicities.* They are also called right and left Weyl spinors, or *half-spinors* (Chevalley) or *reduced spinors* (Penrose and Rindler).

In view of 7.12, we have constructed two irreducible representations of the Lie group $Spin(2n, \mathbb{C})$, γ_+ on S_+ and γ_- on S_-, but also the irreducible representation γ of $Spin(2n+1, \mathbb{C})$ on S.

If we change our orientation of V, the volume element v is replaced by $-v$ and so the roles of the helicities are interchanged.

7.19. The odd dimensions. Let us consider the generating vector space $V = \mathbb{C}^{2n} \times \mathbb{C}$ and a generator e_{2n+1} in $\mathcal{C}\ell_{2n+1}(\mathbb{C})$ with $Q(e_{2n+1}) = 1$, the scalar product on $\mathbb{C}^{2n}$ is as before (the positive definite one works well). Using $\gamma\colon \mathcal{C}\ell_{2n}(\mathbb{C}) \to \mathrm{End}(S)$, we can define two irreducible representations of $\mathcal{C}\ell_{2n+1}(\mathbb{C})$ in S by setting $\gamma'_\pm(x) = \pm\gamma(x)$ for all $x \in \mathbb{C}^{2n} \subset \mathcal{C}\ell_{2n}(\mathbb{C})$, and $\gamma'_\pm(e_{2n+1}) = \pm(\gamma(v'))$ (notation form 7.18). The analogy to v' in dimension $2n+1$ is $v'' = v' * e_{2n+1}$, i.e. $v'' * v'' = 1$. Since $\gamma'_\pm(v'') = \pm\gamma(v') \circ \gamma(v') = \pm\mathrm{Id}$, these representations cannot be faithful. But their direct sum

$$\gamma' = \gamma'_+ \oplus \gamma'_- : \mathcal{C}\ell_{2n+1}(\mathbb{C}) \to \mathrm{End}(S) \oplus \mathrm{End}(S)$$

is a faithful representation in $S \oplus S$. Of course, the representations γ'_+ and γ'_- are equivalent when restricted to the even part $\mathcal{C}\ell^0_{2n+1}$ and then equivalent to the representation γ.

Let us notice that the $\gamma'_\pm$ can be equivalently obtained from the representations $\gamma_\pm$ in the even dimensions using the isomorphism $\mathcal{C}\ell^0_{2n+2}(\mathbb{C}) \simeq \mathcal{C}\ell_{2n+1}(\mathbb{C})$.

7.20. The matrix realization. If we use explicitly the description in 7.7–7.10, we find the important generators of the matrix algebras which realize the isomorphisms with the Clifford algebras. However, their choice can be quite different and we can get different (but equivalent) representations of the Clifford algebras on the spinors. The matrices σ, τ and ν are generators of $\mathrm{Mat}_2(\mathbb{C})$ and satisfy the same relations as the generators e_1, e_2, $e_1 * e_2$ in $\mathcal{C}\ell_2(\mathbb{C})$ where we take the positive definite scalar product.

Consider first the dimension $m = 2n$. Using the above matrices, we can define generators of $\mathrm{Mat}_{2^n}(\mathbb{C})$

$$\begin{aligned}\gamma_{2j-1} &= \tau \otimes \dots \otimes \tau \otimes \sigma \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2\\ \gamma_{2j} &= -\sqrt{-1}\tau \otimes \dots \otimes \tau \otimes \nu \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2\end{aligned}$$

with σ or ν on the j-th place, which satisfy $(\gamma_i)^2 = \mathbb{I}_m$ and $\gamma_i\gamma_k = -\gamma_k\gamma_i$ for all $k \neq i$. Thus we have found a concrete realization of $\mathcal{C}\ell_m(\mathbb{C})$ as a matrix algebra, i.e. one possible explicit form of the Dirac matrices.[22] If we consider the same tensor products of matrices, but we distribute the scalar multiples $\sqrt{-1}$ in another suitable way, we get algebras isomorphic to Clifford algebras corresponding to a prescribed scalar product with any signature. In particular, if there are no $\sqrt{-1}$, we get the so called *neutral Clifford algebras* $\mathcal{C}\ell_m(n, n, \mathbb{C})$. Of course, all these choices lead to isomorphic algebras in the complex case, but they become important if we pass to the real algebras and spinors, see below.

Let us examine how the 2^n equivalent spin representations γ sit in the Clifford algebra. Let us consider the elements $y_i = \sqrt{-1}e_{2i-1} * e_{2i}$, so that the corresponding matrices are $Y_i = \sqrt{-1}\gamma_{2i-1}\gamma_{2i} = \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2 \otimes \tau \otimes \mathbb{I}_2 \otimes \dots \otimes \mathbb{I}_2$ where τ is on the i-th place. The matrices Y_i are diagonal block matrices with $\pm\mathbb{I}_{n-i+1}$ in the blocks regularly changing the signs. Consider the right action of the matrix algebra on itself. This corresponds to the right action of the Clifford algebra on itself by multiplication. Each of the 2^n columns in $\mathrm{Mat}_{2^n}(\mathbb{C})$ is precisely the simultaneous eigen space corresponding to uniquely prescribed sequence of signs ± 1 with respect to this right action. Thus, the spin spaces sit in the Clifford algebra (complex with positive definite scalar product) as the simultaneous ± 1-eigen spaces for the right actions of the elements y_i.

If the dimension is odd, $m = 2n+1$, we need one more generator. We can choose

$$\gamma_{2n+1} = \tau \otimes \dots \otimes \tau$$

which anticommutes with all the generators above and has square one.

7.21. The real spinors. The complexification of each real Clifford algebra $\mathcal{C}\ell_m(p, q)$ is isomorphic to $\mathcal{C}\ell_m(\mathbb{C})$. Hence there is always an injection $\mathcal{C}\ell_m(p, q) \to \mathcal{C}\ell_m(\mathbb{C})$ of algebras and so each representation of the complex Clifford algebra can

[22] The procedure leading to such explicit representations consists in choosing a way how to pass from a representation of $\mathcal{C}\ell_{2n}(\mathbb{C})$ on $S = \mathbb{C}^{2^n}$ to a representation of $\mathcal{C}\ell_{2n+2}(\mathbb{C})$ on $S \oplus S = \mathbb{C}^{2^{n+1}}$. There are several well known extension procedures, let us mention the Brauer-Weyl extension, the Cartan extension, the Dirac extension. We have used the latter one.

be restricted to the real one. This yields representations of $\mathcal{C}\ell_m(p,q)$, but in complex vector spaces. These restrictions are irreducible if the complex representations are irreducible which means that there are no invariant complex subspaces in the representation space.

If we start with a real representation of $\mathcal{C}\ell_m(p,q)$, we can complexify it to obtain a complex representation of the complexification $\mathcal{C}\ell_m(p,q)\otimes\mathbb{C}$. However, if we have started with the complex representation, the restriction to the real algebra may or may not admit an invariant real subspace in the representation space. Let us indicate very briefly what can happen if we restrict the spin representations of the Clifford algebras. Much more details can be found in [Budinich, Trautman, 88, Section 7.2].

Let us consider the dimension $m=2n=p+q$. To each complex space W we associate the complex conjugate space $\bar{W}$ which is the same as W if viewed as a real vector space but the scalar multiplication by $a\in\mathbb{C}$ differs from W by taking $\bar{a}$. If we write w for the elements of W then $\bar{w}$ are the elements in $\bar{W}$, $w\mapsto\bar{w}$ is the identity of the real spaces. Each linear map $f: W_1\to W_2$ induces a linear map $\bar{f}\colon \bar{W}_1\to\bar{W}_2$, $\bar{f}(\bar{w})=\overline{f(w)}$. The correspondence $f\mapsto\bar{f}$ is not linear as $\lambda f\mapsto\bar{\lambda}\bar{f}$. The bar mapping is compatible with the duals, i.e. $\overline{W^*}\simeq(\bar{W})^*$ and the Hermitean conjugate map to f is defined by $\bar{f}^t: \bar{W}_2^*\to\bar{W}_1^*$.

Consider now the restriction of the spin representation $\gamma\colon\mathcal{C}\ell_m(p,q)\to\mathrm{End}_{\mathbb{C}}S$ and the conjugate $\bar{\gamma}\colon\mathcal{C}\ell_m(p,q)\to\mathrm{End}_{\mathbb{C}}\bar{S}$. We shall write $\mu=p-q \bmod 8$. Since the center of the Clifford algebra is the field of the scalars, there is a $\mathbb{C}$-linear isomorphism $C\colon S\to\bar{S}$ which intertwines the representations γ and $\bar{\gamma}$, and which satisfies either $\bar{C}C=\mathrm{Id}$ if $\mu=0$ or 2, or $\bar{C}C=-\mathrm{Id}$ if $\mu=4$ or 6 (this needs of course a proof). The first case is called real while the other one quaternionic. In the real case, $\gamma=\gamma^+ +\gamma^-$ decomposes and $\gamma^\pm:\mathcal{C}\ell_m(p,q)\to\mathrm{End}_{\mathbb{R}}S^\pm$ are two real equivalent representations. The elements of $S^\pm$ are called the *Majorana spinors* (of the first kind). They can be also characterized by $S^\pm=\{s\in S;\ \bar{C}(\bar{s})=\pm s\}$. The restriction of γ to $\mathcal{C}\ell_m^0(p,q)$ decomposes even in the complex case into the eigen spaces of the action of the suitable multiple of the volume element v'. The same takes place for the complex conjugate space $\bar{S}$ and $\bar{v}'$. One computes $\bar{\gamma}(\bar{v}')\circ C=(-1)^{\mu(\mu-1)/2}C\circ\gamma(v')$ and so C respects the helicity if $\mu=0$ or 4, but changes the helicity if $\mu=2$ or 6. If $\mu=2$, then there are $\mathbb{C}$-linear isomorphisms $F_\pm\colon S_\pm\to S^\pm$ constructed by means of v' and C, but $S_\pm\cap S^\pm$ is zero for all combinations of signs. If $\mu=0$ or $\mu=6$ then we can find another decomposition of the Dirac spinors, $S=S_i^+\oplus S_i^-$ with $S_i^\pm=\{s\in S;\ \bar{C}\bar{v}'(\bar{s})=\pm s\}$ (notice $\bar{C}\bar{v}'Cv'=\mathrm{Id}$ if $\mu=0$ or $\mu=6$). But these spinors, called Majorana spinors of the second kind, are invariant under the action ${}_i\gamma(s)=\sqrt{-1}\gamma(s)$ of $\mathcal{C}\ell_m(q,p)$ but not under the action γ of $Cl_m(p,q)$. If $\mu=6$, they are equivalent to the Majorana spinors of the first kind for the algebra $\mathcal{C}\ell_m(q,p)$ and we have once more the $\mathbb{C}$-linear isomorphisms $F_\pm$. The representations $\gamma^\pm$ are intertwined by the multiplication by $\sqrt{-1}$. If $\mu=0$, then all representations γ, $\gamma_\pm$ are real and there are non-zero intersections of $S^\pm$ and $S_i^\pm$. Thus, the real form of S decomposes into four real 2^{n-1}-dimensional space $S_\pm^\pm$, the so called Weyl-Majorana spinors. If $\mu=4$, there are no Majorana spinors.

7.22. Dirac operators on the Weyl spinors. Let us consider an even dimension $m=2n$ and let us specialize the Clifford multiplication from 7.14 to the Clifford modules $S_\pm$ of Weyl spinors viewed as the complex representations of the real algebras. We get

Proposition. *The Clifford multiplication* $\bullet\colon \mathbb{R}^m \otimes S \to S$ *interchanges the helicities, i.e. it restricts to the mappings* $\bullet\colon \mathbb{R}^m \otimes S_\pm \to S_\mp$. *The same holds for the complex Clifford multiplication.*

Proof. It suffices to check the mappings on the generators. Let us remember that the Weyl spinors are ± 1-eigen spaces for the v', where v' is either the volume v or $\sqrt{-1}v$. For all $e_i \in \mathbb{R}^m$, $s \in S_+$ we get $v'.(e_i \bullet s) = \gamma(v' * e_i)(s) = a\gamma(e_1 * \dots * e_m * e_i)(s) = a(-1)^{m-i}\gamma(e_1 * \dots {}^{\wedge_i} \dots * e_m)(s) = (-1)^{i-1}(-1)^{m-i}\gamma(e_i * v') = -e_i \bullet (v'.s)$ since m is even (a is either 1 or $\sqrt{-1}$). $\square$

Let us also write Δ and $\Delta_\pm$ for the (real) bundles over pseudo-Riemannian manifolds corresponding to the (complex) spin representations. Since the Riemannian covariant derivative is a natural operator, it must respect subbundles coming from $Spin(2n)$-invariant submodules. Hence the Dirac operator $D\colon \Delta \to \Delta$ decomposes as

$$D_\pm : \Delta_\pm \to \Delta_\mp$$

in the even dimensions. We claim that the operators D and $D_\pm$ are the operators discussed in the Example 6.22. In order to see this explicitly, we have to find the highest weights of the basic spin representations and for that reason we need a good description of the Lie algebra.

7.23. The Lie algebra $\mathfrak{o}(m+1, \mathbb{C})$. Write $m+1 = 2n$ or $m+1 = 2n+1$ for the dimension. Let us consider the usual positive definite scalar product, hence the Lie algebra is generated (as a vector space) by the matrices $A_{ij} = E_{ij} - E_{ji}$ and their commutators are (remember $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$)

$$[A_{ij}, A_{kl}] = \delta_{jk}A_{il} + \delta_{il}A_{jk} - \delta_{jl}A_{ik} - \delta_{ik}A_{jl}.$$

The matrices A_{ij} admit two eigen values, ± 1, the commutative subalgebra $\mathfrak{h}$ generated by $A_{12}, A_{34}, \dots, A_{2n-1,2n}$ is the Cartan subalgebra. A general element in $\mathfrak{h}$ has the form $X = \tau_1 A_{12} + \dots + \tau_n A_{2n-1,2n}$. The element $(m_1, \dots, m_n) \in \mathbb{C}^{n*}$ is a weight of a representation φ if all $\varphi(A_{2i-1,2i})$ admit a common eigen vector such that the corresponding eigen value for $H_i = A_{2i-1,2i}$ is $\sqrt{-1}m_i$, i.e. the eigen value for X is $\sqrt{-1}(m_1\tau_1 + \dots + m_n\tau_n)$.

If we choose a (weak) order in $\mathfrak{h}$, then the highest weights are those ones with weight vectors under trivial action of the positive root elements, or equivalently the maximal ones in the chosen order. The multiplication of the weights by $\sqrt{-1}$ corresponds to the isomorphism which transforms the scalar product we use now, to the scalar product we use in 10.10–10.11. Hence the fundamental weights remain unchanged.

Consider now $Spin(m+1, \mathbb{C}) \subset \mathcal{C}\ell^0_{m+1}(\mathbb{C}) \simeq \mathcal{C}\ell_m(\mathbb{C})$. We shall identify $\mathfrak{o}(m+1, \mathbb{C})$ with a subspace in $\mathcal{C}\ell_m(\mathbb{C})$. Let us define the bracket $[\ ,\ ]$ on $\mathcal{C}\ell_m(\mathbb{C})$ by $[x, y] = x * y - y * x$, i.e. $[e_i, e_j] = 2e_i * e_j$, $i \neq j$, for the generators, and write $\alpha_{0j} = e_j = -\alpha_{j0}$, $\alpha_{ij} = [e_i, e_j]$. Hence the $\frac{1}{2}n(n+1)$ elements α_{jk} with $j < k$ are linearly independent and $\alpha_{jk} = -\alpha_{kj}$. An elementary computation leads to the commutators

$$\begin{aligned} [\alpha_{ij}, \alpha_{kl}] &= 4(\delta_{jk}\alpha_{il} + \delta_{il}\alpha_{jk} - \delta_{jl}\alpha_{ik} - \delta_{ik}\alpha_{jl}) && \text{for all } i,j,k,l \text{ non-zero} \\ [\alpha_{0j}, \alpha_{kl}] &= 4(\delta_{jk}\alpha_{0l} - \delta_{jl}\alpha_{0k}) && \text{for } j,k,l \text{ non-zero} \\ [\alpha_{0j}, \alpha_{0k}] &= \alpha_{jk} && j,k \neq 0 \end{aligned}$$

We would like to have generators X_{ij}, $0 \le i < j \le m$, which satisfy the same commutator relations as the above generators A_{ij} of $\mathfrak{o}(m+1, \mathbb{C})$. First of all the commutators have the right form in the case of indices different from zero, up to the multiple $\frac{1}{4}$. Further, if $i = k$ then we need $[X_{ij}, X_{kl}] = -X_{jl}$, so the $\alpha_{0j} = e_j$ must be multiplied by some pure imaginary scalar. Finally, the second and third rows in the above commutators suggest $\pm\frac{\sqrt{-1}}{2}$ for this scalar factor and we shall use the minus sign to fit with the earlier choice of the Dirac matrices γ_i. Now one checks by elementary computations that the choice of generators

$$\begin{aligned} X_{0j} &= -\frac{\sqrt{-1}}{2}\alpha_{0j} = -\frac{\sqrt{-1}}{2}e_j \\ X_{jk} &= \frac{1}{4}\alpha_{jk} = \frac{1}{2}e_j * e_k \end{aligned}$$

leads really to a Lie algebra sitting in the Clifford algebra $\mathcal{C}\ell_m(\mathbb{C})$ which is isomorphic to $\mathfrak{o}(m+1, \mathbb{C})$. The bracket in this algebra is precisely the commutator and so there is the analogy to Proposition 7.12:

Proposition. *Each representation of the Clifford algebra $\mathcal{C}\ell_m\mathbb{C}$ induces the representation of the Lie algebra $\mathfrak{o}(m+1, \mathbb{C})$ given by the restriction.*

Proof. The generators of $\mathcal{C}\ell_m(\mathbb{C})$ are contained in $\mathfrak{o}(m+1, \mathbb{C})$ and every representation of the Clifford algebra respects the commutators by the definition. □

In fact there is the other part of the proposition which we shall not need in general: each representation of $\mathfrak{o}(m+1, \mathbb{C})$ induces a representation of the spin group (for the latter is simply connected) and therefore a representation of the Clifford algebra $\mathcal{C}\ell_m(\mathbb{C})$ as well. However, the resulting representation may fail to be an extension of the original one. We shall need this correspondence of representations only for the spinors. This is the identical representation of the corresponding matrix algebra and so it remains the same as a representation of the spin group.

7.24. The weights of spin representations. Let us consider first the group $Spin(2n+1, \mathbb{C})$ and its faithful irreducible representation on the spinors $S = \mathbb{C}^{2^n}$ (unique up to equivalence). The weights are evaluated from the corresponding representation of the Lie algebra $\mathfrak{o}(2n+1, \mathbb{C})$. If we view $Spin(2n+1, \mathbb{C})$ as a subgroup in the matrix algebra $\mathrm{Mat}_{2^{2n}}(\mathbb{C})$ then the representation is the identical one and so the induced representation of the Lie algebra is also the standard identical one. First of all we have to find the expression for the elements $H_i \in \mathcal{C}\ell_{2n}(\mathbb{C})$ from the Cartan algebra as elements in the corresponding matrix algebra. We shall use the explicit representation of $\mathcal{C}\ell_m(\mathbb{C})$ as a matrix algebra from 7.20 (consult [Boerner, 67, Chapter VIII] for more details here or below if necessary). Let us recall the generators $\gamma_{2j-1} = \tau\otimes\cdots\otimes\tau\otimes\sigma\otimes\mathbb{I}_2\otimes\cdots\otimes\mathbb{I}_2$ and $\gamma_{2j} = -\sqrt{-1}\tau\otimes\cdots\otimes\tau\otimes\nu\otimes\mathbb{I}_2\otimes\cdots\otimes\mathbb{I}_2$ where τ and ν are at the j-th place and there are altogether n (2×2)-matrices in the expressions.

Now, we use the above description of the Cartan algebra with the zero index replaced by $2n+1$ (in fact the elements X_{0j} will not appear explicitly at all, for they do not belong to the Cartan subalgebra). Then

$$H_i = \frac{1}{2}\gamma_{2j-1}\gamma_{2j} = -\frac{\sqrt{-1}}{2}\mathbb{I}_2\otimes\cdots\otimes\mathbb{I}_2\otimes\tau\otimes\mathbb{I}_2\otimes\cdots\otimes\mathbb{I}_2,$$

see the identities in 7.7. Thus, the H_i are diagonal matrices with the same number of the entries $\frac{1}{2}\sqrt{-1}$ and $-\frac{1}{2}\sqrt{-1}$. Inspecting the distribution of the signs, we see that a common eigen vector in $\mathbb{C}^{2^n}$ can involve only one non-zero entry. Hence the weights are precisely of the form $(\pm\frac{1}{2},\dots,\pm\frac{1}{2})$ and according to our choice of the order, the highest among them is the weight $(\frac{1}{2},\dots,\frac{1}{2})$. This shows that the spin representation is really the remaining representation among those corresponding to the fundamental weights, see the last section. Thus, for the odd dimension $m=2n+1$, all representations of $Spin(m,\mathbb{C})$ are involved in tensor products of the exterior forms of degrees less then n and the spin representation on $S=\mathbb{C}^{2^n}$. Since we know that all representations of the real spin groups are obtained from suitable complexifications, see 6.4, we can use the above result for the real case as well (but it is not simple at all to get concrete results, cf. 7.21).

7.25. The even dimensions. Consider now $Spin(2n+2,\mathbb{C})$, so we have to study the representation of $\mathcal{C}\ell_{2n+1}(\mathbb{C})$. We can proceed analogously, but the Cartan algebra contains now additionally the matrix H_{n+1} which has a quite different form, for it corresponds to the generator $X_{0,2n+1} = -\frac{\sqrt{-1}}{2}e_{2n+1}$, see 7.23. In 7.19, we defined the two representations $\gamma'_\pm$ of $\mathcal{C}\ell_{2n+1}(\mathbb{C})$ on S. On the $(n+1)$-st generator they were defined through the volume element with the proper scalar multiple. If we perform the necessary identification with a matrix in $\mathrm{Mat}_{2^n}(\mathbb{C})$ we get the action of

$$H'_{n+1} = \pm\frac{\sqrt{-1}}{2}\tau\otimes\cdots\otimes\tau.$$

This is also a diagonal matrix with the entries of the form $\pm\frac{\sqrt{-1}}{2}$. If we inspect once more the distribution of the signs, we conclude that the highest weights are precisely $(\frac{1}{2},\dots,\frac{1}{2},\pm\frac{1}{2})$ and they correspond to the Weyl spinors with positive and negative helicities.

7.26. Tensor products of spin representations. We know from the representation theory that all irreducible representations of the spin groups must appear in the tensor products of the spin representations and the exterior forms, cf. 10.11. Let us describe the situation more explicitly in the even dimensional case, $m=2n$. We shall write Λ for the complexification $\Lambda\mathbb{R}^m\otimes\mathbb{C}$ (i.e. $\Lambda=\mathcal{C}\ell_m(\mathbb{C})$ as a vector space), Λ_e and Λ_o for the even and odd forms, while $\Lambda_\pm$ are the eigen spaces of the action of the suitable multiple v' as in the definition of the Weyl spinors. The left multiplication by the volume element plays the role of the Hodge star operator, in particular, the splitting of the exterior form of degree n coincides with the splitting

$\Lambda^n_\pm$ discussed in the last section. Beside these homogeneous forms, the spaces $\Lambda_\pm$ are generated by linear combinations from $\Lambda^p \oplus \Lambda^{m-p}$.

If we let act the spin group on the Clifford algebra by right multiplication by inverse elements, we get an equivalent representation and the decomposition into a sum of 2^n equivalent spin modules S exactly as in the decomposition in 7.16. This is best seen on the matrix realization: The generators $Y_i = \gamma_{2i-1} * \gamma_{2i}$ of the even part of the Clifford algebra are symmetric (see 7.17) and the remaining $\gamma_{2i} * \gamma_{2i+1}$ equals to $\sqrt{-1}\mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2 \otimes \sigma \otimes \sigma \otimes \mathbb{I}_2 \otimes \cdots \otimes \mathbb{I}_2$, hence is also symmetric. If we apply the transposition to these generators they should change the signs, but this corresponds to the transposition of the corresponding matrix generators. Thus, the transposition $(e_{i_1} * \cdots * e_{i_{2p}})^t$ on $\mathcal{C}\ell^0_m(\mathbb{C})$ corresponds to the transposition of the corresponding matrix accomplished with suitable sign depending on p mod 2. The spin representations as right $\mathcal{C}\ell^0_m(\mathbb{C})$-modules are the rows in the matrices with the right multiplication by the matrices from the algebra. We define a linear mapping $f\colon S \otimes S \to \mathcal{C}\ell_m(\mathbb{C})$ by $f(u \otimes v) = u * v^t$, i.e. we view $S \otimes S$ as the tensor product of one left and one right $Spin(m, \mathbb{C})$ module. This is a linear isomorphism, which is easily seen on the matrices ($E_{j1}E_{1k} = E_{jk}$, $1 \le j, k \le 2^n$, and so f is surjective, but the dimensions of Λ and $S \otimes S$ coincide). The (twisted) adjoint representation of $Spin(m, \mathbb{C})$ on $\mathbb{C}^m \subset \mathcal{C}\ell_m(\mathbb{C})$ is precisely the usual standard representation of $SO(m, \mathbb{C})$ and its extension to the whole algebra coincides with the standard representation of $SO(m, \mathbb{C})$ on the exterior forms Λ. By the definition, f intertwines the action of $\mathcal{C}\ell_m(\mathbb{C})$ on $S \otimes S$ and the adjoint action (warning: the right-hand S is the right $\mathcal{C}\ell_m$ module). We can also get information on the behavior of subspaces:

Proposition. *There are the following equivalences of representations:*

$$\begin{aligned}
\Lambda &= S \otimes S \\
\Lambda_+ &= (S_+ \otimes S_+) \oplus (S_+ \otimes S_-) \\
\Lambda_- &= (S_- \otimes S_+) \oplus (S_- \otimes S_-) \\
\Lambda_e &= \begin{cases} (S_+ \otimes S_+) \oplus (S_- \otimes S_-) & \text{if } n \text{ is even} \\ (S_- \otimes S_+) \oplus (S_+ \otimes S_-) & \text{if } n \text{ is odd} \end{cases} \\
\Lambda_o &= \begin{cases} (S_- \otimes S_+) \oplus (S_+ \otimes S_-) & \text{if } n \text{ is even} \\ (S_+ \otimes S_+) \oplus (S_- \otimes S_-) & \text{if } n \text{ is odd} \end{cases}
\end{aligned}$$

Proof. The first equivalence has been already proved, the isomorphism is $u \otimes v^t \mapsto u * v$. By the definition, $\Lambda_\pm$ are the eigen spaces of the left multiplication by v', hence $\Lambda_+ = S_+ \otimes S$ and $\Lambda_- = S_- \otimes S$. The volume element v' satisfies $v'^t = (-1)^{2n(2n-1)/2} v' = (-1)^n v'$. In the proof of 7.22 we derived that each generator e_i commutes with v' with the change of its sign. Thus, the odd elements w in Λ are precisely those with $v' * w * (v')^t = (-1)^n v' * w * v' = (-1)^{\text{degree of } w} (-1)^n w$ and this implies the description of the odd and even forms. □

7.27. The inner products on spinors. Consider the space S of Dirac spinors with the faithful representation γ of $\mathcal{C}\ell_{2n}(\mathbb{C})$ or $\mathcal{C}\ell^0_{2n+1}(\mathbb{C})$ and its dual space S^* with the representation $\gamma^t(a) = (\gamma(a^t))^t$. If restricted to the spin group, this is precisely the contragredient representation. Since these representations must be equivalent, there is a linear isomorphism $\varepsilon\colon S \to S^*$ intertwining these representations. This defines a bilinear non-degenerate form $\varepsilon(s_1, s_2) = \varepsilon(s_1)(s_2)$ denoted by the same symbol. If we define $\varepsilon^t(s_1)(s_2) = \varepsilon(s_2, s_1)$ we get a mapping which must be proportional to ε by the Schur's lemma. Since $\varepsilon^{tt} = \varepsilon$, the multiple must be ± 1. This means ε is either symmetric or skew. We can check which of the possibilities takes place by evaluating $\varepsilon(\gamma(a)(s), s)$ with suitable elements $a \in \mathcal{C}\ell_{2n}(\mathbb{C})$ and $s \in S$. Let us pass to the matrix realization and choose $a = \tau \otimes \cdots \otimes \tau$, i.e. a volume element, and s be the column vector with only the first entry non-zero. Hence $\gamma(a)(s) = s$, $a^t = (-1)^{2n(2n-1)/2} a = (-1)^n a$ and we get $\varepsilon(s, s) = \varepsilon(\gamma(a)s, s) = \gamma(a^t)^t(\varepsilon(s))(s) = \varepsilon(s, \gamma(a^t)(s)) = (-1)^n \varepsilon(s, s)$. Therefore, the inner product ε is symmetric if $n = 0 \bmod 2$, while ε is skew if $n = 1 \bmod 2$.

The next question is: what about an inner product on the Weyl spinors? The Weyl spinors are ± 1-eigen spaces for the multiplication by the proper volume element v' and the same is true for the duals. We have seen $(v')^t = (-1)^n v'$. Hence $\gamma(v')^t \circ \varepsilon = (-1)^n \varepsilon \circ \gamma(v')$ and the inner product ε restricts to the Weyl spinors if n is even. We shall denote the products by ε_+ and ε_-.

7.28. The four-dimensional case. Let us work out more explicit formulas in the (most interesting) case of dimension $m = 2n = 4$. The Dirac spinors are complex 4-component. In the above identification, the Dirac matrices are ($i = \sqrt{-1}$)

$$\gamma_1 = \sigma \otimes \mathbb{I}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \gamma_2 = -i\nu \otimes \mathbb{I}_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

$$\gamma_3 = \tau \otimes \sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad \gamma_4 = -i\tau \otimes \nu = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}$$

The volume element is then

$$\nu = \tau \otimes \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \nu * \nu = \mathbb{I}_4.$$

Hence the Weyl spinors S_+ and S_- of positive and negative helicities are precisely $(a, 0, 0, b) \in \mathbb{C}^2 \subset \mathbb{C}^4$ and $(0, a, b, 0) \in \mathbb{C}^2 \subset \mathbb{C}^4$. We have found that these representations are irreducible and their highest weights are $\rho^+ = \frac{1}{2}(e^1 + e^2)$ and $\rho^- = \frac{1}{2}(e^1 - e^2)$, see 10.11 and 6.22 for the notation. The tensor product $S_+ \otimes S_-$ must involve the representation corresponding to the dominant weight $\rho^+ + \rho^- = e^1$ with multiplicity one. But the dimension of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ is exactly the dimension of $\mathbb{C}^4$ which corresponds to the weight e^1. Thus, the tensor product of the two different half-spin representations is equivalent to the identical representation on $\mathbb{C}^4$. This shows how all tensor representations of $SO(m, \mathbb{C})$ arise from the fundamental ones, i.e. from the spin representations. In Proposition 7.26 we proved $\Lambda_o = (S_+ \otimes S_-) \oplus (S_- \otimes S_+)$, hence the second summand corresponds to Λ^3. The product $S_+ \otimes S_+$ contains the invariant subspace with highest weight $e^1 + e^2$. However, its dimension is only 3.[23] These are the positive exterior two-forms Λ^2_+. The remaining one dimensional space corresponds to the trivial representation on the field of scalars, Λ^0. Similarly $S_- \otimes S_-$ splits to one dimensional representation Λ^4 and the other half Λ^2_- of Λ^2.

In conformal geometry, one often meets elements from tensor products of several copies of S_+, S_- and their duals S^*_-, S^*_+, or even mixed with tensors. Similarly as with the tensors in the previous text, we shall use the Penrose's abstract index notation. We have chosen the small italics superscripts (with possible further indices like a_1, b_p, etc.) as labels for distinct but isomorphic copies of $\mathbb{K}^m$, while the same labels as subscripts indicate always copies of $\mathbb{K}^{m*}$. If we want a similar notation for spinors, we need two further kinds of labels. We choose the capital italic superscripts (with possible further indices) for copies of S_+ and the same subscripts for S^*_+. The same labels with primes will indicate the spaces S_- and S^*_-. In view of the above description of the tensor products of spinors, this becomes a very powerful notation (in the dimension 4). Let us add some further conventions. We have proved $t^{AA'} = t^a$ (i.e. tensor product of S_+ and S_- is $\mathbb{C}^4$) and we shall adopt this convention also for general expressions like $\dots C_j C'_j \dots = \dots c_j \dots$. The skew inner products ε defined in 7.27 are elements ε_{AB}, $\varepsilon_{A'B'}$, ε^{AB}, $\varepsilon^{A'B'}$, antisymmetric in the indices. These elements allow rising and lowering of indices similarly to that induced by a metric on tensors, but since they are antisymmetric we have to fix the usage of the indices: $s^{\dots A \dots} = \varepsilon^{AB} s^{\dots}{}_{B}{}^{\dots}$ where the dots can involve both subscripts and superscripts. In particular, $\varepsilon_A{}^B = \varepsilon^{BC}\varepsilon_{AC} = -\varepsilon^{BC}\varepsilon_{CA} = -\varepsilon^B{}_A$.

The tensor product $\varepsilon_+ \otimes \varepsilon_-$ is a linear isomorphism $\mathbb{C}^4 \to \mathbb{C}^{4*}$ which intertwines the standard representations and so it corresponds to the original scalar product g on $\mathbb{C}$. This is expressed by $\varepsilon_{AB}\varepsilon_{A'B'} = g_{ab}$. As seen on $\varepsilon^A{}_B = -\varepsilon_B{}^A$, we have to be very careful to preserve the order of the primed and unprimed indices (including superscripts and subscripts) separately, while the relative order of the primed and unprimed ones is not important. The symmetrizations and alternations in some entries are denoted on the indices exactly as with the tensor indices.

A special convection concerns the pseudo-Riemannian covariant derivative ∇. This is an operator with one vector argument, hence we have denoted it by ∇_a and its value on a tensor was $\nabla_a t^{\dots}_{\dots}$, understand as one symbol. Now we can use the covariant derivative on all spinors and write $\nabla_{AA'} t^{\dots}_{\dots}$ where the dots may involve all three types of indices. Moreover, we can rise and lower all indices, e.g. $\nabla_A{}^{A'} t^{\dots}_{\dots}$. Let us notice that this is a very effective notation. For example, $T_{ab} = T_{ABA'B'}$ for

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[23] There is the famous Weyl's degree formula: The dimension of an irreducible representation corresponding to a dominant form λ is

$$d_\lambda = \frac{\prod_{\alpha>0}\langle \alpha, \lambda + \delta\rangle}{\prod_{\alpha>0}\langle \alpha, \delta\rangle}$$

where the products go over all positive roots and δ is half the sum of all positive roots.

In our case the positive roots are chosen as $e^1 - e^2$ and $e^1 + e^2$, hence $\delta = e^1$. The Killing form is the Euclidean scalar product up to a scalar multiple which does not play any role in the formula. For $\lambda = e^1 + e^2$ we get immediately the dimension 3.

every twice covariant tensor but $T_{(AB)(A'B')}$ is a simple expression for its symmetric trace-free part! This follows from the antisymmetry of $\varepsilon^{AB} = -\varepsilon^{BA}$ which is used in the trace.

In 6.22, we found two other operators beside the Dirac operators. They are defined on the sections of the bundles Δ_+ and Δ_- of the Weyl spinors and their values are in the bundles corresponding to the representations $S_+ \otimes S_+ \otimes S_-$ and $S_- \otimes S_+ \otimes S_-$. Now we are able to write down a simple formula for these operators:

$$D(s^B) = \nabla_{A'}^{(A} s^{B)}, \qquad D(s^{B'}) = \nabla_A^{(A'} s^{B')}.$$

They have values in the required spaces, symmetric in the unprimed or primed indices and trace-free. The whole $S_+ \otimes \mathbb{C}^4$ decomposes into S_- and another space corresponding to the weight $\frac{1}{2}(3e^1 + e_1)$. (Its dimension is six as easily computed using the Weyl's degree formula.) Similarly we get the other case. The first operator is called the *twistor operator*, its solutions are called the (global) *twistors*.

8. Verma modules and natural operators

In this section we present the complete classification of natural linear operators on first order natural vector bundles on locally flat conformal manifolds, which is achieved by means of the methods from representation theory. Our inspiration is [Baston, 90], and [Baston, Eastwood, 90], however we succeed also in the case of singular infinitesimal characters and we present complete (and rather elementary) proofs. In particular, we correct some claims of the latter survey paper. Some basic notions and results from representation theory are outlined in the Appendix.

8.1. The main idea. Each locally flat conformal manifold is locally isomorphic to the sphere, so that we shall restrict ourselves to the homogeneous bundles over the (pseudo-) spheres without loss of generality. Let us fix two such bundles $E = E_\lambda$ and $F = F_\rho$ corresponding to irreducible representations V_λ, V_ρ, for two weights λ, ρ of $\mathfrak{g} = \mathfrak{o}(m+2, \mathbb{C})$, dominant for the Poincaré conformal subalgebra $\mathfrak{b}$, i.e. V_λ and V_ρ are (real or complex) representation spaces either for the Poincaré conformal group B or for its simply connected covering. This notation is different form that used in Section 6, where the weights were dominant weights of $\mathfrak{o}(m, \mathbb{C})$ and the remaining information was involved in the conformal weight. The explanation of the present notation is in 10.13 and 10.14. Let us remind that all linear representations of orthogonal groups are completely reducible and the action of the nilpotent part must be trivial in each irreducible representation of the Poincaré algebra. Thus, the above restriction to the irreducible representations means in fact that we will describe operators on all first order natural bundles.

In fact we used the general idea in the first order case in the proof of 6.10, cf. 6.13. According to the non-linear version of the Peetre theorem, each local operator $D\colon C^\infty(E_\lambda M) \to C^\infty(F_\rho M)$ on sections of bundles $E_\lambda M$ and $F\rho M$ over the same base M factors through a mapping $\tilde{D}\colon J^\infty(E_\lambda M) \to F_\rho M$, see [Slovák, 88] or [Kolář, Michor, Slovák, 93]. For a linear operator we get even the finiteness of the order and a smooth $\tilde{D}\colon J^k(E_\alpha M) \to F_\rho M$ (this is the classical Peetre theorem).

The locally flat conformal manifolds are homogeneous enough to apply the general theory of natural bundles and operators, see Section 2. In particular, the whole operator is completely determined by the equivariant mapping $\tilde{D}_M\colon J^k_x(EM) \to F_xM$ for an arbitrary point $x \in M$, with respect to the group of locally defined conformal isomorphisms at x keeping x fixed.

Thus, in order to classify linear natural operators $D\colon C^\infty(E_\lambda) \to C^\infty(F_\rho)$ on locally flat conformal manifolds, we have to find all B-equivariant linear mappings $D\colon J^k_0E_\lambda \to (F_\rho)_0 = V_\rho$, where 0 is the coset in G/B containing the unit e. Dualizing this mapping, we get a B-equivariant mapping $D^*\colon (V_\rho)^* \to (J^k_0E)^*$. Since $(V_\rho)^*$ is irreducible, all such mappings are uniquely determined by the highest weight vectors in $(J^k_0E)^*$ with the same weight as $(V_\rho)^*$. Then the mapping D is the dual mapping to the corresponding inclusion. The main technical step is a suitable identification of $(J^k_0E)^*$. In Section 6 we derived the action only up to the first order. Now, the most effective way is to deal with the direct limit of $(J^k_0E)^*$ which will be identified with a generalized Verma module.

More exactly, we shall solve the whole classification problem on the Lie algebra level, i.e. we shall discuss the equivariance with respect to the action of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on the duals of the jet spaces. Let us recall that this is an equivalent formulation of the problem as shown in 6.7 and 6.8. The passing to the Lie algebras has two big advantages. First, we can forget about the coverings and, which is more important, the derivatives with respect to constant vector fields enable us to work still in a single fiber but to involve the translations into the equivariance conditions at the same time.

8.2. The $\mathfrak{U}(\mathfrak{g})$-module $(J^\infty_0E)^*$. As usual the sections of the homogeneous bundle E are identified with B-equivariant mappings $G \to V_\lambda$ and the jets of sections form a submanifold in $J^k_e(G, V_\lambda)$. Then the action of G is given by the composition with the left translation by the inverse, see 2.10. Let us identify the real universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ with the Lie derivatives with respect to right invariant vector fields on G and consider an element $x \otimes v^* \in \mathfrak{U}(\mathfrak{g}) \otimes (V_\lambda)^*$. An element $X \in \mathfrak{g} \subset \mathfrak{U}(\mathfrak{g})$ is identified with the Lie derivative $\mathcal{L}_{-X}$ with respect to the right invariant vector field on G, see 6.1. This identification is extended to the actions $\mathcal{L}_x$ for all $x \in \mathfrak{U}(\mathfrak{g})^k$. Then we can associate an element in $(J^k_0E)^*$ to each $x \otimes v^* \in \mathfrak{U}(\mathfrak{g})^k \otimes V^*_\lambda$ acting on $j^k_e s$ by $(x \otimes v^*)(j^k_e s) = \langle \mathcal{L}_x s(e), v^* \rangle$. However this identification is not one-to-one since for $x \in \mathfrak{U}(\mathfrak{b})$ where $\mathfrak{b} \subset \mathfrak{g}$ is the Lie algebra of B, we get the same action of $x \otimes v^*$ and $1 \otimes x.v^*$ where $x.v^*$ is the contragredient action. Let us write $I \subset (\mathfrak{U}(\mathfrak{g}) \otimes V^*_\lambda)$ for the left $\mathfrak{U}(\mathfrak{g})$-submodule generated by all $x \otimes v^* - 1 \otimes x.v^*$ with $v^* \in V^*_\lambda$, $x \in \mathfrak{U}(\mathfrak{b})$ and define

$$M_\mathfrak{b}(V^*_\lambda) = (\mathfrak{U}(\mathfrak{g}) \otimes V^*_\lambda)/I = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V^*_\lambda.$$

This is the generalized Verma module corresponding to the weight λ dominant for the parabolic subalgebra $\mathfrak{b} \subset \mathfrak{g}$, see 10.18.

We have $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}$ and $\mathfrak{b}_{-1} = \mathbb{C}^m$ or $\mathfrak{b}_{-1} = \mathbb{R}^m$ is abelian. Hence by the properties of the enveloping algebras

$$M_\mathfrak{b}(V^*_\lambda) = \mathfrak{U}(\mathfrak{b}_{-1}) \otimes V^*_\lambda = \sum_{k=0}^{\infty} S^k(\mathfrak{b}_{-1}) \otimes V^*_\lambda$$

with the grading induced from that of the symmetric algebra. Choosing basis ∂_i of the Lie algebra $\mathfrak{b}_{-1}$ and completing it into a basis of $\mathfrak{g}$, we get the normal coordinates on a neighborhood of $e \in G$ and we see immediately that $M_{\mathfrak{b}}(V_\lambda^*)$ coincides with $(J_0^\infty E)^*$ as a vector space. But the left actions of $\mathfrak{U}(\mathfrak{g})$ coincide by the definition. Let us write this action down explicitly. For this reason, we define for every multi index $\alpha = i_1 \dots i_{|\alpha|}$, $i_1 \le \dots \le i_{|\alpha|}$, the linear map

$$\ell_\alpha \colon J_0^k E \to V_\lambda, \qquad \ell_\alpha(j_e^k s) = (\mathcal{L}_{-\partial_{i_1}} \circ \dots \circ \mathcal{L}_{-\partial_{i_{|\alpha|}}} s)(e).$$

Since the elements in $\mathfrak{g}_{-1}$ commute, we can view the elements in $S^{|\alpha|}(\mathfrak{g}_{-1})$ as linear combinations of maps ℓ_α. This is precisely the above identification. Let us denote $\ell_i = \mathcal{L}_{-\partial_i} \in \mathfrak{b}_{-1}^* = S^1(\mathfrak{b}_{-1})$, so the elements ℓ_α can be viewed as $\ell_\alpha = \ell_{i_1} \circ \dots \circ \ell_{i_{|\alpha|}} \in S^{|\alpha|}(\mathfrak{b}_{-1})$ and we have $\ell_\alpha = 0$ if $|\alpha| > k$. Further, for every $X \in \mathfrak{g}$ we shall denote $\mathrm{ad}\ell_\alpha.X = (-1)^{|\alpha|}[\partial_{i_1}, [\dots[\partial_{i_{|\alpha|}}, X]\dots]]$.

Lemma. *The action of elements $X_q \in \mathfrak{b}_q$ on $\ell_\alpha \otimes v^* \in S^p \otimes V_\lambda^*$ is*

$$X_{-1}.(\ell_\alpha \otimes v^*) = \ell_\alpha \circ X_{-1} \otimes v^*$$

$$X_0.(\ell_\alpha \otimes v^*) = -\sum_{\substack{\beta+1_i=\alpha\\ 1\le i\le m}} (\ell_\beta \circ [\partial_i, X_0]) \otimes v^* + \ell_\alpha \otimes X_0.v^*$$

$$X_1.(\ell_\alpha \otimes v^*) = \sum_{\substack{\beta+\gamma=\alpha\\ |\gamma|=1}} \ell_\beta \otimes (\mathrm{ad}\ell_\gamma.X_1).v^* + \sum_{\substack{\beta+\gamma=\alpha\\ |\gamma|=1+1}} (\ell_\beta \circ (\mathrm{ad}\ell_\gamma.X_1)) \otimes v^*$$

Proof[24]**.** We compute with $\ell = j_0^k X \in \mathfrak{b}_q$

$$\ell.(\ell_\alpha \otimes v^*)(j_e^k s) = -(\ell_\alpha \otimes v)(\ell.j_e^k s) = (\ell_\alpha \otimes v)(j_e^k(\mathcal{L}_X s)) = \langle (\ell_\alpha \circ \mathcal{L}_X s)(e), v^* \rangle$$

Since $\ell_j \circ \mathcal{L}_Y = \mathcal{L}_Y \circ \ell_j + \mathcal{L}_{[-\partial_j, Y]}$ for all $Y \in \mathfrak{g}$, $1 \le j \le n$, and $[\partial_j, \mathfrak{b}_l] \subset \mathfrak{b}_{l-1}$, we get

$$\ell.(\ell_\alpha \otimes v^*)(j_e^k s) = \langle \ell_{i_1} \dots \ell_{i_{p-1}} \mathcal{L}_X \ell_{i_p} s(0), v^* \rangle + \langle \ell_{i_1} \dots \ell_{i_{p-1}} \mathcal{L}_{[-\partial_{i_p}, X]} s(0), v^* \rangle$$

[24] The last formula also applies to the action of the isomorphism groups of other geometric structures (like the symplectic or unimodular manifolds or simply all manifolds) on the duals of jets of sections of natural bundles in the sense of 2.12. More explicitly, this formula with $q = 1$ replaced by a general $q \ge 0$ describes the action of the Lie algebra of all vector fields on the sections of the natural bundles. The natural linear operators are just those commuting with the action of these vector fields, see [Kolář, Michor, Slovák, 93, Section 34] for more details. This formula is the main ingredient of the classification of all linear natural operators on all manifolds, unimodular manifolds, symplectic manifolds, derived (with quite different aim) in [Rudakov, 74, 75], and the classification of all bilinear natural operators on all manifolds due to [Grozman, 80], see also the excellent survey [Kirillov, 80], or [Kolář, Michor, Slovák, 93, Section 34]. Of course, the methods used for the proofs must be quite different since the groups are infinite dimensional. The idea is to disable first all vectors with non-trivial action of the subalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots$ and then apply the finite dimensional representation theory of $\mathfrak{g}_0$ on the remaining vectors, the so called *singular vectors*. In fact we have described this idea explicitly in 6.13 in the conformal case.

and the same procedure can be applied p times in order to get the Lie derivative terms just at the left hand sides of the corresponding expressions. Each choice of indices among $i_1, \dots, i_p$ determines just one summand of the outcome. Hence we obtain (the sum is taken also over repeating indices)

$$\ell.(\ell_\alpha \otimes v^*)(j^k_e s) = \sum_{\beta+\gamma=\alpha} \langle (\mathrm{ad}\ell_\gamma.\ell).\ell_\beta s(e), v^* \rangle.$$

Further $\mathrm{ad}\ell_\gamma.\ell = 0$ whenever $|\gamma| > q+1$ and for all vector fields $Y \in \mathfrak{b}_0 \oplus \mathfrak{b}_1$ we have

$$\langle (\mathcal{L}_Y \circ \ell_\beta s)(e), v* \rangle = -\langle (\ell_\beta s)(0), \mathcal{L}_Y v^* \rangle$$

so that only the terms with $|\gamma| = q$ or $|\gamma| = q+1$ can survive in the sum. Since $\ell = j^k_e Y \in \mathfrak{b}_0$ acts on (the jet of constant section) v by $\ell.v = \mathcal{L}_{-Y} v(0)$, we get the result. $\square$

The formulas work in both real and complex domains.

8.3. Consider now an equivariant mapping $D^*\colon V^*_\rho \to (J^k_0 E)^*$. The Verma modules $M_\mathfrak{b}(V^*_\rho)$ and $M_\mathfrak{b}(V^*_\lambda)$ are generated by the elements $1 \otimes v^*_\rho$, $1 \otimes v^*_\lambda$ where v^*_ρ and v^*_λ are the highest weight vectors. Thus, the mapping D^* extends uniquely to a homomorphism $D^*\colon M_\mathfrak{b}(V^*_\rho) \to M_\mathfrak{b}(V^*_\lambda)$ of the $\mathfrak{U}(\mathfrak{g})$-modules. On the other hand, each such homomorphism clearly specifies a translational invariant operator. Hence we have proved for both real and complex homogeneous bundles

Theorem. *There is a bijective correspondence between the homomorphisms of the generalized Verma modules and the translational invariant operators on homogeneous bundles.*

8.4. Remark. It might seem that we have successfully reduced our problem to an algebraic task and what remains is only to look somewhere, find the classification of all homomorphisms and interpret them as differential operators. This is very far from the truth. First of all, the description of all homomorphisms is given in terms of the action of the Weyl group and a complete classification is well known only for the classical Verma modules, i.e. for Borel subalgebras B. In the conformal case, we meet the more general parabolic subgroups and here the classification covering all possible bundles has been found only recently. But say, we do not want to know really all operators, it could suffice to be able to find complete lists of them acting on some concrete fixed bundles. Even then the results are not very satisfactory since we find the extreme weight vectors and we know that the operators are the dual mappings to the identical embeddings up to a scalar factor, but we do not get explicit formulas for the operators in this way. Nevertheless, the fact that we can be sure that there is an operator between some given bundles is of great importance, cf. the deriving of the conformal Laplace operator in Section 1.

In the rest of this section we mainly follow [Slovák, 92].

8.5. The use of the infinitesimal character. It turns out to be convenient to prove the classification in the complex setting and at the very end to specify the result to the real case. So we shall treat only complex groups and algebras in the sequel. As explained in the Appendix, if two $\mathfrak{U}(\mathfrak{g})$ modules generated by a single highest weight vector admit a homomorphism, then they must have the same infinitesimal character, see 10.17. Hence we have a rather strong restriction on the possible homomorphisms between the Verma modules. The Harish-Chandra theorem reduces the problem to the study of the affine action of the Weyl group W, see 10.19 and 10.20. Thus, if there should exist an invariant operator $D\colon C^\infty(E_\lambda) \to C^\infty(E_\rho)$, for two weights dominant for $\mathfrak{b} \subset \mathfrak{g}$, then there must be an element $w \in W$ such that $w.\rho^* = \lambda^*$, i.e. $w(\rho^* + \delta) - \delta = \lambda^*$ where ρ^* and λ^* are the weights of the contragredient representations and δ is the lowest form.

Definition. If λ is a weight dominant for $\mathfrak{b}$ such that $\lambda + \delta$ does not lie on a wall of a Weyl chamber, then the infinitesimal character ξ_λ is said to be *regular*. The infinitesimal characters of the weights λ with $\lambda + \delta$ lying on some wall are called *singular*. The infinitesimal characters of weights λ and ρ with the same cardinality of the stabilizers of $\lambda + \delta$ and $\rho + \delta$ in the Weyl group W are called *equisingular*.

In particular, all regular infinitesimal characters are equisingular.

8.6. Notation for natural bundles. In 10.12 and 10.13, we explain the general notation for $\mathfrak{b}$-dominant weights by means of the Dynkin diagrams. We adopt the following convention for natural vector bundles corresponding to such representations:

Definition. A vector bundle corresponding to an irreducible representation which is dual to that one with highest weight λ will be denoted by the Dynkin diagram with the values of $\lambda + \delta$ on the simple coroots inscribed over the corresponding nodes (δ is the sum of fundamental weights as usual).

This seems to be a very strange notation, but the passing to the duals reflects the fact that we are describing the dual mappings to the operators and the shift by δ simplifies heavily our formulas. In fact, the dual representations are distinguished only by their opposite conformal weights (which is, of course, not the same as the inverting of the sign over the crossed node in general). Concrete examples are listed in 10.14 (we have only to take the duals). The expressions $C^\infty(\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{d_1}{\bullet} \cdots \underset{d_{n-2}}{\bullet}\!\!<\!\!{}^{\bullet\, a}_{\bullet\, c})$ and $C^\infty(\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{d_1}{\bullet} \cdots \overset{d_{n-1}}{\bullet}\!\!\Rightarrow\!\!\overset{a}{\bullet})$ mean the corresponding spaces of sections of the homogeneous vector bundles.

8.7. The patterns of natural bundles. We discuss in 10.15 that the elements in the Weyl group which map at least some of the weights dominant for $\mathfrak{b}$ into weights dominant for $\mathfrak{b}$ form the so called parabolic subgraph $W^{\mathfrak{b}}$ of W. Let us describe this explicitly for the orthogonal algebras.

If $m = 4$, we have $\mathfrak{b} = \bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet$ and let s_1, s_2, s_3 be the simple roots as indicated in $\mathfrak{g} = \overset{s_2}{\bullet}\!\!-\!\!\!-\!\!\overset{s_1}{\bullet}\!\!-\!\!\!-\!\!\overset{s_3}{\bullet}$. The Weyl group consists of all permutations of four letters, the generators s_i correspond to the transposition of the i-th and $(i+1)$-st coordinates (in the proper ordering), see 10.10. If $w \in W^{\mathfrak{p}}$ is different from the identity, then its decomposition into the generators must end with s_1. A further discussion yields the parabolic subgraph $W^{\mathfrak{p}}$

$$\begin{tikzcd}[column sep=small]
 & & s_1s_2 \arrow[dr] & & \\
\mathrm{id} \arrow[r] & s_1 \arrow[ur] \arrow[dr] & & s_1s_2s_3 \arrow[r] & s_1s_2s_3s_1 \\
 & & s_1s_3 \arrow[ur] & &
\end{tikzcd}$$

More generally, in the even dimensions $m = 2n$ we can describe $W^{\mathfrak{b}}$ symbolically by

$$\mathrm{id} \longrightarrow s_1 \longrightarrow s_1s_2 \longrightarrow \dots$$

$$\begin{tikzcd}[column sep=small]
 & s_1s_2\dots s_{n-1}s_n \arrow[dr] & & \\
{} \arrow[r] & s_1s_2\dots s_{n-1} \arrow[u] \arrow[d] & s_1s_2\dots s_{n-1}s_ns_{n+1} \arrow[r] & {} \\
 & s_1s_2\dots s_{n-1}s_{n+1} \arrow[ur] & &
\end{tikzcd}$$

$$\longrightarrow s_1s_2\dots s_{n-1}s_ns_{n+1}s_{n-1} \longrightarrow \dots \longrightarrow s_1s_2\dots s_2s_1$$

where the symbols s_i denote the reflections corresponding to the simple roots indicated in the diagram $\overset{s_1}{\times}\!\!-\!\!\overset{s_2}{\bullet}\cdots\underset{s_{n-1}}{\bullet}\!\!<\!\!\begin{smallmatrix}\bullet\ s_n\\ \bullet\ s_{n+1}\end{smallmatrix}$.

If $m = 2n + 1$ we order the simple roots as indicated in $\overset{s_1}{\times}\!\!-\!\!\overset{s_2}{\bullet}\cdots\overset{s_n}{\bullet}\!\Rightarrow\!\overset{s_{n+1}}{\bullet}$ and we get

$$\mathrm{id} \longrightarrow s_1 \longrightarrow \dots \longrightarrow s_1\dots s_{n+1} \longrightarrow$$

$$\longrightarrow s_1\dots s_{n+1}s_n \longrightarrow \dots \longrightarrow s_1s_2\dots s_2s_1$$

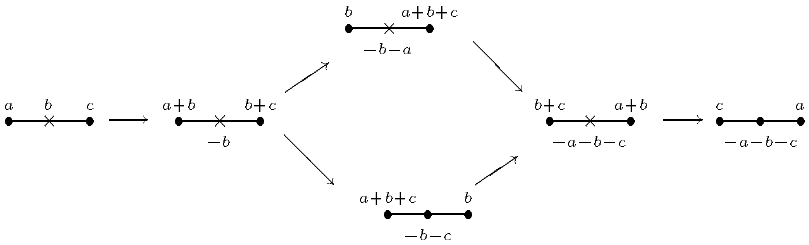
The arrows describe the so called Bruhat order on $W^{\mathfrak{b}}$, for a more detailed description see e.g. [Boe, Collingwood, 85] or [Borho, Jantzen, 77].

If a weight dominant for $\mathfrak{b}$ has all coefficients over the nodes integral then its infinitesimal character is regular if and only if there is a weight ρ with the same infinitesimal character, which is dominant for the whole $\mathfrak{g}$. For such weights with regular infinitesimal characters, the meaning of the above patterns is easy to explain: We take the only weight λ dominant for $\mathfrak{g}$ with the infinitesimal character ξ_λ and we let the elements from $W^{\mathfrak{b}}$ act on $\lambda + \delta$ as indicated in the diagrams. In this way we get just all weights $\rho + \delta$ with ρ dominant for $\mathfrak{b}$ and with the same infinitesimal character ξ_λ.

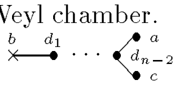
The action of the simple reflections from the Weyl group is described in 10.20. For example, let us consider a dominant weight λ for $\mathfrak{g}$, $\lambda = \overset{a}{\bullet}\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\overset{c}{\bullet}$ with integers a, b, $c > 0$. The action of the reflection $s_1 \in W$ corresponding to simple root denoted by the second node on λ is

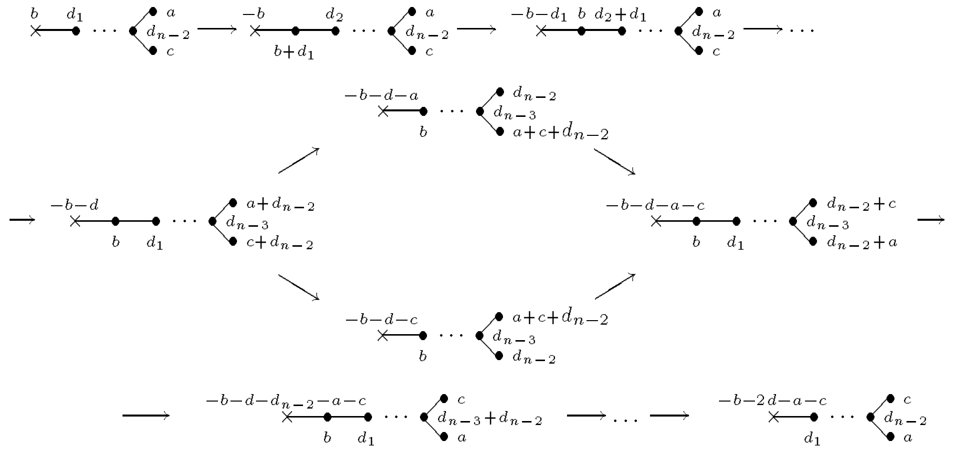
$$s_1.\lambda = s_1(\lambda + \delta) - \delta = \overset{a+b}{\bullet}\!\!-\!\!\overset{-b}{\bullet}\!\!-\!\!\overset{b+c}{\bullet} \ .$$

Similar simple computation yields the action of all elements in the directed graph $W^{\mathfrak{p}}$ from 10.15. Altogether we get the pattern

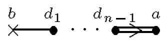


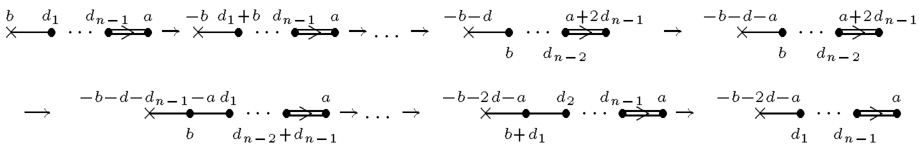
It is a straightforward computation to write down explicitly the patterns in the higher dimensions. We shall do this in a quite formal way, i.e. the only restriction on the coefficients over the nodes of the left most weight is that this should belong to the closed fundamental Weyl chamber.

Let us fix first a weight  with all coefficients non-negative (but not necessarily integral).



where $d = d_1 + \cdots + d_{n-2}$.

The pattern for manifolds of dimension $2n+1$ starts with a weight  with non-negative coefficients



where $d = d_1 + \cdots + d_{n-1}$.

Let us point out once more that the weights λ in the patterns correspond to the duals of the standard fibers of the bundles and the coefficients themselves are the values of $\lambda + \delta$ on the simple corrots. In view of the above discussion we know that all natural operators must appear between two bundles in the same pattern.

8.8. Each position in the pattern corresponds to just one Weyl chamber and the weights λ which determine representations with regular infinitesimal character are those with $\lambda + \delta$ not lying on a wall of a Weyl chamber. Thus, the unique position of every representation with regular infinitesimal character can be read off the coefficients over the nodes. Let us call the non-negative coefficients $a, b, \dots$ over the left-most weight in the pattern the *coefficients of the pattern.*

If some of the coefficients of the pattern are not integral, then a lot of the listed weights are not dominant for $\mathfrak{b}$. If the stabilizer of a weight λ under the affine action of the Weyl group is not trivial, then the pattern degenerates in such a way that some of the weights are not dominant for $\mathfrak{b}$ and the number of occurrences of the remaining weights appearing in the pattern equals to the cardinality of the stabilizer of each of them.

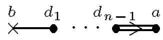
Lemma. *The number of occurences of the $\mathfrak{b}$-dominant weights in the pattern equals to the number of the zeros among its coefficients increased by one.*

Proof. The claim follows from the explicite description of the patterns in 8.7. $\square$

8.9. The order of the operators. The conformal weights are easily computed by means of the coefficients in the Dynkin diagrams as described in 10.14. The conformal weight ω of the representation with the highest weight 

is

$$\omega = b + d_1 + \cdots + d_{n-2} + \frac{1}{2}(a + c) - n$$

while the conformal weight of 

is

$$\omega = b + d_1 + \cdots + d_{n-1} + \frac{1}{2}a - \frac{1}{2}(2n + 1).$$

The conformal weights of the natural bundles corresponding to such diagrams are obtained by taking the negative of the above formulas (this is our duality convention).

If there is a translational invariant operator $D\colon C^\infty((F_\lambda)^*) \to C^\infty((F_\rho^*)$ between the complex bundles over complex pseudo-spheres, then its order is described easily be means of the conformal weights of λ and ρ. Let us remind that D corresponds to the inclusion of the representation space V_ρ into the Verma module $M_{\mathfrak{b}}(\lambda)$. Since each homogeneous component in the grading of the Verma module is a $\mathfrak{g}_0$-submodule, the image of the inclusion must be contained in one homogeneous component. But the degree of this component is exactly the order of the operator D. If ω_1 is the conformal weight of λ, then the conformal weight of all irreducible representations in the i-th homogeneous component in $M_{\mathfrak{b}}(\lambda)$ is $\omega_1 - i$. Thus, the operator D has the order $r = \omega_1 - \omega_2$ where ω_2 is the conformal weight of ρ. This elementary observation will become one of the basic tools for the classification.

8.10. Translation functors. There is a general construction which allows to translate the results on homomorphisms of Verma modules from one pattern to another one, the so called Jantzen-Zuckerman functors, see e.g. [Zuckerman, 77]. As before, let us write V_μ for the finite dimensional irreducible representation with highest weight μ dominant for $\mathfrak{b}$. Further, write V_μ^* for the module contragradient to V_μ, i.e. V_μ^* has the lowest weight $-\mu$. Each $\mathfrak{U}(\mathfrak{g})$-module decomposes completely into submodules with different infinitesimal characters, see e.g. [Zuckerman, 77]. Let us write p_λ for the projections onto the modules with infinitesimal character ξ_λ. Hence given a weight λ dominant for $\mathfrak{b}$ and a weight μ dominant for $\mathfrak{g}$, we can define two functors

$$\varphi^\lambda_{\lambda+\mu} = p_{\lambda+\mu} \circ ((\ ) \otimes V_\mu) \circ p_\lambda$$
$$\psi^{\lambda+\mu}_\lambda = p_\lambda \circ ((\ ) \otimes V^*_\mu) \circ p_{\lambda+\mu}$$

where the action on the morphisms is defined by the tensor product with the identity.

These functors are defined on a large class of $\mathfrak{U}(\mathfrak{g})$-modules involving the generalized Verma modules. For technical reasons, we shall also allow λ to be an arbitrary weight with $s.\lambda$ dominant for $\mathfrak{b}$ for some $s \in W^{\mathfrak{b}}$ (then the projections p_λ and $p_{\lambda+\mu}$ are well defined), but we shall always assume that $\lambda + \delta$ belongs to the closed fundamental Weyl chamber which contains the weights corresponding to the representations appearing in the most left position in the patterns. In particular, this means that λ is dominant for $\mathfrak{g}$ if ξ_λ is regular and λ is integral.

Lemma.

(1) *The functor $\psi^{\lambda+\mu}_\lambda$ is left adjoint to $\varphi^\lambda_{\lambda+\mu}$.*

(2) *If the weights λ and $\lambda+\mu$ are equisingular, then $\psi^{\lambda+\mu}_\lambda(M_{\mathfrak{b}}(s.(\lambda+\mu))) = M_{\mathfrak{b}}(s.\lambda)$ and $\varphi^\lambda_{\lambda+\mu}(M_{\mathfrak{b}}(s.\lambda)) = M_{\mathfrak{b}}(s.(\lambda+\mu))$ whenever $s.\lambda$ is dominant for $\mathfrak{b}$.*

Proof. Since V_μ is finite dimensional, the space of homomorphisms $\mathrm{Hom}(M_{\mathfrak{b}}(s.(\lambda+\mu)) \otimes V^*_\mu, M_{\mathfrak{b}}(s'.\lambda))$ is naturally isomorphic to $\mathrm{Hom}(M_{\mathfrak{b}}(s.(\lambda+\mu)), M_{\mathfrak{b}}(s'.\lambda)\otimes V_\mu)$. In view of 8.5, only the summand $p_\lambda(M_{\mathfrak{b}}(s.(\lambda+\mu))\otimes V^*_\mu)$ can contribute to $\mathrm{Hom}(M_{\mathfrak{b}}(s.(\lambda+\mu))\otimes V^*_\mu, M_{\mathfrak{b}}(s'.\lambda))$ and similarly only $p_{\lambda+\mu}(M_{\mathfrak{b}}(s'.\lambda)\otimes V_\mu))$ contributes to the other homomorphisms. This shows the required natural equivalence

$$\mathrm{Hom}(\psi^{\lambda+\mu}_\lambda(M_{\mathfrak{b}}(s.(\lambda+\mu))), M_{\mathfrak{b}}(s'.\lambda)) \simeq \mathrm{Hom}(M_{\mathfrak{b}}(s.(\lambda+\mu)), \varphi^\lambda_{\lambda+\mu}(M_{\mathfrak{b}}(s'.\lambda))).$$

The other assertion is more difficult to prove. A general theorem reads that if the weights λ and $\lambda+\mu$ are equisingular, then the functors $\psi^{\mu+\lambda}_\lambda$ and $\varphi^\lambda_{\lambda+\mu}$ are the mutually inverse natural equivalences on their definition domains, see [Zuckerman, 77]. If we fix such weights λ and $\lambda+\mu$, then for each $s\in W^{\mathfrak{b}}$ the weights $s.\lambda$ and $s.(\lambda+\mu)$ determine representations appearing at the same position in the patterns starting with λ and $\mu+\lambda$. The infinitesimal characters are the same ones for the whole pattern and so the projection p_λ is the identity on $M_{\mathfrak{b}}(s.\lambda)$. Further

$$\begin{aligned} M_{\mathfrak{b}}(V_{s.\lambda})\otimes V_\mu &= \bigoplus_{i=0}^{\infty}(S^i(\mathfrak{g}_{-1})\otimes(V_{s.\lambda}\otimes V_\mu)) \\ &= \bigoplus_{i=0}^{\infty}(S^i(\mathfrak{g}_{-1})\otimes(\oplus_{j=1}^k V_{\nu_j})) = \bigoplus_{j=1}^{k} M_{\mathfrak{b}}(V_{\nu_j}) \end{aligned}$$

The weights ν_j appearing in the tensor product and their multiplicities can be determined using one of the consequences of the Weyl character formula, e.g. the well known Brower's formula or Klimyk's formula. Finally, the projection $p_{\lambda+\mu}$ selects just those ν_j which lead to the prescribed infinitesimal character $\xi_{\mu+\lambda}$.

So we see that the value of $\varphi^{\lambda}_{\lambda+\mu}$ on a generalized Verma module must be a sum of generalized Verma modules. If we replace V_μ and λ by V^*_μ and $\lambda+\mu$, we get the same result for the functor $\psi^{\lambda+\mu}_{\lambda}$. But since $\psi^{\lambda+\mu}_{\lambda}\circ\varphi^{\lambda}_{\lambda+\mu}$ is naturally equivalent to the identity, the values can always consist of only one generalized Verma module. But there is certainly the weight $\nu = s.(\lambda+\mu)$ involved among the weights ν_j and this appears with multiplicity one. Thus for all $s\in W^{\mathfrak{b}}$ we have $\varphi^{\lambda}_{\lambda+\mu}(M_{\mathfrak{b}}(s.\lambda)) = M_{\mathfrak{b}}(s.(\lambda+\mu))$ if $s.\lambda$ is dominant for $\mathfrak{b}$.

Similarly we can analyze the functor $\psi^{\lambda+\mu}_{\lambda}$ with μ and λ replaced by $-\mu$ and $\mu+\lambda$ and we get $\psi^{\lambda+\mu}_{\lambda}(M_{\mathfrak{b}}(s.(\lambda+\mu))) = M_{\mathfrak{b}}(s.\lambda)$. $\square$

As a consequence of the lemma, we can pass from one pattern to another one by adding integral weights with regular infinitesimal character. In particular, once we describe all operators between the representations in one pattern, we can get all operators in many other patterns by applying the above translations.

8.11. The operators on exterior forms. All linear natural operators on Riemannian manifolds which do not disappear on flat manifolds and which behave well with respect to constant rescaling of the metric were described in 4.23. Those which are natural on conformally flat manifolds are indicated in the following two diagrams. In the even dimension $m=2n$ they are all composed from the exterior differential d and the Hodge star operator $*$.

$$
\begin{tikzcd}
 & & & & \Omega^n_+ \arrow[dr, "d"] & & & & \\
\Omega^0 \arrow[r, "d"] \arrow[rrrrrrrr, bend right=50, "D_0=d\circ(*d)^{m-1}"'] & \Omega^1 \arrow[r, "d"] \arrow[rrrrrr, bend right=40, "D_1=d\circ(*d)^{m-3}"'] & \cdots \arrow[r, "d"] & \Omega^{n-1} \arrow[ur, "d_+"] \arrow[dr, "d_-"'] \arrow[rr, bend right=30, "D_{p-1}=d*d=d\circ d_+-d\circ d_-"'] & & \Omega^{n+1} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \Omega^{2n-1} \arrow[r, "d"] & \Omega^{2n} \\
 & & & & \Omega^p_- \arrow[ur, "d"'] & & & &
\end{tikzcd}
$$

The odd-dimensional case $(m=2n+1)$ coincides with the de Rham resolvent:

$$\Omega^0 \xrightarrow{\quad d\quad} \Omega^1 \xrightarrow{\quad d\quad} \cdots \xrightarrow{\quad d\quad} \Omega^{m-1} \xrightarrow{\quad d\quad} \Omega^m$$

All of them are natural on locally flat conformal manifolds and there are no other natural linear operators there. In view of the translation procedure and the form of our patterns, this solves the existence problem for operators which act between bundles determined by integral weights with regular infinitesimal character. In particular, there is at most one operator between any two such bundles up to constant multiples.

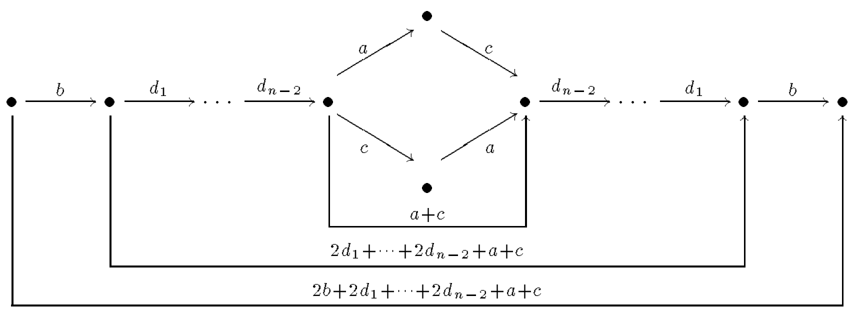
8.12. Powers of the Laplace operator. We shall list more natural operators on functions with conformal weights which appear in the patterns with singular infinitesimal character or in patterns with non-integral coefficients. The coefficients of the Dynkin diagram correspond to a function space if and only if all of them equal to one except the coefficient over the crossed node. Inspecting one of the patterns with singular infinitesimal character from 8.7 which involves such an entry, we see immediately that either the coefficients of the pattern are non-integral or some of them are zero. We shall omitt now the general discussion on all possibilities since we have to do this more complex in the proof of the main theorem below, but we shall describe the existing operators. In fact the translation procedure described above will produce all natural operators from those on exterior froms described in 8.11, those on functions described below and the conformally invariant Dirac operators on Weyl spinors derived in 6.22.

In the even dimension $m = 2n$ we have for each $0 \le i \le n-2$ the translational invariant operator $D\colon C^\infty(\overset{-i}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!<^{\bullet\,1}_{\bullet\,1}) \to C^\infty(\overset{-2n+i+2}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!<^{\bullet\,1}_{\bullet\,1})$. This is the so called conformally invariant $(n-i-1)$-st power of the Laplacian which is defined by the complete contraction of the suitable iteration of the covariant derivative. Its uniqueness is clear from the considerations in the category of Riemannian manifolds (by evaluation in the Euclidean metric we exclude the curvatures but then the only possibility to end in functions is to take a complete contraction of iterated covariant derivative), its invariance is a matter of a direct evaluation of the effect of the rescaling of the metric. In particular, the choice $i = n-2$ yields the well known conformally invariant Laplace operator.

In the odd dimensions $m = 2n+1$ we also have only the powers of the Laplace operators. More explicitely, for each $0 \le i \le n-2$ there is the translational invariant operator $D\colon C^\infty(\overset{-i+\frac12}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{1}{\bullet}) \to C^\infty(\overset{-2n+i+\frac12}{\times}\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!\Rightarrow\!\!\overset{1}{\bullet})$. The invariance has to be verified by direct computation, the uniqueness follows from the Riemannian invariance just as above.

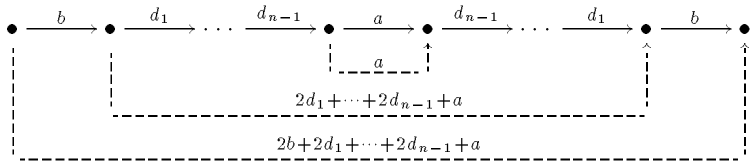
8.13. Theorem. *For every two weights ρ, λ dominant for $\mathfrak{b}$, the space of the natural linear operators $D\colon C^\infty(F_\lambda M) \to C^\infty(F_\rho M)$ acting on smooth sections of complex natural vector bundles over complex conformal Riemannian manifolds is at most one dimensional. All such non-trivial operators, i.e. those different from constant multiples of the identities, are indicated in the patterns below. The labels over the arrows indicate their orders.*

Let $\dim M = 2n$, $n > 1$. *The pattern starting with the weight* $\overset{b}{\times}\!\!-\!\!\overset{d_1}{\bullet}\cdots\overset{d_{n-2}}{\bullet}\!\!<^{\bullet\,a}_{\bullet\,c}$, *where all* $b, d_1, \dots, d_{n-2}, a, c \ge 0$, *is*



All arrows in the diagram which join integral weights dominant for $\mathfrak{b}$ describe a non-zero linear natural operator on conformally flat manifolds and there are no other ones.

If the dimension of M is $2n+1$, $n>0$, then the non-zero linear natural operators act between bundles corresponding to weights with integral and half-integral coefficients. If the pattern starts with $\overset{b}{\times}\!\!-\!\!\overset{d_1}{\bullet}\cdots\overset{d_{n-1}}{\bullet}\!\Leftarrow\!\overset{a}{\bullet}$ and all the coefficients are positive integers, then the operators are exhausted exactly by those which are indicated by the solid arrows in the diagram



while if some of the coefficients are half-integral and the infinitesimal character is regular, then we get exactly those operators indicated by the dashed arrows which join weights dominant for $\mathfrak{b}$. If the infinitesimal character of the pattern is singular, then there are no non-trivial operators in odd dimensions.

Exactly the same classification applies to natural linear operators acting on smooth sections of real natural vector bundles over conformal Riemannian manifolds with an arbitrary signature (m',n'), $m'+n'=2n\geq 4$ or $m'+n'=2n+1\geq 3$.

Proof. The description of the general patterns and the computation of the conformal weights in 8.8 and 8.9 yield the possible orders of natural operators as indicated on the labels over the arrows in the diagrams above. Since the order must be a non-negative integer, a careful inspection of the general patterns from 8.7 shows that the coefficients of the patterns must be half-integral. Moreover, if these coefficients are not integral and the dimension is even, then the only possibility to find a weight dominant for $\mathfrak{b}$ is either to choose b half-integral or to take two half-integral coefficients over the adjacent nodes in the left-most weight or the couple (a,c) or the triple (d_{n-2},a,c) must be half-integral, while all other coefficients must be integral. The proof of this claim consist of an elementary discussion based on the form of the patterns from 8.7. But now, in view of the translation principle we can choose the half-integral coefficients to be $\frac{1}{2}$ while the integral can be set to one. In the case (a,c) is half-integral, the only two weights dominant for $\mathfrak{b}$ are the two weights just in the middle, which are different but the order should be zero. Thus there is no non-zero operator available in this case. In all other cases listed above, the operator should transform complex functions with suitable conformal weights into complex functions with another conformal weight, but the orders should be odd. However, if we apply the methods leading to the description of the Riemannian invariants in Section 4, then we see that there is no such non-zero operator in the even dimensional case. The reason is that the evaluation in the Euclidean metric excludes all curvatures and after applying an odd number of covariant derivatives we get into an odd tensor power of the covectors, but then there is no way how to come to functions using the orthogonal invariant tensor operations. Hence there are no non-zero linear natural operators acting between bundles with non-integral coefficients in the even dimensions.

In order to finish the description of the even dimensional case, we have now to discuss case by case the infinitesimal characters by means of the translations between the equisingular ones. If the infinitesimal character of the pattern is regular, then the assertion of the theorem follows from 8.11. We have seen in 8.8 that two patterns have equisingular infinitesimal characters if and only if they posses the same number of zeros among their coefficients. On the other hand, if there should exist a weight dominant for $\mathfrak{b}$ in the pattern, then there can appear at most one zero, except the case $a = c = 0$, see 8.6.

Assume first $d_i = 0$ for some $0 < i \leq n-2$, or $b = 0$. Then there are only two weights dominant for $\mathfrak{b}$. Let us choose all other coefficients equal to one. Hence the operator should be defined on complex functions $C^\infty(\overset{-i}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!<\!\!\begin{smallmatrix}\bullet^{1}\\ \bullet_{1}\end{smallmatrix})$ with values in $C^\infty(\overset{-2n+i+2}{\times}\!\!-\!\!\!-\!\!\underset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!<\!\!\begin{smallmatrix}\bullet^{1}\\ \bullet_{1}\end{smallmatrix})$ (we set $i = 0$ if we have chosen $b = 0$). Such operators do exist and they are unique up to scalar multiples, see 8.12.

Now, let us choose $a = 0$ and suppose all other coefficients equal to one. Then we have also only two weights which are dominant for $\mathfrak{b}$ in the pattern. The corresponding operator $C^\infty(\overset{-n+1}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!<\!\!\begin{smallmatrix}\bullet^{1}\\ \bullet_{2}\end{smallmatrix}) \to C^\infty(\overset{-n}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\!<\!\!\begin{smallmatrix}\bullet^{2}\\ \bullet_{1}\end{smallmatrix})$ exists and is unique up to constant multiples. It is just the conformally invariant Dirac operator. The choice $c = 0$ leads to the other Dirac operator on the basic spin representations. The last choice, $a = c = 0$ yields four identical weights and operators of order zero. This finishes the discussion on the even dimensions.

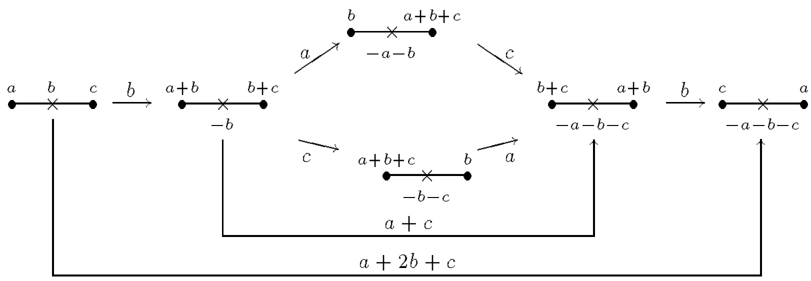
A quite different situation appears in the odd dimensions. There we must admit also the half-integral weights. If we combine our knowledge of the possible orders with the requirement that the arrows which could indicate a natural operator must join the nodes with weights dominant for $\mathfrak{b}$, we see that the only possibility is either to consider b half-integral or b and d_1 half integral or two adjacent coefficients d_i, d_{i+1} half-integral or d_{n-1} half-integral. But then either the orders indicated over the solid arrows are not integral or the weights are not dominant for $\mathfrak{b}$, so they are all excluded. Now we can discuss the individual positions of the pattern for functions with suitable half-integral conformal weights. The whole discussion is quite similar to the above description of the sigular patterns in even dimensions. Let us first show this procedure on the case of the longest arrow. We consider the weight $\overset{\frac{1}{2}}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}$, i.e. the operator should act on the complex functions with conformal weight $\frac{1}{2}$. The order $r = 2n$ of the operator is now even and the complete contraction of the r-th iterated covariant derivative is just the n-th power of the Laplacian which is conformally invariant on flat manifolds as an operator acting on functions with conformal weight $\frac{1}{2}$ with values in functions with conformal weight $\frac{1}{2} + 2n$. The uniqueness up to constant multiples is proved easily in the category of Riemannian manifolds. Similarly we obtain $(n-i)$-th powers of the Laplacians $C^\infty(\overset{-i+\frac{1}{2}}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\Rightarrow\!\overset{1}{\bullet}) \to C^\infty(\overset{-2n+i+\frac{1}{2}}{\times}\!\!-\!\!\!-\!\!\bullet\cdots\overset{1}{\bullet}\!\Rightarrow\!\overset{1}{\bullet})$ in the remaining cases listed above. The last possibility is $d_{n-1} = \frac{1}{2}$ and it leads to the unique operator $C^\infty(\overset{-n+\frac{1}{2}}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\Rightarrow\!\overset{2}{\bullet}) \to C^\infty(\overset{-n-\frac{1}{2}}{\times}\!\!-\!\!\!-\!\!\overset{1}{\bullet}\cdots\overset{1}{\bullet}\!\Rightarrow\!\overset{2}{\bullet})$ which is the conformally invariant Dirac operator on the basic spin representation.

If the dimension is three, the whole pattern of weights starting with the functions with conformal weight $\frac{1}{2}$ survives and the middle arrow corresponds to the conformally invariant Dirac operator acting on spinors with conformal weight one.

If the pattern has a singular infinitesimal character, then the weights must be integral. Indeed, with some half-integral coefficient we need the summation to neglect it, but then we cannot get off the zeros among the coeffiecients. Similarly, there can appear only one zero among the coefficients. If all non-zero coefficients equal one, then independent of our choice of the zero, we should find a non-trivial operator acting on complex functions with an odd order. This is not possible for the reason discussed above. Thus, there are no non-trivial operators acting between bundles with singular infinitesimal character in the even dimensions.

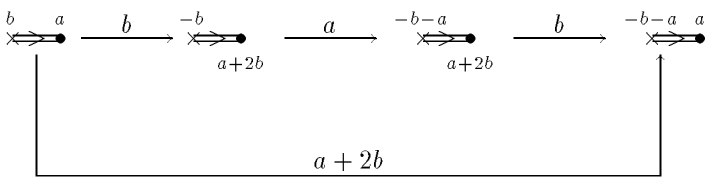
If we want to describe the natural operators in the real setting, then we also have to describe the singular highest weight vectors, but in the real generalized Verma modules, see 8.3. But if we complexify the duals to the jet spaces, then either we obtain the same set of highest weight vectors or some of them can be doubled. In any case no new singular highest weights appear. Since the spaces of the natural operators are always at most one-dimensional in the complex case, either the highest weight vector generating the whole Verma module is doubled, or no other one can be doubled. Thus we may look for the singular highest weight vectors in the complex $\mathfrak{U}(\mathfrak{g})$-module $M_{\mathfrak{b}}(\lambda)$. This also implies the pleasant fact that the existence of the operators and some of their characteristics do not depend on the signature (m', n'). $\square$

8.14. Examples. Let us write down the complete patterns with the orders of the operators inscribed above the arrows, which exhaust all operators in dimension four. If some weights are not dominant for $\mathfrak{b}$ they have to be ignored involving all adjacent arrows.



All coefficients are non-negative integers. All linear natural operators on locally flat conformal manifolds are involved.

In dimension three we start with $\overset{b}{\times}\!\!\Rightarrow\!\!\overset{a}{\bullet}$ with all coefficients integral or half-integral and non-zero. If the order is not integral we have to omit the corresponding arrow.



Using the general patterns, we can sometimes answer rather general questions. For example, if we want to find all linear natural operators, say of order two, on conformal manifolds of dimension $2n$ such that their source and target bundles coincide up to conformal weights, then they must correspond to the 'long' arrows in our patterns and $a = c$, cf. [Branson, 89, Theorem 3.14]. Now the exact formulas for the orders yield lists of possible sources. In particular, we find the operators $D_{2,k}$ discovered by Branson for $k < n$. The operators $D_{2,n}$ appear in the central diamond, e.g. $D_{2,2}\colon C^\infty(\overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{3}{\bullet}) \to C^\infty(\overset{3}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet})$ in the pattern which should start with $\lambda = \overset{0}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{2}{\bullet}$, cf. [Branson, 89].

8.15. Examples of the highest weights. In order to get some feeling how concrete calculations work, let us discuss some examples in dimension four. For this reason we fix the generators of the Lie algebra $\mathfrak{gl}(4,\mathbb{C})$ as indicated by the position in the matrix

$$\begin{pmatrix} H_1 & X_1 & x_2 & x_4 \\ Y_1 & H_2 & x_1 & x_3 \\ y_2 & y_1 & H_3 & X_2 \\ y_4 & y_3 & Y_2 & H_4 \end{pmatrix}$$

The generators off the diagonal together with $h_i = H_i - H_{i+1}$, $i = 1,2,3$, generate the Lie algebra $\mathfrak{sl}(4,\mathbb{C}) \simeq \mathfrak{o}(6,\mathbb{C})$. Then the summands in $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{l} \oplus \mathfrak{n}$ are generated as follows: $\mathfrak{n} = \langle x_i \rangle$, $\mathfrak{l} = \langle X_i, Y_j \rangle$, $\mathfrak{n}^- = \langle y_i \rangle$. The simple root elements are $\alpha_1 = X_1$, $\alpha_2 = x_1$, $\alpha_3 = X_2$. In the concrete calculations we shall need the commutators of the root elements:

	$[h_1,$	$[h_2,$	$[h_3,$	$[X_1,$	$[x_1,$	$[X_2,$
$Y_1]$	$-2Y_1$	Y_1	0	h_1	0	0
$Y_2]$	0	Y_2	$-2Y_2$	0	0	h_3
$y_1]$	y_1	$-2y_1$	y_1	0	h_2	0
$y_2]$	$-y_2$	$-y_2$	y_2	$-y_1$	Y_1	0
$y_3]$	y_3	$-y_3$	$-y_3$	0	$-Y_2$	y_1
$y_4]$	$-y_4$	0	$-y_4$	$-y_3$	0	y_2

Let us seek first for maximal weight vectors in $M_{\mathfrak{b}}(V_\lambda^*)$ with $\lambda = \overset{1}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{1}{\bullet}$, i.e. we shall describe invariant operators on functions. Let us recall that the maximal weight vectors are the weight vectors for the Cartan algebra which are annihilated by the simple root elements from $\mathfrak{b}$ (i.e. by X_1 and X_2) and also by the whole $\mathfrak{n}$ (i.e. we have to verify the vanishing of the action of x_1 and the rest will follow). Hence we can consider the elements $P(y_i, Y_i) \in \mathfrak{U}(\mathfrak{g})$ given by 'polynomial expressions' in y_i, $i = 1,2,3,4$, and Y_j, $j = 1,2$, let them act on the generating highest weight vector $v \in M_{\mathfrak{b}}(V_\lambda^*)$ and look which of the values have the desired properties. The simplest possibility is to consider $y_1.v$. Then

$$\begin{aligned} X_j.y_1 \otimes v &= [X_j, y_1] \otimes v + y_1 \otimes X_j.v = [X_j, y_1] \otimes v = 0, \qquad j = 1,2 \\ x_1.y_1 \otimes v &= [x_1, y_1] \otimes v + y_1 \otimes x_1.v = 1 \otimes [x_1, y_1].v = 1 \otimes h_2.v = 0 \end{aligned}$$

so that $y_1 \otimes v$ is a good candidate for a maximal weight vector. It remains to compute

$$h_i.y_1 \otimes v = [h_i, y_1] \otimes v + y_1 \otimes h_i.v = [h_i, y_1] \otimes v = \begin{cases} y_1 \otimes v & i = 1 \\ -2y_1 \otimes v & i = 2 \\ y_1.v & i = 3 \end{cases}$$

and so $y_1.v$ generates (as a maximal weight vector) a subspace in $M_{\mathfrak{b}}(V_\lambda^*)$ isomorphic to $\overset{2}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{2}{\bullet}$. The standard fiber of the target bundle of the corresponding operator is the dual, hence we get the bundle of 1-forms as the target of the operators.

Let us notice that the same computation yields also the operators corresponding to $\overset{p+1}{\bullet}\!-\!\overset{1}{\times}\!-\!\overset{r+1}{\bullet} \;\to\; \overset{p+2}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{r+2}{\bullet}$. In particular, $p = r = 1$ determines a first order operator on vector fields with values in symmetric two-forms with suitable conformal weight and its null-space consists of conformal vector fields, cf. [Hitchin, 80], while $p = 1$, $r = 0$ leads to the local twistor operator defined on spinors, cf. 6.22 and 7.28.

Similar direct computations show

$$
\begin{aligned}
X_1(y_1y_4 - y_2y_3) &= -y_1y_3 + y_1y_4X_1 + y_1y_3 - y_2y_3X_1 = (y_1y_4 - y_2y_3)X_1\\
X_2(y_1y_4 - y_2y_3) &= y_1y_2 + y_1y_4X_2 - y_1y_2 - y_2y_3X_2 = (y_1y_4 - y_2y_3)X_2\\
x_1(y_1y_4 - y_2y_3) &= y_4h_2 + y_4 + y_3Y_1 + y_2Y_2 + (y_1y_4 - y_2y_3)x_1\\
h_i(y_1y_4 - y_2y_3) &= \begin{cases} -2(y_1y_4 - y_2y_3) + (y_1y_4 - y_2y_3)h_2 & \text{if } i = 2\\ 0 & \text{if } i = 1,3.\end{cases}
\end{aligned}
$$

If we choose $\lambda = \overset{1}{\bullet}\!-\!\overset{0}{\times}\!-\!\overset{1}{\bullet}$, the y_4 entries in the third row cancel each other and the Y_j, $j = 1,2$, act trivially on the highest weight vector. Hence we obtain a second order operator with values in the bundle corresponding to the dual of $\overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet}$, i.e. the conformal Laplacian on the flat manifolds. If we replace the weight λ by $\lambda = \overset{1}{\bullet}\!-\!\overset{-1+q}{\times}\!-\!\overset{1}{\bullet}$, we get $x_1(y_1y_4 - y_2y_3)^q.(1 \otimes v) = 0$ and the actions of X_i, $i = 1,2$, and h_j, $j = 1,3$, remain trivial. The action of h_2 yields that the resulting operator has the values in the bundles corresponding to $\overset{1}{\bullet}\!-\!\overset{-1-q}{\times}\!-\!\overset{1}{\bullet}$. These operators are called the powers of the Laplace operator, in particular, the case with $q = 2$ can be viewed as the square of the Laplace operator $\Box^2$ acting on functions (with weight zero) with values in the functions with weight four, i.e. the longest arrow in the diagram in 8.13.

The root elements Y_1, Y_2 can also appear in the polynomials but they do not increase the order. For example, $(-y_3+y_1Y_2)(-2y_3+y_1Y_2)(-3y_3+y_1Y_2)$ determines a third order operator $\overset{2}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{4}{\bullet} \;\to\; \overset{5}{\bullet}\!-\!\overset{-4}{\times}\!-\!\overset{1}{\bullet}$, $\lambda^{A'}_{(ABC)} \mapsto \nabla^A_{(A'}\nabla^B_{B'}\nabla^C_{C'}\lambda_{D')ABC}$, see [Baston, 90].

8.16. The Bernstein-Gelfand-Gelfand resolution. The original study of homomorphisms between Verma modules was made for a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, i.e. in the case of classical Verma modules. A complete classification of them was derived by [Verma, 68] and [Bernstein, Gelfand, Gelfand, 71]. The result (translated into the language of differential operators) states: Let $B \subset G$ be a connected and simply connected subgroup in a connected and simply connected semisimple complex Lie group G with a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. *If λ is a dominant weight for $\mathfrak{g}$, then there is a translational invariant operator $D\colon C^\infty(E_{w.\lambda}) \to C^\infty(E_{w'.\lambda})$ acting on homogeneous bundles on G/B if and only if $w \le w'$ in the Weyl group W of* $\mathfrak{g}$, see 10.15 for the notation.

In our conformal case, the Poincaré conformal subgroup B is far from being a Borel subgroup $B_0 \subset SO(m+2, \mathbb{C})$, but some of the homomorphisms can be derived from the Borel case using the natural fibration $\tau \colon G/B_0 \to G/B$ where B_0 is chosen to be contained in B. Let us give a rough idea.

By the means of the latter fibration, we can lift bundles and their sections and we can apply the result for the Borel case, however it might happen that the operator acting on the sections of homogeneous bundles on G/B_0 vanishes on the pullbacks of the original sections and so the invariant operator obtained in this way happens to be the zero one. The operators obtained from the Borel case via this construction are called the *standard operators*. But if this construction fails, there can still exist non-trivial invariant operators. Such operators are called *non-standard*. The Laplace operator $\overset{1}{\bullet}\!\!-\!\!\overset{0}{\times}\!\!-\!\!\overset{1}{\bullet} \to \overset{1}{\bullet}\!\!-\!\!\overset{-2}{\times}\!\!-\!\!\overset{1}{\bullet}$ is an example pointed out in [Baston, 90] bringing also slightly more details on the latter construction.

In our conformal case, the operators denoted by the straight arrows are the standard operators, the other ones are non-standard. Without the non-standard operators, this pattern is known as the *Bernstein-Gelfand-Gelfand resolution* which generalizes the de Rham resolution.[25] The operators corresponding to the longest arrow in our patterns are called the *long operators* (they correspond to the longest element in the Weyl group). Only the long operators in these patterns might fail to admit curved analogues, cf. Section 9.

It is a difficult problem (and probably unsolved in full generality) to specify all homomorphisms of the generalized Verma modules in the general parabolic case. However, the problem was solved for many cases with regular infinitesimal characters se e.g. [Boe, Collingwood, 85a,b].

9. The conformal connection and operators on curved manifolds

In the last section of this text we want to comment on natural operators on the whole category of conformal manifolds. We shall only indicate some of the known results, we provide the reader with further references and we sketch some directions of possible development in the near future. We shall not mention all of the known constructions of invariant operators on curved conformal manifolds, a detailed survey with many references can be found in [Baston, Eastwood, 90].

We discussed in Section 4 how all Riemannian invariants are constructed by means of the Levi-Cività connection. In the conformal case, we can use first the Riemannian invariance, then to build general formulas in terms of the covariant derivatives, and then to discuss which of them give rise to a conformally invariant operator, i.e. to a natural operator on the category of conformal manifolds. This is the approach used by many authors, see e.g. [Branson, 85], [Ørsted, 81], [Wünsch,

[25] On general manifolds, the de Rham sequence is used to resolve the sheaf of constant functions. On homogeneous manifolds we can resolve this constant sheaf in a more efficient way. The point is, on a homogeneous manifolds M we have a natural choice of a distribution D in the tangent bundle T such that $[D, D] = T$ and so it suffices to use the vector fields tangent to D in order to recognize the constants. More details of this point of view are found in [Baston, Eastwood, 89].

86]. They developed sophisticated infinitesimal methods for checking the invariance of such operators.

Another possibility is to use a canonical connection which does exist on the conformal manifolds, the so called Cartan connection. This can be viewed as a general connection on a suitable fiber bundle (i.e. not right-invariant) with very special properties. This is the approach we would like to indicate in more details. We have to begin with the description of the Cartan connections. But first we need to find the bundles where it lives, the first prolongations of the conformal structures. This will also complete our development from Section 5.

9.1. For every closed Lie subgroup $B \subset G^r_m$, the B-structures on m-dimensional manifolds were defined in 2.11. In Section 5, we identified the conformal structures on the pseudo-spheres with such a structure of order two, while the conformal structures were defined as first order structures in general, see 5.1. We have promised to clarify how are these two kinds of structures related.

Roughly speaking, the second order conformal structure is the first prolongation of the first order one. In order to make this idea more precise, we need to discuss a little the prolongations of the first order structures. Usually, the latter means a tower of $B^{(k)}$-structures $F^{(k)}$ on $F^{(k-1)}$ such that $F^{(k)} \subset P^1(F^{(k-1)})$ is a first order structure and the morphisms $f\colon M \to M$ of these structures coincide (f is a morphism of $F^{(1)}$ if $P^1(P^1f)(F^{(1)}) \subset F^{(1)}$). Such prolongations always exist but they are not canonically defined. For a detailed exposition of this theory see e.g. [Kobayashi, 72, Chapter I]. However, our aim is to get the prolongation as a reduction of the higher order frame bundle which is not so easy in general. The reader who likes to believe that the two definitions of conformal structures coincide (or prefers to define the conformal structures as second order ones) can skip the next text up to 9.4.

First we have to describe the prolongation $B^r \subset G^{r+1}_m$ of the Lie group $B \subset G^1_m = GL(m,\mathbb{R})$. The group $\mathcal{B}_M \subset \mathrm{Diff}M$ of the diffeomorphisms f satisfying $P^1f(FM) \subset FM$, cf. 2.12, determines the Lie subalgebra of the so called *infinitesimal automorphisms of the B-structure* in the algebra of all vector fields, which consists of the vector fields X with flows Fl^X_t in $\mathcal{B}_M$ for small parameters t. A B-structure is called *flat* if $FM \simeq M \times B$, the trivial bundle. Let us consider a flat B-structure and a fixed point $x \in M$. Then we have a subgroup $\mathcal{B}_0 \subset \mathcal{B}$ of automorphisms fixing the point x and the Lie algebra $\mathfrak{b}$ of infinite jets of the infinitesimal automorphisms at x (a subalgebra in the Lie algebra of the so called formal vector fields). As a Lie subalgebra of the infinite jets of all vector fields at x, the latter carries a canonical grading $\mathfrak{b} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \dots$. In particular $\mathfrak{b}_0 \subset \mathfrak{gl}(m,\mathbb{R})$ is the Lie algebra of B. The jets $j^{r+1}_0 f$ of the automorphisms $\mathcal{B}_0$ keeping the fixed point $0 \in \mathbb{R}^m$ form the Lie groups $B^r \subset G^{r+1}_m$. Their Lie algebras are the algebras $\mathfrak{b}_0 \oplus \mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \dots \oplus \mathfrak{b}_r$ with grading. The simplest way how to describe the Lie groups B^r is to study these Lie algebras, since the nonlinear parts of the polynomial expressions for the jets of morphisms in $\mathcal{B}$ can be identified with the polynomial expressions for the elements in the subalgebra $\mathfrak{b}_1 \oplus \mathfrak{b}_2 \oplus \cdots$.

Without loss of generality, we may assume $M = \mathbb{R}^m$ with the standard coordinates and $x = 0$, the origin. Then the elements in homogeneous components $\mathfrak{b}_q$ of $\mathfrak{b}$ have distinguished polynomial representatives $X_q(x) = \sum_{\underline{\alpha},\underline{i}} a^{\underline{i}}_{\underline{\alpha}} x^{\underline{\alpha}} \frac{\partial}{\partial x^{\underline{i}}}(x)$.

The condition on a vector field X to belong to $\mathfrak{b}$ is $P^1(\mathrm{Fl}^X_t)(F\mathbb{R}^m) \subset F\mathbb{R}^m$ for all small t, which means in local coordinates $\frac{\partial}{\partial x^j}(\mathrm{Fl}^X_t)^i \in \mathfrak{b}_0$. If we differentiate with respect to t we get the condition on the coefficients $a^i_{\underline{\alpha}}$ in the form $\frac{\partial}{\partial x^j}(\sum_{\underline{\alpha}} a^i_{\underline{\alpha}} x^{\underline{\alpha}}) \in \mathfrak{b}_0 \subset \mathfrak{gl}(m,\mathbb{R})$. But this condition is equivalent to the requirement that the matrices $(a^i_{j\underline{j}_1 \dots \underline{j}_q})$ are elements in $\mathfrak{b}_0$ for all fixed indices $\underline{j}_1, \dots, \underline{j}_q$. Since the coefficients a^i_α are symmetric in the subscripts, we have obtained an identification of $\mathfrak{b}_q$ with a subset in $S^{q+1}(\mathbb{R}^{m*}) \otimes \mathbb{R}^m$ of symmetric $(q+1)$-linear mappings s satisfying $s(\ ,v_1, \dots, v_q) \in \mathfrak{b}_0 \subset \mathbb{R}^{m*} \otimes \mathbb{R}^m$ for all fixed elements $v_1, \dots, v_q$. The linear subspaces $\mathfrak{b}_q$ are called the q-th prolongation of the Lie algebra $\mathfrak{b}_0$. If $\mathfrak{b}_q = 0$, then $\mathfrak{b}_r = 0$ for all $r \geq q$, by the definition. The smallest q with $\mathfrak{b}_q = 0$ is called the *order of the Lie algebra* $\mathfrak{b}_0$. If $\mathfrak{b}_q \neq 0$ for all q, then $\mathfrak{b}_0$ is said to be of infinite type.

9.2. Examples. In order to illustrate the above procedure, let us discuss the Lie algebras $\mathfrak{o}(m', n, \mathbb{R})$ and $\mathfrak{co}(m', n, \mathbb{R})$, $m' + n = m$, just now. Let us assume $X \in \mathfrak{b}_1$ is a polynomial field in the first case. Then its coefficients a^i_{jk} can be viewed as elements a_{ijk} by means of the isomorphism provided by the pseudo-metric. But then we have the anti-symmetry $a_{ijk} = -a_{jik}$ for all signatures. Since $a_{ijk} = a_{ikj}$, we get

$$a_{ijk} = -a_{jik} = -a_{jki} = a_{kji} = a_{kij} = -a_{ikj} = -a_{ijk}$$

and so $a_{ijk} = 0$. Thus, the Lie algebra $\mathfrak{o}(m', n, \mathbb{R})$ is of order one.

By the definition of the algebra $\mathfrak{co}(m', n, \mathbb{R})$, the kernel of the homomorphism $\mathfrak{co}(m', n, \mathbb{R}) \to \mathbb{R}$, $A \mapsto \mathrm{Tr}A$, is just the Lie algebra $\mathfrak{o}(m', n, \mathbb{R})$. Since $\mathfrak{o}(m', n, \mathbb{R})$ has order one, the linear mapping $\mathfrak{b}_1 \to \mathbb{R}^{m*}$, $X = (a^i_{jk} x^j x^k \frac{\partial}{\partial x^i}) \mapsto \frac{1}{n} a^i_{ik} \in \mathbb{R}^{m*}$, is injective (the kernel lies in the first prolongation of $\mathfrak{o}(m', n, \mathbb{R})$ and so is zero). On the other hand, each element $q_i \in \mathbb{R}^{m*}$ defines an element $-q_b g^{bi} \delta_{jk} + q_b g^b_k \delta^i_j + q_b g^b_j \delta^i_k$ which belongs to $\mathfrak{b}_1$, cf. 5.10. Thus the latter formula defines the identification $\mathfrak{b}_1 = \mathbb{R}^{m*}$. Let us consider $X \in \mathfrak{b}_2$ with coefficients a^i_{jkl}. For each l fixed we must get an element from $\mathfrak{b}_1$. Hence after lowering all superscripts, we can write

$$a_{ijkl} = -q_{il}\delta_{jk} + q_{kl}\delta_{ij} + q_{jl}\delta_{ik}.$$

Since the coefficients are symmetric in j, k, l, the trace satisfies $a_{bbkl} = mq_{kl} = a_{bblk} = mq_{lk}$. Further we have $a_{bbkl} = a_{bklb} = -q_{bb}\delta_{kl} + q_{lk} + q_{kl}$ and so $-q_{bb}\delta kl = (m-2)qkl$. The trace of this expression yields $(m-2)q_{bb} = -mq_{bb}$ and therefore $q_{bb} = 0$. Then the last but one equality implies $q_{ij} = 0$ if $m \geq 3$. In this way, we have proved that $\mathfrak{co}(m', n, \mathbb{R})$ is of order two in dimensions greater then two. (In dimension two, there is the isomorphism $\mathfrak{co}(2, \mathbb{R}) \simeq \mathfrak{gl}(1, \mathbb{C})$, hence it is an algebra of infinite type.)

A general theorem due to R. Palais claims that if the Lie algebra of all infinitesimal automorphisms of a B-structure on M is finite dimensional, then the group $\mathcal{B} \subset \mathrm{Diff}M$ is a finite dimensional Lie group and the infinitesimal automorphisms form its Lie algebra. In particular, this happens for each B-structure with the Lie algebra $\mathfrak{b}_0$ of B of finite order, see [Kobayashi, 72, Chapter I] for the proofs.

9.3. The first order prolongation. There is the so called *canonical form* $\theta \in \Omega^1(P^1M, \mathbb{R}^m)$ (called also *soldering form*) defined by $\theta(X) = \varphi_*^{-1}(T\pi(X)) \in \mathbb{R}^m$

where $X \in T_u(P^1M)$, $\pi\colon P^1M \to M$ is the bundle projection and $u = j^1_0\varphi$. Equivalently, $\theta(X) = j^1_0\varphi^{-1} \circ \pi \circ c$ if $X = j^1_0 c$.

The B-structure FM is a subbundle in $\pi\colon P^1M \to M$, hence $J^1(FM) \subset J^1(P^1M)$. If we choose a horizontal subspace $H \subset T_uP^1M$, then $\theta|H$ is an isomorphism. Now, each $y = j^1_x s \in J^1(FM)$ determines a horizontal subspace $H_y \in T_{s(x)}(FM)$ and an isomorphism $\mathbb{R}^m \oplus \mathfrak{b}_0 \to T_{s(x)}(FM)$ given by $(X, Y) \mapsto \zeta_Y + X'$ where $\theta(X') = X$, $X' \in H_y$ and ζ_Y is the fundamental field corresponding to Y. Hence we can view the one-jets of the sections as elements in $P^1(FM)$. The actions of the isomorphisms $f\colon M \to M$ on $J^1(FM)$ depend on the second derivatives and we shall try to find a subbundle in $J^1(FM)$ carrying the structure of the principal fiber bundle with the structure group B^1, which is preserved by the action of second jets of the automorphisms of the B-structure. This can be constructed by means of the differential $d\theta$ restricted to the tangent spaces to sections. Let us start with the notion of the torsion. The *torsion t of the B-structure FM* is the smooth function t on $J^1(FM)$ with values in $\operatorname{Hom}(\Lambda^2\mathbb{R}^m, \mathbb{R}^m))$ defined by

$$t(y)(\theta(X_1) \wedge \theta(X_2)) = d\theta(X_1, X_2), \qquad y = j^1_x s,\ X_1, X_2 \in H_y \subset T_{s(x)}(FM).$$

The torsion t is equivariant with respect to the action of the vector group $\mathbb{R}^{m*} \oplus \mathfrak{b}_0$ with respect to the following actions. The transitive action on the bundle $J^1(FM) \to FM$ is defined by means of the above identification $\mathbb{R}^m \oplus \mathfrak{b}_0 \simeq T_{s(x)}(FM)$ determined by $j^1_x s$, while the action on $\operatorname{Hom}(\Lambda^2\mathbb{R}^m, \mathbb{R}^m)$ is given by $A(w) = w + \partial A$, where $A \in \operatorname{Hom}(\mathbb{R}^m, \mathfrak{b}_0)$ and $\partial\colon \operatorname{Hom}(\mathbb{R}^m, \mathfrak{b}_0) \to \operatorname{Hom}(\Lambda^2\mathbb{R}^m, \mathfrak{b}_0)$, $(\partial f)(v_1, v_2) = -f(v_2)v_1 + f(v_1)v_2$, is the Spencer operator. Hence we can factorize t by these action of $\mathbb{R}^{m*} \oplus \mathfrak{b}_0$ and we get a mapping

$$c\colon FM \to \operatorname{Hom}(\Lambda^2\mathbb{R}^m, \mathbb{R}^m)/(\mathbb{R}^{m*} \oplus \mathfrak{b}_0)$$

which is called the *structure function* of B.

The space $\mathbb{R}^m$ is identified with the (abelian) subalgebra of constant vector fields $\mathfrak{b}_{-1}$ and so each value of t can be viewed as a cochain in $C^{0,2}(\mathfrak{b}_{-1}; \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \cdots)$ in the Spencer bigraded complex. All cochains in $C^{0,q}$ are closed (since $\mathfrak{b}_{-2} = 0$) and we factorize precisely by the image of the differential ∂, cf. 10.21. Hence the values of c are in the Spencer bigraded cohomology space $H^{0,2}(\mathfrak{b}_{-1}; \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \cdots)$.

If the structure function is zero, then there is a canonical way of the prolongation of the B-structure: The first jet prolongation $J^1(FM)$ is embedded into the bundle of second semi-holonomic frames $\bar{P}^2M$ and the vanishing of c is a necessary and sufficient condition for the existence of a holonomic subbundle $F^1M = i(J^1(FM) \cap P^2M)$. The latter is then the first prolongation with all required properties, see [Kolář, 85] for details. Let us remark that the structure function is defined in the latter paper by a nice geometrical construction using the difference tensor on semiholonomic second frame bundle.

In the conformal case, we compute in 10.21 that $H^{0,2}(\mathfrak{b}_{-1}; \mathfrak{g}) = 0$ and so the structure function must be always zero. Thus there is the canonical second order structure $F^1M \subset P^2M$ on conformal manifolds which is the first prolongation of the conformal structure $FM \subset P^1M$.[26]

[26] If the structure function is not zero, the torsion still helps to get the usual (but not canonical)

9.4. Another construction of the prolongations is based on the torsion free connections on FM. We shall need several technical tools.

The second frame bundle, is equipped with a generalization of the soldering form, a form $\theta^{(2)} \in \Omega^1(P^2M, \mathbb{R}^m \oplus \mathfrak{g}^1_m)$ defined as follows[27]. Each element $u \in P^2_x M$, $u = j^2_0\varphi$, determines a linear isomorphism $\tilde{u}\colon \mathbb{R}^m \oplus \mathfrak{g}^1_m \to T_{\pi^2_1(u)}P^1M$ (in fact $T_0(P^1\varphi)\colon T_{(0,e)}(\mathbb{R}^m \times G^1_m) \to TP^1M$). Now if $X \in T_uP^2M$ then $\theta^{(2)}(X) = \tilde{u}^{-1}(T\pi^2_1(X))$, i.e. $\theta^{(2)}(X) = j^1_0(P^1\varphi^{-1} \circ \pi^2_1 \circ c)$ if $X = j^1_0 c$. This canonical form decomposes as $\theta^{(2)} = \theta_{-1} \oplus \theta_0$ where θ_{-1} is the pullback of the soldering form θ on P^1M, $\theta_{-1} = (\pi^2_1)^*\theta$, while θ_0 is $\mathfrak{g}^1_m$-valued. The values of $\theta^{(2)}$ can be viewed as elements in the Lie subalgebra of constant and linear vector fields in the Lie algebra of formal vector fields.

Lemma. (1) *For each* $X \in \mathfrak{g}^2_m$, $\theta^{(2)}(\zeta_X) = T\pi^2_1(X) \in \mathfrak{g}^1_m = \mathfrak{gl}(m)$.
(2) *For each* $g \in G^2_m$, $(r_g)^*\theta^{(2)} = Ad(g^{-1})\theta^{(2)}$.
(3) *There is the structure equation* $d\theta_{-1} + [\theta_0, \theta_{-1}] = 0$.

Proof. The first two statements follow easily from the definition of $\theta^{(2)}$. Let us prove the last one. We shall use the canonical local coordinates u^i, u^i_j, u^i_{jk} on $P^2\mathbb{R}^m = \mathbb{R}^m \times GL(m,\mathbb{R}) \times \mathbb{R}^m \otimes S^2\mathbb{R}^{m*}$. The coordinate expression of (3) is $d\theta^i = -\theta^i_k \wedge \theta^k$, where $\theta_{-1} = \theta^i \otimes e_i$, $\theta_0 = \theta^i_k \otimes e^k_i$ are the expressions with respect to usual bases $e_i \in \mathfrak{b}_{-1} = \mathbb{R}^m$, $e^k_i \in \mathfrak{gl}(m,\mathbb{R})$ so that $[e^k_i, e_j] = \delta^k_j e_i$. The definition of $\theta^{(2)}$ provides us with the coordinate expression for the differentials of the coordinate functions u^i, u^i_j on P^1M

$$\begin{aligned} du^i &= u^i_j\theta^j \\ du^i_j &= u^i_h\theta^h_j + u^i_{hj}\theta^h. \end{aligned}$$

Applying the differential to the first equality we get

$$0 = du^i_j \wedge \theta^j + u^i_j d\theta^j = u^i_j d\theta^j + u^i_h\theta^h_j \wedge \theta^j + u^i_{hj}\theta^h \wedge \theta^j$$

where the last term is zero, for u^i_{hj} is symmetric in the subscripts. If we multiply by the inverse matrix function v^k_i to u^i_j on the left, we obtain $d\theta^k = -\theta^k_j \wedge \theta^j$ as required. □

9.5. A section of the bundle $\pi^2_1\colon P^2M \to P^1M$ is called admissible if $s(u.g) = s(u).g$ for all $u \in P^1M$ and $g \in G^1_m \subset G^2_m$. The admissible sections are precisely sections of $P^2M/G^1_m \to M$.

Lemma. *There is a bijective correspondence between local torsion-free connections* Γ *on* P^1M *and local admissible sections* s_Γ *given by* $\Gamma = s^*_\Gamma\theta_0$.

Proof. Given any local admissible section $s\colon P^1M \to P^2M$, the Lie algebra valued one form $\Gamma = s^*\theta_0$ is a local principal connection. One verifies easily in local

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prolongations as mentioned at the very beginning. Every choice of a complementary subspace $C \in \mathrm{Hom}(\Lambda^2\mathbb{R}^m, \mathbb{R}^m)$ to the subspace $\partial(\mathrm{Hom}(\mathbb{R}^m, \mathfrak{g}))$ determines the subspace $t^{-1}(C) \subset J^1(FM)$. The bundle $t^{-1}(C) \to FM$ has the proper structure group (corresponding to the Lie subalgebra $\mathfrak{b}_1 \subset \mathfrak{b}_0 \oplus \mathfrak{b}_1$).

[27] In general, a similar definition yields a form $\theta^{(k)} \in \Omega^1(P^kM, \mathbb{R}^m \oplus \mathfrak{g}^k_m)$ where $\mathfrak{g}^k_m$ is the Lie algebra of G^k_m

coordinates that Γ is without torsion. On the other hand, the coordinate expression shows that each locally defined principal connection without torsion defines a local section $M \to P^2M/G^1_m$, see [Kobayashi, 72, Proposition 7.1] for more details if necessary. $\square$

The value $s_\Gamma(u)$ depends only on the restriction of Γ to T_uP^1M. Hence if we consider a connection Γ of a B-structure $FM \subset P^1M$, then the admissible section s_Γ defines the B-principal subbundle $s_\Gamma(FM) \subset P^2M$. Now, we can take the orbit $B^1(s_\Gamma(FM)) \subset P^2M$ which is a B^1-principal subbundle. Hence the problem which remains is how to determine whether two different torsion-free connections give rise to the same second order B^1-structure. Such connections are called equivalent and the set of all equivalence classes of connections belonging to certain B-structures is parameterized by sections of the associated bundles of the B-principal bundle FM with respect to the representation of B on $H^{1,1}(\mathfrak{b}_{-1};\mathfrak{g})$, cf. [Ochiai, 70] or [Baston, 90]. This applies in particular to the conformal case, where the first prolongation B^1 of the structure is just the Poincaré conformal group and we compute in 10.21 that the above mentioned cohomology is zero. In other words, all torsion-free connections on conformal manifolds are equivalent, see also [Kobayashi, 72] for a more elementary direct treatment. Thus the local prolongations do not depend on our choice of the connections and hence they can be glued into a unique reduction of the second frame bundle P^2M to the Poincaré conformal group. In particular, we can use the Levi-Cività connection with respect to any pseudo-Riemannian metric from the conformal class.

9.6. The Cartan connections. We have established the existence of a canonical subbundle in the second order frame bundle on each conformal manifold, let us denote this principal bundle PM. We are interested in some analogy to the Levi-Cività connection for conformal manifolds. We shall see, that there exists a canonical Cartan connection which is unfortunately not a connection but more an analogy of the Maurer-Cartan form on Lie groups.

Definition. Let G be a Lie group with a closed subgroup B and let $\dim G/B = m$. A *Cartan connection* ω on a principal bundle P with m-dimensional base manifold and structure group B is a $\mathfrak{g}$-valued one-form on P ($\mathfrak{g}$ is the Lie algebra of G) with the properties

(1) $\omega(\zeta_X) = X$ for all $X \in \mathfrak{b}$
(2) $(r_g)^*\omega = \mathrm{Ad}(g^{-1})\omega$ for each $g \in B$
(3) $\omega(Y) \neq 0$ for each non-zero $Y \in TP$

As already mentioned, the Maurer-Cartan form on G is the simplest example of a Cartan connection on the principal bundle $G \to G/B$. The Cartan connection ω on P can be viewed as a principal connection on the principal bundle $P \times_B G$ with structure group G.

Similarly to the usual principal connections, we can write down the structure equation

$$d\omega = -\frac{1}{2}[\omega, \omega] + \Omega$$

where Ω is some $\mathfrak{g}$-valued 2-form. This 2-form is called the *curvature form of the Cartan connection* ω.

9.7. Our next aim is to find canonical Cartan connections on the canonical subbundles PM in P^2M on conformal manifolds. We shall follow the elementary treatment from [Kobayashi, 72] using local coordinates but we shall review the whole story in the language of the Spencer cohomologies later on.

A Cartan connection ω and its curvature Ω on the canonical bundle PM can be always decomposed as $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$ and $\Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$. But this decomposition is only $\mathfrak{b}_0$-invariant and not $(\mathfrak{b}_0 \oplus \mathfrak{b}_1)$-invariant.

Lemma. (1) *the restriction ω_{-1} to each fiber of PM vanishes*

(2) *the restriction of the $\mathfrak{b}$-component, i.e. $\omega_0 \oplus \omega_1$, to each fiber is the Maurer-Cartan form of $\mathfrak{b}$*

(3) *the curvature is a horizontal 2-form, i.e. $\Omega(X, Y) = 0$ if X is vertical*

(4) *if $\omega_{-1} = \omega^1 \otimes e_1 + \dots + \omega^m \otimes e_m$ for some fixed base of $\mathfrak{b}_{-1}$, then the curvature admits an expression $\Omega = \sum_{\underline{i},\underline{j}} \frac{1}{2} K_{\underline{ij}} \omega^{\underline{i}} \wedge \omega^{\underline{j}}$ where $K_{\underline{ij}}$ are $\mathfrak{g}$-valued functions.*

Proof. The assertions (1) and (2) follow directly form the definition of the Cartan connections. Then the structure equation, restricted to any fiber, yields (3). Since each Cartan connection ω defines an absolute parallelism on PM, the components ω^i, ω^i_j, ω_j of $\omega_{-1} \oplus \omega_0 \oplus \omega_1$ with respect to basis of the components of the Lie algebra generate the whole algebra of the exterior forms $\Lambda(M)$. But then obviously (1)–(3) imply (4). $\square$

9.8. Admissible Cartan connections. The restriction of the canonical form $\theta^{(2)}_M \in \Omega^1(P^2M, \mathbb{R}^m \oplus \mathfrak{g}^1_m)$ on an m-dimensional conformal manifold M to the principal subbundle $P \subset P^2M$ with structure group B is an $(\mathbb{R}^m \oplus \mathfrak{b}_0)$-valued form, we shall denote it by $\theta_P \in \Omega^1(PM, \mathbb{R}^m \oplus \mathfrak{b}_0)$. This decomposes as $\theta_P = \theta_{-1} \oplus \theta_0$ where $\theta_{-1} \in \Omega^1(P, \mathfrak{b}_{-1})$ and $\theta_0 \in \Omega^1(P, \mathfrak{b}_0)$. We have

$$\theta_0(\zeta_{X_0+X_1}) = X_0 \qquad \text{for each } X_0 + X_1 \in \mathfrak{b}_0 \oplus \mathfrak{b}_1 \tag{1}$$

$$(r^g)^*(\theta_{-1} \oplus \theta_0) = \operatorname{Ad}(g^{-1})(\theta_{-1} \oplus \theta_0) \tag{2}$$

$$\theta_{-1}(Y) = 0 \text{ if and only if } Y \text{ is vertical} \tag{3}$$

$$d\theta_{-1} + [\theta_0, \theta_{-1}] = 0 \tag{4}$$

and so there can exist Cartan connections $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$ on P where ω_1 is subject of a free choice. Such Cartan connections are called *admissible.*

The Maurer-Cartan equations of $O(m+1, 1, \mathbb{R})$ can be easily read off 5.9 if we decompose the bracket

$$[\omega, \omega] = [\omega_{-1}, \omega_0] + ([\omega_{-1}, \omega_1] + [\omega_0, \omega_0]) + [\omega_1, \omega_0].$$

The structure equation for a Cartan connection ω consists then of the same terms together with the curvature components:

$$d\omega^i = -\omega^i_k \wedge \omega^k + \Omega^i \tag{5}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j - \omega^i \wedge \omega_j - \omega_i \wedge \omega^j + \delta^i_j \omega_k \wedge \omega^k + \Omega^i_j \tag{6}$$

$$d\omega_j = -\omega_k \wedge \omega^k_j + \Omega_j. \tag{7}$$

If ω is admissible, then $\Omega^i = 0$ by the definition and (5). Now, applying the exterior differential d to (5) we get

$$0 = d(\omega^i_j \wedge \omega^j) = d\omega^i_j \wedge \omega^j - \omega^i_j \wedge d\omega^j.$$

If we substitute from (5) and (6) we obtain the *Bianchi identity* $\Omega^i_j \wedge \omega^j = 0$. In the expression $\Omega^i_j = \frac{1}{2} K^i_{jkl} \omega^k \wedge \omega^l$ this means

$$K^{\underline{i}}_{\underline{jkl}} + K^{\underline{i}}_{\underline{klj}} + K^{\underline{i}}_{\underline{ljk}} = 0.$$

9.9. Theorem. *Let P be a principal bundle over an m-dimensional manifold M, $m \geq 3$, with structure group B, the Poincaré conformal group. If $\omega_{-1} \in \Omega^1(P, \mathfrak{b}_{-1})$ and $\omega_0 \in \Omega^1(P, \mathfrak{b}_0)$ are two 1-forms satisfying the equalities 9.8.(1)–(3) and the structure equation 9.8.(4), then there is a unique Cartan connection $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$, such that the curvature $\Omega = \Omega_{-1} \oplus \Omega_0 \oplus \Omega_1$ satisfies $\Omega_{-1} = 0$ and Ω_0 is in the trace-free part of the space of $\mathfrak{b}_0$-valued 2-forms. In the standard basis of the components of the Lie algebra, the latter means $\Omega = (0, \Omega^i_j, \Omega_j)$ with*

$$\Omega^{\underline{i}}_{\underline{j}} = \sum \frac{1}{2} K^{\underline{i}}_{\underline{jkl}} \omega^{\underline{k}} \wedge \omega^{\underline{l}}, \qquad \sum K^{\underline{i}}_{\underline{jil}} = 0.$$

Proof. Let us first prove the uniqueness. Consider two admissible Cartan connections ω, $\bar{\omega}$ with the properties required in the theorem. Then the $\mathfrak{b}_{-1}$-component and $\mathfrak{b}_0$-component of the difference $\bar{\omega} - \omega$ are zero by the definition and

$$\bar{\omega}_{\underline{j}} - \omega_{\underline{j}} = \sum_{\underline{k}} A_{\underline{jk}} \omega^{\underline{k}}$$

for suitable functions $A_{\underline{jk}}$ on the principal bundle P. Now, direct computation using 9.8.(5)–(7) yields the expression for the difference of the curvatures $\bar{\Omega}^{\underline{i}}_{\underline{j}} - \Omega^{\underline{i}}_{\underline{j}} = \frac{1}{2} \sum_{\underline{kl}} (\bar{K}^{\underline{i}}_{\underline{jkl}} - K^{\underline{i}}_{\underline{jkl}}) \omega^{\underline{k}} \wedge \omega^{\underline{l}}$ with

$$\bar{K}^{\underline{i}}_{\underline{jkl}} - K^{\underline{i}}_{\underline{jkl}} = -\delta^{\underline{i}}_{\underline{l}} A_{\underline{jk}} + \delta^{\underline{i}}_{\underline{k}} A_{\underline{jl}} + \delta^{\underline{i}}_{\underline{j}} A_{\underline{kl}} - \delta^{\underline{i}}_{\underline{j}} A_{\underline{lk}}.$$

Thus, the traces are

$$\sum_{\underline{i}} (\bar{K}^{\underline{i}}_{\underline{jil}} - K^{\underline{i}}_{\underline{jil}}) = (m-2) A_{\underline{jl}} + \delta_{\underline{jl}} \sum_{\underline{i}} A_{\underline{ii}}$$
$$\sum_{\underline{ij}} (\bar{K}^{\underline{i}}_{\underline{jij}} - K^{\underline{i}}_{\underline{jij}}) = 2(m-1) \sum_{\underline{i}} A_{\underline{ii}}$$

and so $A_{\underline{ij}} = 0$ for all subscripts.

Next we notice that there is a Cartan connection satisfying all requirements if there is at least one Cartan connection with the given components ω_{-1} and ω_0.

The point is, we write $\bar{\omega} - \omega$ as above and we find functions $A_{\underline{ij}}$ such that $\bar{\omega}$ will obey all the required properties. One verifies easily that the right choice is

$$A_{\underline{jk}} = \frac{1}{m-2}\left(\frac{1}{2(m-1)}\delta_{\underline{jk}} \sum_{\underline{i},\underline{l}} K^{\underline{i}}_{\underline{lil}} - \sum_{\underline{i}} K^{\underline{i}}_{\underline{jik}}\right).$$

In order to complete the proof, we have to construct an arbitrary Cartan connection ω with given components ω_{-1} and ω_0. Since local Cartan connections on M can be glued together using the partition of unity on the manifold M, it suffices to construct the connections locally. (Another argument is, each such local Cartan connection gives rise to a local connection with the required properties, but the latter is unique and so we must get a globally well defined object.) If we choose a section σ of P, we can define $\omega_j = 0$ on the tangent spaces to the section and since the values of ω_j are given also on the vertical tangent spaces by the definition and ω_j must be right invariant with respect to action of the Poincaré group B, ω_j is well defined by this choice. Explicitly, each vector $Y \in T_uP$, $u = \sigma(x).g$, decomposes uniquely as $Y = (r^g)_*(X_1) + \zeta_{X_2}(u)$ with $X_1 \in T\sigma(T_xM)$ and $X_2 \in \mathfrak{b}_0 \oplus \mathfrak{b}_1$. Then by the definition

$$\omega(Y) = \mathrm{Ad}(g^{-1})(\omega(X_1)) + X_2 \quad \square$$

and this formula defines the values of ω_j.

9.10. Remark. The local construction from the end of the above proof can be modified to produce a globally defined admissible Cartan connection by means of a torsion-free connection on the 'linear' frame bundle PM/B_1.

Let us consider such a connection Γ and the corresponding admissible section s_Γ from 9.5. Now, we set $\omega_j = 0$ on the image of s_Γ, and we decompose each $Y \in T_uP$, $u = s_\Gamma(x).g$ with $g \in B_1$, uniquely as $Y = (r^g)_*(X_1) + \zeta_{X_2}(u)$ with $X_1 \in T\sigma(T_x(PM/B_1)$ and $X_2 \in \mathfrak{b}_1$. Then $\omega(Y) = \mathrm{Ad}(g^{-1})(\omega(X_1)) + X_2$ defines ω_j. One checks easily that this is an admissible Cartan connection.

9.11. The conformal connection. For each conformal manifold M, we can apply the above theorem to the canonical principal subbundle $PM \subset P^2M$ with structure group B and the restriction θ_P of the canonical two-form $\theta^{(2)}$ on P^2M to PM. Thus, there is the uniquely defined Cartan connection ω_M on PM such that $\omega_M = \theta_{-1} \oplus \theta_0 \oplus (\omega_M)_1$ and $\Omega_M = 0 \oplus (\Omega_M)_0 \oplus (\Omega_M)_1$ with values of $(\Omega_M)_0$ in the trace-free part of $\Lambda^2T^*P \otimes \mathfrak{b}_0$. This connection is called the *normal Cartan connection* on M or the *conformal connection* on M. Usually, we shall omit the subscript M in the sequel.

As mentioned in 9.6, the Cartan connections can be viewed as the usual connections on the extended principal fiber bundle $P \times_B G$ with structure group G and so we get the induced connections on each associated bundle. In particular, we can consider the standard fiber G/B, the sphere. The associated bundle can be viewed as the 'pointwise compactified tangent space' over the base manifold M. The connection on this space is also called the conformal connection on M in the literature.

9.12. The cohomological interpretation. We present briefly an alternative description of the conformal connection and its curvature. We follow [Ochiai, 70] and [Baston, 90].

Let us consider an arbitrary admissible Cartan connection $\omega = \theta_{-1} \oplus \theta_0 \oplus \omega_1$ on the canonical bundle PM over a conformal manifold M and let us write briefly $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 = \mathfrak{o}(m+2)$. Further we shall use the notation $\omega^{-1}(X)$ for the vector field on P corresponding to an element $X \in \mathfrak{g}$. In particular, we can rewrite the structure equation $d\omega = \frac{1}{2}[\omega,\omega] + \Omega$ as

$$\Omega(\omega^{-1}(X), \omega^{-1}(Y)) = [X,Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)]) \tag{1}$$

for all X, $Y \in \mathfrak{g}$ (the values of ω on our particular fields are constant and so the 'Lie derivative part' of the differential disappear).

For each $u \in P$ we define the cochain $W(u) \in C^{1,2}(\mathfrak{b}_{-1}; \mathfrak{g})$ by

$$W(u)(X,Y) = \Omega_0(\omega^{-1}(X), \omega^{-1}(Y)), \quad X, Y \in \mathfrak{b}_{-1}. \tag{2}$$

The differential ∂W is evaluated on three elements from $\mathfrak{b}_{-1}$, and the formula from 10.21 yields

$$\partial W(X,Y,Z) = [\theta_{-1}, \Omega_0](X,Y,Z) \tag{3}$$

so that the Bianchi identity implies $\partial W = 0$. Hence W determines a cohomology class in $H^{1,2}(\mathfrak{b}_{-1}, \mathfrak{g})$.

In the first part in the proof of Theorem 9.8 we proved in fact that this class is independent of our choice of ω_1. The assumption on the values of Ω_0 in Theorem 9.8 mean that we have to adjust ω_1 in such a way that W is the unique harmonic representative of the class. Let us give some more details.

Given any pair ω, $\bar{\omega}$ of admissible Cartan connections, there is the $C^{2,1}(\mathfrak{b}_{-1}; \mathfrak{g})$-valued function f on P defined by

$$\bar{\omega}^{-1}(X) - \omega^{-1}(X) = \bar{\omega}^{-1}(f(X)), \quad X \in \mathfrak{b}_{-1}.$$

Since Ω is a horizontal form, we get

$$\begin{aligned}(\bar{W} - W)(X,Y) &= (\bar{\Omega}_0 - \Omega_0)(\omega^{-1}(X), \omega^{-1}(Y)) \\ &= [\theta_{-1}, \bar{\omega}_1 - \omega_1](\omega^{-1}(X), \omega^{-1}(Y)) \\ &= \partial f(X,Y)\end{aligned} \tag{4}$$

(only the $\mathfrak{b}_1$-valued entry in the structure equation can contribute to the last but one term). This shows that the cohomology class of W is uniquely defined.

We can always construct an admissible Cartan connection ω on PM from local sections, see the proof of 9.8, or equivalently from the Levi-Cività connection of one of the metrics from the conformal class by means of the construction from 9.4. In order to get the right one, we have to find the proper $C^{2,1}(\mathfrak{b}_{-1}; \mathfrak{g})$-valued function f. This is obtained as the solution of the equation

$$\Box f = -\partial^* W \tag{5}$$

where $\Box$ is the Laplace operator on the cochains and ∂^* is the codifferential, see 10.22 for the notation and definitions. Indeed, then we can define

$$\bar{\omega}^{-1}(X) = \omega^{-1}(X) + \omega^{-1}(f(X))$$

for all $X \in \mathfrak{b}_{-1}$ and so

$$\bar{W} = \partial f + W,$$

see (4). Now, $\partial^* \bar{W} = \partial^* \partial f + \partial^* W = \Box f + \partial^* W = 0$ and so $\Box \bar{W} = 0$.

Since we know from 10.21 that $H^{2,1}(\mathfrak{b}_{-1};\mathfrak{g}) = 0$ in the conformal case, this solution f is unique and we have recovered the uniqueness and existence of the normal conformal connection.

9.13. The conformal curvature. In the coordinate like description, the components of the curvature of the normal Cartan connection are $\Omega^{\underline{i}}_{\underline{j}} = \frac{1}{2}\sum K^{\underline{i}}_{\underline{jkl}}\omega^{\underline{k}} \wedge \omega^{\underline{l}}$ and $\Omega_{\underline{j}} = \sum K_{\underline{jkl}}$. In the proof of 9.8 we deduced the Bianchi identity for Ω^i_j. An analogous computation leads to the equalities

$$\sum_{\underline{i}} \omega^{\underline{i}} \wedge \Omega_{\underline{i}} = 0, \qquad \omega^{\underline{i}} \wedge \Omega_{\underline{j}} - \omega^{\underline{j}} \wedge \Omega_{\underline{i}} = 0.$$

A further computation with traces verifies also that Ω_1 vanishes whenever Ω_0 does, provided the dimension is at least four. Hence Ω_0 is the proper obstruction against the integrability of the conformal structures.

Let us represent the $\mathfrak{b}_0$-component Ω_0 of the curvature as a section of a suitable bundle. As mentioned in 10.21, the cohomology spaces $H^*(\mathfrak{b}_{-1},\mathfrak{g})$ carry a canonical $\mathfrak{b}_0$-module structure. Hence the cohomology class of the function W on the canonical bundle PM could represent a section of the associated bundle corresponding to the $\mathfrak{b}_0$-module $H^{1,2}(\mathfrak{b}_{-1},\mathfrak{g})$ (viewed as $(\mathfrak{b}_0 \oplus \mathfrak{b}_1)$ -module via the trivial extension, if it satisfies the proper equivariance condition. Indeed, then we view W as an equivariant smooth mapping with values in the standard fiber, i.e. as a section. But the latter equivariance follows from the fact that Ω_0 is right invariant modulo $\mathfrak{b}_1$.

In 10.21 we compute the highest weights of the representations of $\mathfrak{b}_0$ occurring in $H^{1,2}(\mathfrak{b}_{-1};\mathfrak{g})$ in the conformal case. One finds, that in dimensions greater then four we get the irreducible conformally invariant part of the Riemann curvature tensor, the so called *Weyl curvature tensor*, while in dimension four the latter still splits into two irreducible components.

An interesting fenomenon appears in dimension three, where $H^{1,2}(\mathfrak{b}_{-1};\mathfrak{g}) = 0$ and so the Weyl curvature does not exist and has to be replaced by a third order invariant tensor. See [Baston, 90] for more comments.

Now, let us come back to the natural operators on conformal manifolds.

9.14. Let us first recall the meaning of 'conformally invariant'. In the sense of the general definition of Section 2, the natural operators are systems of operators defined on sections of bundles with distinguished actions of the conformal morphisms and intertwining these actions, one for each conformal manifold. If we deal with spinor bundles, we have to consider the coverings of the morphisms to the spin structures, see 2.14. It has no meaning to restrict this definition to individual manifolds, since in general there may be no conformal morphisms beside the identity, or only very few of them, and in such a case all operators would be 'invariant'. However exactly those constructions on individual manifolds which make no use of some special choices extend into natural operators on all conformal manifolds.

All our operators are local and so we do not take care of the spin structures which always exist locally. The effect is that the operators we obtain might not exist on some manifolds simply because of the lack of the definition domains.

9.15. The 'curved' translation principle. The translation procedure which was heavily used in Section 8 in the conformally flat case, was worked out in [Baston, 90] by means of the normal Cartan connection for general conformal manifolds of all dimensions greater then two.

The inverse of the Cartan connection ω on the canonical bundle $PM \to M$ on a conformal manifold M is an injective linear mapping

$$\omega^{-1}\colon \mathfrak{g} \to C^\infty(TPM)$$

where $G = SO(m+2, \mathbb{C})$ and $\mathfrak{g}$ is its Lie algebra. The right invariance of ω with respect to the action of the conformal Poincaré group B has the infinitesimal form

$$\omega^{-1}([X,Y]) = [\omega^{-1}(X), \omega^{-1}(Y)], \qquad X \in \mathfrak{b},\ Y \in \mathfrak{g}.$$

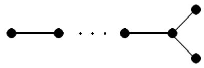
Let us now fix two weights λ and ρ dominant for $\mathfrak{b}$ and write as usual V_λ, V_ρ for the corresponding representation spaces. They define the associated bundles $E_\lambda M = PM \times_\lambda V_\lambda$ and $E_\rho = PM \times_\rho V_\rho$ on all conformal (spin) manifolds (in the 'spin case' PM means the lift to the double covering of the canonical bundle, see 2.14).

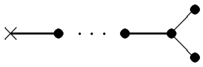
In order to find an invariant linear operator $D_M\colon C^\infty(E_\lambda M) \to C^\infty(E_\rho M)$, we have to describe the dual mapping to its action on the infinite jets of sections of the bundles. If we fix a point $u \in PM$, the latter should be an invariantly defined mapping $\{u\} \times_\rho V_\rho^* \to (J_u^\infty(PM, V_\lambda)^B)^*$. Now we can employ the Cartan connection. The domain of this map is a $\mathfrak{U}(\mathfrak{b})$-module generated by a highest weight vector but the codomain is , with the help of ω, too.

Let us write $\mathfrak{A}(\mathfrak{g})$ for the quotient $T(\mathfrak{g})/\langle X \otimes Y - Y \otimes X - [X,Y];\ X \in \mathfrak{b}, Y \in \mathfrak{g}\rangle$ of the tensor algebra over $\mathfrak{g}$ by the indicated ideal. $\mathfrak{A}(\mathfrak{g})$ is a $\mathfrak{U}(\mathfrak{b})$-bimodule and $\mathfrak{U}(\mathfrak{g})$ is a quotient of $\mathfrak{A}(\mathfrak{g})$. As a vector space $\mathfrak{A}(\mathfrak{g}) \simeq T(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{b})$ and the left $\mathfrak{b}$-modules $\mathfrak{A}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_\lambda^*$ cover the generalized Verma modules $M_\mathfrak{b}(V_\lambda^*)$. In particular, the maximal weight vectors are defined in $\mathfrak{A}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_\lambda^*$ and they must cover maximal weight vectors in $M_\mathfrak{b}(V_\lambda^*)$.

Now the point is, the normal Cartan connection identifies the dual of the jet space $(J_u^\infty(PM, V_\lambda)^B)^*$ with a quotient of $\mathfrak{A}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_\lambda^*$, exactly as in the identification in 8.2. Indeed, in Section 8 we made use of the special case of the normal Cartan connection, the Maurer-Cartan form on $\mathfrak{g}$ in the identification of the right invariant vector fields on G with $\mathfrak{U}(\mathfrak{g})$ and this was the crucial point of the identification of the dual jet spaces. Now we can do the same, but we are allowed only to use commutators of the form $[X,Y]$ with $X \in \mathfrak{g}$, $Y \in \mathfrak{b}$.

If we find a maximal weight vector with weight ρ in $\mathfrak{A}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_\lambda^*$, then we obtain a uniquely defined mapping $\{u\} \times_\rho V_\rho^* \to (J_u^\infty(PM, V_\lambda)^B)^*$ and since we deal with jets of right-invariant mappings, the latter cannot depend on our choice of u in the fiber. Once such maximal weight vector exists in one fiber, we get it in all other ones as well and we obtain an invariant operator in this way. Each such maximal weight vector covers a maximal weight vector in $M_\mathfrak{b}(V_\lambda^*)$ and so the corresponding operator can be viewed as an extension of the invariant operator on the locally flat conformal manifolds. Hence, in our algebraic reformulation, the question whether the invariant operators on flat manifolds admit curved analogues reads: *do the maximal weight vectors in $M_{\mathfrak{b}}(V_\lambda^*)$ lift to maximal weight vectors in $\mathfrak{A}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_\lambda^*$?* A partial answer is given in [Baston, 90]:

9.16. Lemma. *Let λ be an integral dominant weight for $\mathfrak{g} =$* *,*

let $\mathfrak{b} =$  *and let $w,\ w' \in W^{\mathfrak{b}}$. If $D\colon M_{\mathfrak{b}}(V_{w'.\lambda}^*) \to M_{\mathfrak{b}}(V_{w.\lambda}^*)$ is a homomorphism of Verma modules, then the image of $M_{\mathfrak{b}}(V_{w'.\lambda}^*)$ is generated by a maximal weight vector v which can be lifted to a maximal weight vector in $\mathfrak{A}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} V_{w.\lambda}^*$, unless $w = id$ and $|w'| = 2n$, the full dimension.*

In the formulation of the Lemma, we use terminology and notation introduced in the Appendix. In particular, $W^{\mathfrak{b}}$ means the parabolic subgraph, $|w'|$ the length of its element. In order to prove the lemma one has to find an expression for the maximal weight vector v as a sum of terms of the form $P \otimes v'$ with $P \in T(\mathfrak{n}_-)$ and $v' \in V_{w.\lambda}^*$ such that its maximality can be proved only by means of commutators of the form $[X, Y]$, $X \in \mathfrak{b}$, $Y \in \mathfrak{g}$. The complete proof is available in [Baston, 90] and is based heavily on the translation principle.

The latter author claims also that an analogous lemma holds in odd dimensions. As a consequence, we get immediately the following general result on the existence of natural operators.

9.17. Theorem. *All natural operators between natural vector bundles with regular infinitesimal characters on flat conformal m-dimensional manifolds, $m = 2n$ even, extend to bundles on curved m-dimensional conformal manifolds except the long operators, i.e. those corresponding to the longest arrow in the diagram from 8.13.*

In odd dimensions, all natural operators on locally flat conformal manifolds extend to a natural operator on the whole category.

Though the proof of Lemma 9.16 consists in certain inductive construction, it provides us with no direct method for writing down the formulas for the operators, cf. the situation in the flat case, Remark 8.4. Nevertheless, there is a general reason for which all these formulas are expression in the Levi-Cività connection with highest order term coinciding with the flat case, accomplished with certain lower order correction terms. The correction terms are expressed only through the Ricci curvature of the Levi-Cività connections and their formal expressions do not depend on the choice of the metric in the conformal class.

9.18. Remark. The problem which of the so called long operators admit curved analogues seems to be still unsolved, in general. There is the theorem due to [Graham, 90] which shows that the cube of the Laplace operator in dimension four has no curved analogue. (The proof consists of twenty nine pages of careful elimination of all possible correction terms!) On the other hand, the operator $\Delta^n\colon \Omega^0 \to \Omega^{2n}$ on $2n$-dimensional manifolds is a long operator which admits a curved analogue. There is a conjecture that this is the only long operator which does, see [Baston, Eastwood, 90].

Another unsolved problem is to clarify how far is the extension unique. For example, we can add multiples of $\varphi_a \mapsto B_d{}^a\varphi_a$ to the invariant operator $\mathcal{O}_a[1] \to \mathcal{O}_d[-3]$, where $B_d{}^a$ is the so called Bach tensor.

9.19. Explicit formulas. If we choose a metric from the conformal class, we get the admissible Cartan connection $\bar{\omega}$ constructed from the Levi-Cività connection, see 9.10. Let ω be the normal Cartan connection. For each element $X \in \mathfrak{n}_-$ we define the vector fields $\bar{X} = \bar{\omega}^{-1}(X)$ and $X^* = \omega^{-1}(X)$. The two admissible Cartan connections define the $C^{2,1}(\mathfrak{b}_{-1}; \mathfrak{g})$-valued f on the canonical bundle PM such that $X^* = \bar{X} + \omega^{-1}(f(X))$, see 9.12. A homomorphism of Verma modules $\varphi\colon M_{\mathfrak{b}}(V_\rho^*) \to M_{\mathfrak{b}}(V_\lambda^*)$ is determined by the proper maximal weight vector in the target which must be of the form $\sum_i P^i(X_j)\kappa_i$ where the elements κ_i form a weight basis of V_λ^*, the X_j's form a root space basis of $\mathfrak{n}_-$ and P^i are homogeneous polynomials. These polynomials must be chosen according to Lemma 9.16 and in order to obtain a differential operator, each occurrence of X_j must be replaced by the vector field X_j^*. Thus, in order to get differential operators in terms of the Levi-Cività connection we have to substitute X_j^* in terms of $\bar{X}$, ω and f. Then the monomials in $\bar{X}$ will induce the differential operator obtained from projecting $\nabla_{a_1} \cdots \nabla_{a_n} s$ into its irreducible factor corresponding to the target bundle of the operator in question and the terms $\omega^{-1}(f(X_j))$ will build certain correction terms. A more careful study of the two Cartan connections involved enables to express f as $f = -\Box^{-1}\partial^* r(\Gamma)$ where Γ is the Levi-Cività connection and $\partial^* r(\Gamma)$ is the Ricci curvature of Γ, if viewed as a section of the appropriate induced bundle.

The algorithm which leads to the explicit correction terms goes quite quick out of hand with increasing order. In [Baston, 90], the correction terms were computed in general for second order operators (with some particular examples of higher order operators involved). We add only two general remarks concerning this algorithm: If $\{Y_j\}$ is a basis of the negative root spaces in $\mathfrak{g}_0$, then in an expansion in terms of $\bar{X}_i$ of an expression of the form $X_{i_1}^* X_{i_2}^* \dots X_{i_n}^*(Y_{j_1}Y_{j_2}\dots Y_{j_p}.v)$

(1) the first element $X_{i_n}^*$ gives rise to no correction terms
(2) for each occurrence of a curvature correction term in the expansion, there are two fewer occurrences of $\bar{X}_i$'s in the result then X_i^*'s in the original expression

The point (1) recovers the result form Section 6 where we proved that the first order invariant operators always extend to the whole category without changing the formal expression. From (2) it follows immediately, that the highest order correction terms are of order at least two less than that of the leading term.

9.20. Some other methods. The Gover's idea how to find explicit formulas of some invariant operators is to apply the standard technique of the twistor theory, the double fibration $\mathcal{A} \xleftarrow{\ \mu\ } \mathcal{G} \xrightarrow{\ \nu\ } \mathcal{M}$ where $\mathcal{G}$ is the bundle of null directions on a sufficiently small region $\mathcal{M}$ of a conformal manifold, $\mathcal{A}$, the ambitwistor space, is the space of null geodesics of $\mathcal{M}$ and μ, ν are the obvious projections. In the flat four-dimensional case, we have the homogeneous space $\mathcal{M} = SL(4,\mathbb{C})/(\bullet\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\bullet)$, the space of full flags in $\mathbb{C}^4$ $\mathcal{G} = SL(4,\mathbb{C})/(\times\!\!-\!\!\!-\!\!\times\!\!-\!\!\!-\!\!\times)$ and $\mathcal{A} = SL(4,\mathbb{C})/(\times\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\times)$. The twistor theory studies in detail the relations between the homogeneous bundles on $\mathcal{G}$ and $\mathcal{M}$, in particular, it is well known how to induce operators acting on bundles over $\mathcal{M}$ from the operators acting on bundle over $\mathcal{G}$. It turns out that all operators on bundles over $\mathcal{G}$ which involve only differentiation in the directions of fibers of μ descend to non-trivial operators on bundles on $\mathcal{M}$. Such operators are called *horizontal operators.* [Gover, 89] proves that all horizontal operators on the homogeneous space $\mathcal{G}$ in the flat case have curved analogues and he also gives explicit method how to find the formulas for the correction terms. Comparing these results with the discussion from Section 8 one finds that what we get in this way are precisely the standard operators and nothing else. For explicit formulas and details see [Gover, 89], a geometric description of this method in terms of the canonical projective structures on curves in conformal manifolds is given in [Baston, Eastwood, 90].

Let us further mention the methods related to Lie algebra cohomology and the Fefferman-Graham method, cf. [Feffeman, Graham, 85] and [Baston, Eastwood, 90]. A lot of the methods which were elaborated for the classification of the conformal invariants are efficient also for some other, higher order geometries. The so called almost Hermitian symmetric structures are treated in [Baston, preprint, 90].

9.21. Possible development. We shall mention only a few of areas where the interested reader could find a lot of possibilities for his own activity.

First, the representation theory provides the necessary background for similar classifications in different geometric categories with finite dimensional spaces of morphisms. A lot of activity is visible in the literature in this direction. It seems, that even the specialists in the representation theory could profit from the geometric reformulations of their problems.

Second, the construction of the operators on the curved manifolds should be expressed in more geometric terms and some analogy to the general theory for Riemannian invariants could be achieved. The general theory of connections could be a good tool for that. One of the crucial questions reads: *Are all natural operators built of the above mentioned extensions of those living on the conformally flat manifolds and the Weyl conformal curvature?*

Further, the infinitesimal naturality could be weakened by dropping the locality assumption. *Are all such operators obtained by integration of local ones?* In the category of all manifolds and mappings the answer to an analogous question is, yes, cf. [Cap, Slovák, to appear].

Next, any effective algorithm for concrete formulas for the operators would be highly appreciated, even in the conformally flat case (in fact we need the curvature correction terms even in the conformally flat case and may be that the contents of the above extension construction is that the same formulas apply).

10. Appendix

This is a rather sketched overview of some basic facts concerning representations of Lie algebras and Lie groups used in the main text. The main sources are: [Samelson, 89], [Knapp, 86], [Zhelobenko, 70], [Naymark, 76], [Baston, 90], [Lepowsky, 77], [Zuckerman, 77].

10.1. A *representation* π of a (real or complex) Lie group G on a finite dimensional (real or complex) vector space V is a Lie group homomorphism $\pi\colon G \to GL(V)$. Analogously, a representation of a Lie algebra $\mathfrak{g}$ on V is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(V)$. For every representation $\pi\colon G \to GL(V)$ of a Lie group, the tangent map at the identity $T\pi\colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of its Lie algebra. Given two representations π_1 on V_1 and π_2 on V_2 of a Lie group G, a linear map $f\colon V_1 \to V_2$ is called a *G-module homomorphism* if $f(\pi_1(a)(x)) = \pi_2(a)(f(x))$ for all $a \in G$ and all $x \in V$. Analogously we define the *$\mathfrak{g}$-module homomorphisms*. We say that the representations π_1 and π_2 are *equivalent*, if there is a G-module isomorphism (or $\mathfrak{g}$-module isomorphism) $f\colon V_1 \to V_2$.

A linear subspace $W \subset V$ in the representation space V is called *invariant* if $\pi(a)(W) \subset W$ for all $a \in G$ (or $a \in \mathfrak{g}$) and π is called *irreducible* if there is no proper invariant subspace $W \subset V$. A representation π is said to be *completely reducible* if V decomposes into a direct sum of irreducible invariant subspaces. A decomposition of a completely reducible representation is unique up to the ordering and equivalences.

A representation π of a connected Lie group G is irreducible, or completely reducible if and only if the induced representation $T\pi$ of its Lie algebra $\mathfrak{g}$ is irreducible, or completely reducible, respectively.

10.2. The commutator of two elements a_1, a_2 of a Lie group G is the element $a_1 a_2 a_1^{-1} a_2^{-1}$ in G. The closed subgroup $K(S_1, S_2)$ generated by all commutators of elements $s_1 \in S_1 \subset G$, $s_2 \in S_2 \subset G$ is called the commutator of subsets S_1 and S_2. In particular, $G' := K(G, G)$ is called the *derived group* of the Lie group G. We get two sequences of closed subgroups $G^{(n)}$ and $G_{(n)}$, $n \in \mathbb{N}$, defined by $G^{(0)} = G = G_{(0)}$, $G^{(n)} = (G^{(n-1)})'$, $G_{(n)} = K(G, G_{(n-1)})$. A Lie group G is called *solvable* if $G^{(n)} = \{e\}$ for some $n \in \mathbb{N}$, G is called *nilpotent* if $G_{(n)} = \{e\}$ for some $n \in \mathbb{N}$. Since always $G_{(n)} \supset G^{(n)}$, every nilpotent Lie group is solvable.

The Lie bracket determines in each Lie algebra $\mathfrak{g}$ two analogous sequences of Lie subalgebras: $\mathfrak{g} = \mathfrak{g}^{(0)} = \mathfrak{g}_{(0)}$, $\mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}]$, $\mathfrak{g}_{(n)} = [\mathfrak{g}, \mathfrak{g}_{(n-1)}]$. The sequence $\mathfrak{g}_{(n)}$ is called the *descending central sequence* of $\mathfrak{g}$. A Lie algebra $\mathfrak{g}$ is called *solvable*, or *nilpotent* if $\mathfrak{g}^{(n)} = 0$, or $\mathfrak{g}_{(n)} = 0$ for some $n \in \mathbb{N}$, respectively. Every nilpotent Lie algebra is solvable. If $\mathfrak{b}$ is an ideal in $\mathfrak{g}^{(n)}$ such that the factor $\mathfrak{g}^{(n)}/\mathfrak{b}$ is commutative, then $\mathfrak{b} \supset \mathfrak{g}^{(n+1)}$. Consequently, a Lie algebra $\mathfrak{g}$ is solvable if and only if there is a sequence of subalgebras $\mathfrak{g} = \mathfrak{b}_0 \supset \mathfrak{b}_1 \supset \cdots \supset \mathfrak{b}_l = 0$ where $\mathfrak{b}_{k+1} \subset \mathfrak{b}_k$ is an ideal, $0 \le k < l$, and all factors $\mathfrak{b}_k/\mathfrak{b}_{k+1}$ are commutative.

A connected Lie group is solvable or nilpotent if and only if its Lie algebra is solvable or nilpotent, respectively.

Each Lie algebra $\mathfrak{g}$ contains a unique maximal solvable ideal, the so called *radical* $\mathfrak{r}$ of $\mathfrak{g}$. Similarly, there is a unique maximal nilpotent ideal, we call it the *nilradical* $\mathfrak{n}$. A Lie algebra $\mathfrak{g}$ is called *semisimple*, if its radical is zero and its dimension is positive, $\mathfrak{g}$ is called *simple* if it contains no non-trivial ideals.

The quotient $\mathfrak{g}/\mathfrak{r}$ is always semisimple or trivial and we get the exact sequence

$$0 \to \mathfrak{r} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{r} \to 0.$$

The *Levi-Malcev theorem* states this sequence splits, i.e. *each Lie algebra is a direct sum $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ with $\mathfrak{r}$ solvable and $\mathfrak{s}$ semisimple or trivial.*

The *Engel's theorem* claims: *A Lie subalgebra* $\mathfrak{g} \subset \mathfrak{gl}(m, \mathbb{C})$ *consisting entirely of nilpotent operators is a nilpotent Lie algebra.*

A Lie algebra with a completely reducible adjoint representation is called *reductive.* If $\mathfrak{g}$ is reductive, then its radical $\mathfrak{r}$ coincides with the center $\mathfrak{z}$. The *Levi decomposition* $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{n}$ is a decomposition with $\mathfrak{l}$ reductive while $\mathfrak{n}$ nilpotent.

10.3. The *Killing form* κ on the Lie algebra $\mathfrak{g}$ is the symmetric bilinear form defined by $\kappa(X, Y) = \mathrm{Tr}(\mathrm{ad}X \circ \mathrm{ad}Y)$, the trace of the composition of the adjoint actions. A Lie algebra is semisimple if and only if its Killing form is non-degenerate and its dimension is positive. A Lie algebra is solvable if and only if its Killing form vanishes identically on the derived algebra $\mathfrak{g}'$.

10.4. Cartan subalgebra. A nilpotent Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ which is equal to its own normalizer is called a *Cartan subalgebra.* If $\mathfrak{g}$ is complex and semisimple this is equivalent to $\mathfrak{h}$ maximal abelian with $\mathrm{ad}H$ diagonizable for all $H \in \mathfrak{h}$. If $\mathfrak{g} = \mathfrak{gl}(m, \mathbb{C})$ we take the subalgebra of all diagonal matrices for $\mathfrak{h}$. The dimension l of $\mathfrak{h}$ does not depend on the choice and we call it the *rank of* $\mathfrak{g}$.

10.5. Roots and weights. Consider a representation ρ of a Lie algebra $\mathfrak{g}$ in a vector space V. An element $\lambda \in \mathfrak{g}^*$ is called a *weight* if there is a non zero vector $v \in V$ such that $\rho(x)v = \lambda(x)v$ for all $x \in \mathfrak{g}$. Then v is called the *weight vector* (corresponding to λ). Every representation of a nilpotent algebra decomposes as a sum of its *weight spaces* V_λ of weight vectors corresponding to the weights λ.

If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, then the weights α of the adjoint representation of $\mathfrak{h}$ in $\mathfrak{g}$ are called *roots* of the algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$. The corresponding weight vectors X_α are called the root elements (with respect to $\mathfrak{h}$), the weight spaces are called the *root spaces.* Since $\mathfrak{h}$ is nilpotent, the whole algebra $\mathfrak{g}$ splits as a sum of the root spaces $g = \sum_\alpha \mathfrak{g}_\alpha$.

In the sequel we shall assume $\mathfrak{g}$ is complex and semisimple. Let us consider a representation ρ of $\mathfrak{g}$. Then there are the weight vectors corresponding to the restriction of ρ to the Cartan subalgebra. Let us write V_λ for the subspace consisting of the zero vector and all weight vectors corresponding to a weight $\lambda \in \mathfrak{h}^*$. Since the Cartan subalgebra is nilpotent (even abelian), the whole representation space V is spanned by the weight vectors $v \in V_\lambda$. So $V = \sum_\lambda V_\lambda$ and there is only a finite number of V_λ non-zero. The set of weight vectors is always invariant under the action of the root elements in $\mathfrak{g}$, i.e. $X_\alpha . V_\lambda \subset V_{\lambda+\alpha}$. In particular, this applies to the splitting of a complex semisimple Lie algebra $\mathfrak{g}$ into root spaces $\mathfrak{g}_\alpha$ so that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

A maximal solvable subalgebra $\mathfrak{b}$ in a Lie algebra $\mathfrak{g}$ is called a *Borel subalgebra.* Each Borel subalgebra contains a maximal commutative subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the property that all operators $\mathrm{ad}X$, $X \in \mathfrak{h}$, are diagonal in $\mathfrak{g}$, i.e. a Cartan subalgebra. The roots with root elements belonging to the chosen Borel subalgebra are called *positive roots.* Those positive roots which are not linear combinations of two different positive roots with positive coefficients are called *simple roots* (or *fundamental roots*). Choosing an order on the simple roots, we get a weak order (sometimes called lexicographic) on the set of all roots of $\mathfrak{g}$. The set of all roots is denoted by Δ, the space of positive roots by $\Delta^+ \subset \Delta$. The set of all simple roots will be denoted by Δ_0^+. We always have $-\Delta = \Delta$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.

The real vector subspace in $\mathfrak{h}^*$ generated by the roots is called the real part $\mathfrak{h}_0^*$ of $\mathfrak{h}^*$. For semisimple algebras, the Killing form is non-degenerate and also its restriction to $\mathfrak{h}$ is non-degenerate. Thus we get the induced isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$. Using the induced isomorphism with the dual we obtain the real part $\mathfrak{h}_0 \subset \mathfrak{h}$. The restriction of the Killing form to the real part $\mathfrak{h}_0$ is positive definite and so we find for each $\lambda \in \mathfrak{h}_0^*$ a unique element $h_\lambda \in \mathfrak{h}_0$ such that $\langle h_\lambda, X\rangle = \lambda(X)$ for all $X \in \mathfrak{h}$. If $X \in \mathfrak{g}_\lambda$, $Y \in \mathfrak{g}_{-\lambda}$ and $\langle X, Y\rangle = 1$ then $[X, Y] = h_\lambda$. The elements $H_\lambda = \frac{2}{\langle h_\lambda, h_\lambda\rangle} h_\lambda$ are called the *coroots*. The reason for this definition of H_λ will be clear in 10.9.

The simple roots form a basis of $\mathfrak{h}^*$ and so each other root is a real linear combination $\sum a_i\varphi_i$ of the simple ones and, moreover, a root is positive if and only if all coefficients a_i are non-negative. For all roots, the coefficients $a_1, \dots, a_l$, where is the rank of $\mathfrak{g}$, are integral. In particular, all weights of a representation belong to the real part $\mathfrak{h}_0^*$. A weight λ of a representation ρ is called the *highest weight* if there is no positive root α such that $\lambda + \alpha$ is a weight of ρ.[28]

Let us denote $\mathfrak{n}_+$ the derived algebra $[\mathfrak{b}_+, \mathfrak{b}_+]$ of the chosen Borel subalgebra (the subalgebra of upper triangular matrices with zeros on the diagonal in the $\mathfrak{gl}(m, \mathbb{C})$ case). A vector v in a $\mathfrak{g}$-module V is the *highest weight vector* (with respect to $\mathfrak{b}_+$) if it is a weight vector with highest weight. This happens if and only if there is a weight $\lambda \in \mathfrak{h}^*$ such that $x.v - \lambda(x)v = 0$ for all $x \in \mathfrak{h}$ and $x.v = 0$ for all $x \in \mathfrak{n}_+$, i.e. v is a weight vector with the trivial action of $[\mathfrak{b}^+, \mathfrak{b}^+]$. (The latter condition shows that λ is the highest weight of the representation as defined above).

The highest weight vectors always exist for complex finite dimensional representations of complex semisimple algebras (and some more general ones) and they are uniquely determined for the irreducible ones. The procedure of complexification allows to use this for the real case as well.

10.6. Examples. In order to have some simple examples, let us take $\mathfrak{g} = \mathfrak{gl}(m, \mathbb{C})$. The irreducible representations coincide in fact with irreducible representations of $\mathfrak{sl}(m, \mathbb{C})$, see 3.13. We start with the highest weight of the identical representation on $\mathbb{R}^m$ corresponding to the tangent bundle T. The action of $a = (a_l^k)$, $a_l^k = \delta_j^k \delta_l^i$ for some $j < i$, (corresponding to the action of $X = x^i \frac{\partial}{\partial x^j}$ given by the negative of the Lie derivative) on a highest weight vector v must be zero, so that only its first coordinate can be nonzero. Hence the weight is $e^1 \in \mathbb{R}^{m*}$.

For the irreducible modules $\Lambda^p \mathbb{R}^{m*}$ we can express the action of $X = x^i \frac{\partial}{\partial x^j}$ on (constant) form ω through the Lie derivative $\mathcal{L}_{-X}\omega$. Since $\mathcal{L}_X dx^l = \delta_j^l dx^i$ we get that if $X.\omega = 0$ for all $j < i$ then ω is a constant multiple of $dx^{m-p+1} \wedge \cdots \wedge dx^m$. Further, the action of $\mathcal{L}_{-x^i/\partial x^i}$ on $dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ is minus identity if i appears among the indices i_j and zero if not. Hence the highest weight is $-e^{m-p+1} - \cdots - e^m$. Similarly we compute the highest weight of the dual $\Lambda^p \mathbb{R}^m$ $e^1 + \cdots + e^p$ and the highest weight vector of $S^p \mathbb{R}^{m*}$ which is the symmetric tensor product of p copies of dx^m and the weight is $-pe^m$.

10.7. Abstract root systems. The roots of a semisimple complex algebra form a geometric object with a very strong and nice geometric properties. Let us forget

[28] Sometimes, the highest weights are also called 'extreme' but we use this term for all weights in the orbit of the highest weight under the Weyl group, see below.

for a moment about the Lie algebras endowed with the Killing form and let us focus on the roots themselves.

An (abstract) *root system* in a vector space V with respect to a definite bilinear form $\langle\ ,\ \rangle$ is a finite non-empty subset $R \subset V \setminus \{0\}$ which satisfies

(1) For all α, $\beta \in R$, $a_{\beta\alpha} := \frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}$ is an integer.
(2) For all α, $\beta \in R$, the vector $\beta - a_{\beta\alpha}.\alpha$ belongs to R.
(3) If $\alpha \in R$ and $a.\alpha$ are both in R, then $a = \pm 1$.

Sometimes, this is also called *reduced root system* while the unreduced root systems are defined by dropping condition (3).

We can express the conditions (1) and (2) more geometrically: Let us denote $S_\mu(\lambda) = \lambda - \frac{2\langle\lambda,\mu\rangle}{\langle\mu,\mu\rangle}\mu$, i.e. S_μ is the reflection in V with respect to the hyperplane orthogonal to μ. The first two conditions are equivalent to

(1') For all α, $\beta \in R$, the difference $S_\alpha(\beta) - \beta$ is an integral multiple of α.
(2') The set of all roots is invariant under the action of all S_α, $\alpha \in R$.

The group of isometries of V which preserves the root system R is generated by the refelections S_μ and is called the *Weyl group* of the (abstract) root system R.

10.8. Weyl group. Let us come back to complex semisimple Lie algebras. The reflections S_α corresponding to the root system of the Lie algebra $\mathfrak{g}$ generate the *Weyl group* W *of* $\mathfrak{g}$. This is a group of isometries in $\mathfrak{h}_0^*$. The set Δ of roots is invariant under the action of the Weyl group. The hyperplane orthogonal to α in $\mathfrak{h}_0^*$ is called the singular plane of α (of height zero), we shall denote it by $(\alpha, 0)$. Clearly $(\alpha, 0) = (-\alpha, 0)$. The Weyl reflection S_α is identity on $(\alpha, 0)$ and interchanges the two half-spaces determined by $(\alpha, 0)$. We denote by $D' = \cup_{\alpha\in\Delta^+}(\alpha, 0)$. The complement $\mathfrak{h}_0^* \setminus D'$ is an open subset. Its connected components are bounded by parts of some singular planes $(\alpha, 0)$, the so called *walls*. These connected components are called the *Weyl chambers* of Δ. The Weyl group W permutes the Weyl chambers and if an element from W leaves one chamber fixed (as a set), then it is the identity. Moreover, for each $\alpha \in \Delta$, the orbit $W.\alpha$ meets each Weyl chamber in exactly one point.

The union of the singular planes defines the (infinitesimal) *Cartan-Stiefel diagram* D'.

10.9. Dominant weights. Consider a Borel subalgebra $\mathfrak{b}$ in a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, and choose an order on the simple roots. The set of all simple roots is called the *fundamental system*. Recall that every positive root is a linear combination of the simple roots with non-negative integral coefficients and the fundamental system is linearly independent. Hence the number of simple roots equals the rank of the algebra. The coroots corresponding to the simple roots are called the *fundamental coroots*.

Let α_i form the fundamental system of roots and write H_i for the fundamental coroots. Then the set $\{\lambda \in \Delta; \langle\alpha_i, \lambda\rangle \geq 0,\ 1 \leq i \leq l\}$ forms a Weyl chamber, the so called *fundamental Weyl chamber*. We consider the Weyl group as an abstract group acting on $\mathfrak{h}_0^*$. By the duality, the Weyl group acts also on $\mathfrak{h}_0$ with the contragredient representation. Then the coroots form a congruent root system with the fundamental coroots as the simple roots. The fundamental Weyl chamber consists just of all $H \in \mathfrak{h}_0$ with $\alpha_i(H)$ positive.

The lattice of all elements λ in $\mathfrak{h}_0^*$ with $\lambda(H_\alpha)$ integral for all coroots H_α, $\alpha \in \Delta$, is called the lattice of *integral forms*. The dual basis λ_i to the simple coroots H_i is formed by the *fundamental weights* (or fundamental forms) of $\mathfrak{g}$. The integral weights λ which satisfy $\lambda(H_i) \geq 0$, i.e. $\lambda(H_\alpha) \geq 0$ for all $\alpha \in \Delta^+$, are called *dominant*. The set of all dominant weights is an Abelian semigroup generated by the fundamental weights. Each highest weight of a representation of a complex semisimple Lie algebra is dominant and each dominant weight is a highest weight of some irreducible representation. Since the tensor product of two irreducible representations always contains an irreducible representation with highest weight equal to the sum of the two highest weights, the so called *Cartan product* of the two representations, all irreducible representations are generated by those corresponding to the dominant weights (more explicitly, they live in their tensor products).

The sum of all fundamental weights $\delta = \lambda_1 + \cdots + \lambda_l$ is called the *lowest weight* (or lowest dominant form). It holds $\delta - S\delta$ is the sum of those positive roots that become negative under S^{-1}, $S \in W$, and δ is half the sum of all positive roots.

As already mentioned, a representation space V of a complex semisimple Lie algebra splits into subspaces generated by the weight vectors. The weights are always integral forms and the set of all weights of a representation φ is invariant under the action of the Weyl group. In fact, together with λ, all the forms $\lambda, \lambda - \operatorname{sgn}(\lambda(H_\alpha))\alpha, \lambda - 2\operatorname{sgn}(\lambda(H_\alpha))\alpha, \dots, \lambda - \lambda(H_\alpha)\alpha$ are weights of φ. The multiplicities of the weights of φ are invariant with respect to the action of the Weyl group, i.e. $m_\lambda = m_{S\lambda}$, $S \in W$.

For each finite dimensional representation, there is precisely one orbit $W(\lambda)$ under the Weyl group containing the highest weight. The elements ρ from this orbit are called the *extremal weights* of the representation, they are independent of the choice of the positive roots and they can be characterized by $\langle \rho, \rho \rangle \geq \langle \mu, \mu \rangle$ for all weights μ of the representation (the equality takes place if and only if μ is extremal). On the other hand, for each integral weight λ there is precisely one dominant weight in its orbit. Hence each integral weight is an extremal weight of a uniquely defined finite dimensional representation.

10.10. Orthogonal algebras. The properties of the orthogonal algebras differ essentially for even and odd dimensions. Moreover the dimensions $m = 3$, $m = 4$ and $m = 6$ are exceptional, for the corresponding algebras are isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{sl}(2, \mathbb{C}) \oplus \overline{\mathfrak{sl}(2, \mathbb{C})}$ and $SL(4, \mathbb{C})$ (the bar means the complex conjugation).

(i) $m = 2l + 1$. We take the quadratic form defining the orthogonal group in the form $x^T \mathbb{J} x = x_0^2 + 2(x_1 x_2 + x_3 x_4 + \cdots + x_{2l-1} x_{2l})$, i.e. $\mathbb{J} = E_{00} + E_{12} + E_{21} + \cdots + E_{2l-1,2l} + E_{2l,2l-1}$. The symbol E_{ij} means a matrix with just one non-zero element placed in the i-th row and j-th column, e_i are the elements from the standard basis from $\mathbb{R}^m$ or $\mathbb{C}^m$, e^i the dual basis in the dual space. The abelian subalgebra $\mathfrak{h} = \mathbb{C}^l$ of diagonal matrices with $(0, a_1, -a_1, \dots, a_1, \dots, a_l, -a_l)$ is the Cartan subalgebra and the real subspace of diagonal matrices of the same form in

$\mathfrak{h}$ is the real subalgebra $\mathfrak{h}_0$.[29] The roots and root elements are

$$\begin{array}{lll} e^i & \sqrt{2}(E_{2i-1,0}-E_{0,2i}) & 1\le i\le l \\ -e^i & \sqrt{2}(E_{0,2i-1}-E_{2i,0}) & 1\le i\le l \\ e^i-e^j & E_{2i-1,2j-1}-E_{2j,2i} & i\ne j \\ e^i+e^j & E_{2j-1,2i}-E_{2i-1,2j} & i<j \\ -e^i-e^j & E_{2i,2j-1}-E_{2j,2i-1} & i<j \end{array}$$

We choose e^i and $e^i \pm e^j$ with $i<j$ for the positive roots. The simple roots (fundamental system) are $\{e^1-e^2, e^2-e^3, \dots, e^{l-1}-e^l, e^l\}$. The fundamental coroots are $H_1 = e_1 - e_2, \dots, H_{l-1} = e_{l-1} - e_l$, $H_l = 2e_l$. The fundamental Weyl chamber is defined by $a_1 > a_2 > \dots > a_l > 0$ and the maximal root is e^1+e^2. The Killing form is the Euclidean $\sum(e^i)^2$, up to a factor. The Weyl group contains the exchange of any two axes (reflexion with respect to e^i-e^j) and the changes of signs of any axis (corresponds to e^i), i.e. W is the group of all permutations and changing of signs on l variables.

(ii) $m=2l$**.** We consider the quadratic form $x^T\mathbb{J}x = 2(x_1x_2+\dots+x_{2l-1}x_{2l})$, i.e. $\mathbb{J} = E_{12}+E_{21}+\dots$, the Cartan algebra $\mathfrak{h}$ consists of diagonal matrices given by $(a_1,-a_1,\dots,a_l,-a_l)$. The roots and root elements are

$$\begin{array}{lll} e^i-e^j & E_{2i-1,2j-1}-E_{2j,2i} & i\ne j \\ e^i+e^j & E_{2i-1,2j}-E_{2j-1,2i} & i<j \\ -e^i-e^j & E_{2i,2j-1}-E_{2j,2i-1} & i<j \end{array}$$

The order in $\mathfrak{h}_0^*$ is defined by $(l-1, l-2, \dots, 0)$ and the positive roots are the e^i-e^j and e^i+e^j, $i<j$. The simple roots are $e^1-e^2, \dots, e^{l-1}-e^l, e^{l-1}+e^l$, the corresponding coroots are $H_1 = e_1-e_2, \dots, H_{l-1} = e_{l-1}-e_l, H_l = e_{l-1}+e_l$. The fundamental Weyl chamber is $a_1 > a_2 > \dots > a_{l-1} > |a_l|$. The maximal root is e^1+e^2. The Killing form is the Euclidean $\sum(e^i)^2$, up to a factor. The Weyl group contains the exchange of any two axes and the exchange of an arbitrary pair of axes coupled with the change of their signs. Thus W is the group of all permutations and even number of sign changes in l variables.

(iii) The algebras $\mathfrak{sl}(l+1,\mathbb{C})$. Here the situation is most simple. The Cartan algebra is the subalgebra of diagonal matrices with trace zero, the roots are $\alpha_{ij} = e^i-e^j$, $i\ne j$, the E_{ij}, $i\ne j$ are the corresponding root elements. The positive roots are α_{ij} with $i<j$ and the simple roots are $\alpha_{12}, \alpha_{23}, \dots, \alpha_{l,l+1}$ (the corresponding coroots are $e_1-e_2, \dots, e_l-e_{l+1}$). The fundamental Weyl chamber consists of elements with $a_1 > \dots > a_{l+1}$ and the maximal root is e^1-e^{l+1}. The Killing form is also the Euclidean form up to a factor. The Weyl group W is the group of all permutations in $l+1$ variables.

[29] Of course, the usual quadratic form must lead to the same relations, however let us notice that then the real Cartan subalgebra does not consist of diagonal matrices, and involves purely imaginary entries.

10.11. Representations of the complex orthogonal groups. All the groups except $SO(4,\mathbb{C})$ are simple. An irreducible representation of a direct sum of two semisimple Lie algebras is a tensor product of irreducible representations of the summands.

The sum $\lambda+\rho$ of highest weights of two irreducible representations of a semisimple Lie algebra is the highest weight in the tensor product of the two representations and occurs with multiplicity one. The irreducible representation with the highest weight $\lambda+\rho$ is called the *Cartan product* of the original two representations. In this way, the irreducible representations form a semigroup which is isomorphic to the set of dominant weights. The dominant weights are (freely) generated by the fundamental weights. Let us list briefly these fundamental representations and some more information for the three types of algebras discussed in 10.10.

(i) $\mathfrak{o}(m)$, $m = 2l+1$**.** The fundamental weights are $\lambda_1 = e^1$, $\lambda_2 = e^1 + e^2$, ..., $\lambda^{l-1} = e^1 + \dots + e^{l-1}$, $\lambda_l = \frac{1}{2}(e^1 + \dots + e^l)$. The corresponding representations to the first $l-1$ weights are the (complex) exterior forms of degrees $1,\dots,l-1$, the remaining representation is called the spin representation, we shall discuss it in the next section. (Notice, the Hodge star identifies some of the remaining exterior forms, but still there is the degree l missing and so this must be expressed using the two-valued spin representation.)

The set of dominant weights consists of all forms $\lambda = \sum_1^l \alpha_i e^i$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_l \geq 0$ and either all α_i are integral or all α_i half-integral. The numbers $(\alpha_1,\dots\alpha_l)$ are called the *signature* of the irreducible representation λ. The signature of the exterior forms of degree k is $(1,\dots,1,0,\dots,0)$ with k ones, $k \leq l$.

(ii) $\mathfrak{o}(m)$, $m = 2l$**.** The fundamental weights are $\lambda_i = e^1 + \dots + e^i$, $1 \leq i \leq l-2$, and $\lambda_{l-1} = \frac{1}{2}(e^1 + \dots + e^{l-1} - e^l)$, $\lambda_l = \frac{1}{2}(e^1 + \dots + e^{l-1} + e^l)$. The corresponding representations to the first $l-2$ weights are as before the (complex) exterior forms of degrees $1,\dots,l-2$, the remaining representations are called the half-spin representation, see the next section. The set of dominant weights consists of all forms $\lambda = \sum_1^l \alpha_i e^i$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq |\alpha_l|$ and either all α_i are integral or all α_i half-integral.

(iii) $\mathfrak{sl}(l+1)$**.** The fundamental weights are $\lambda_i = e^1 + \dots + e^i$, $i = 1,\dots,l$, the corresponding representations are the exterior forms (the representation on the highest degree forms is trivial). The dominant forms are $\lambda = \sum_1^l \alpha_i e^i$ with $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_l \geq 0$ integral.

All these facts are more or less easily obtained from the above description of the structure of the algebras in question (the Killing form is proportional to the Euclidean metric, so that it is easy to find the coroots and their dual basis). Let us also notice, that we can use the above description of both the structure and representations also in the extreme dimensions, see e.g. [Jacobson, 62], if we omit the objects which do not make sense. So for example, all representations must be generated by the two spin representations for dimension four. This is the basic ingredient of the 'two-spin' formalism which we shall mention later on.

It is important to know all weights involved in a given representation. This is easy for the forms: the weights of λ_r, $r \leq l$, are simply $e^{i_1} + \dots + e^{i_r}$, $1 \leq i_1 < \dots < i_r \leq l+1$. These must be all involved as they form the orbit under the Weyl group. On the other hand their number equals the dimension.

10.12. Parabolic subalgebras. Let us fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ in a complex reductive Lie algebra. Each subalgebra $\mathfrak{p}$ containing $\mathfrak{b}$, i.e. $\mathfrak{b} \subset \mathfrak{p} \subset \mathfrak{g}$ is called a *parabolic subalgebra*. There is only a finite number of parabolic subalgebras containing a fixed Borel algebra. All parabolic subalgebras (up to conjugation) are constructed by a simple procedure:

Let us write $\mathfrak{n}^{\pm}$ for the subalgebras generated by the positive or negative root elements respectively, i.e. $\mathfrak{n}^{+} = [\mathfrak{b}, \mathfrak{b}]$. The whole algebra is a sum

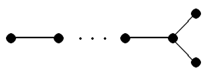
$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha) = \mathfrak{h} \oplus \mathfrak{n}^{+} \oplus \mathfrak{n}^{-} = \mathfrak{n}^{-} \oplus \mathfrak{b}.$$

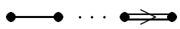
Let us fix a set $\Sigma \subset \Delta_0^{+}$ of simple roots and write Δ_Σ for its span in the set of all roots. Now we define the subalgebras

$$\mathfrak{l} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha), \quad \mathfrak{n} = \oplus_{\alpha \in \Delta^{+} \setminus \Delta_\Sigma} \mathfrak{g}_\alpha, \quad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$$

By the definition, $\mathfrak{p}$ contains the whole Borel algebra $\mathfrak{b}$ and the algebra $\mathfrak{g}$ splits as a vector space direct sum of Lie subalgebras $\mathfrak{g} = \mathfrak{n}^{-} \oplus \mathfrak{p}$. The subalgebra $\mathfrak{l}$ is reductive, $\mathfrak{n}$ is nilpotent. Hence $\mathfrak{l}$ is the reductive Levi factor of the parabolic subalgebra $\mathfrak{p}$. The semisimple factor is $[\mathfrak{l}, \mathfrak{l}] = \oplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha$ and $\mathfrak{l} = \mathfrak{h}_\Sigma \oplus (\oplus_{\alpha \in \Delta_\Sigma} \mathfrak{g}_\alpha)$ where $\mathfrak{h}_\Sigma$ is the linear subspace in $\mathfrak{h}$ corresponding to $\Sigma \subset \mathfrak{h}^*$.

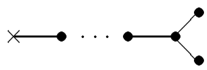
The parabolic subalgebras in semisimple complex algebras can be effectively denoted by means of the Dynkin diagrams if we replace the nodes corresponding to the simple roots which are not in Σ by a cross. In the main text we need the algebras $SO(m+2, \mathbb{C})$ with $m \geq 3$. The Dynkin diagrams are ($SO(6, \mathbb{C}) \simeq SL(4, \mathbb{C})$)

$\mathfrak{g} =$  if $m = 2n$

$\mathfrak{g} =$  if $m = 2n + 1$

$\mathfrak{g} =$ if $m = 4 = 2n$

where all diagrams have $n + 1$ nodes. The explicit description of the Poincaré conformal subalgebra $\mathfrak{b} \subset \mathfrak{o}(m+2, \mathbb{C})$, see 5.9, shows that $\mathfrak{b}$ is a parabolic subalgebra, for the maximal solvable subalgebra in $\mathfrak{b}$ must be maximal in the whole $\mathfrak{o}(m+2, \mathbb{C})$ as well. Looking at the list of roots and root elements in 10.10, one can see that this parabolic subalgebra contains all root spaces corresponding to the negatives of the simple roots, except the first one. Hence

$\mathfrak{b} =$  if $m = 2n$

$\mathfrak{b} =$  if $m = 2n + 1$

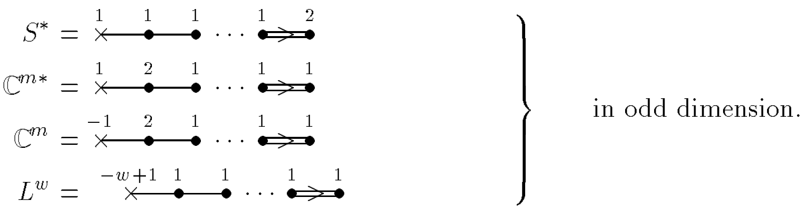
$\mathfrak{b} =$ if $m = 4 = 2n$.

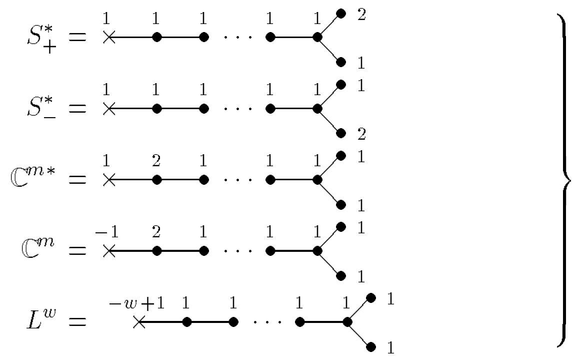
10.13. Representations of parabolic subalgebras. In general, the representations of the parabolic subalgebras of semisimple algebras need not be completely reducible. But we shall still restrict ourselves to the irreducible ones. Let us fix a parabolic algebra $\mathfrak{p} \subset \mathfrak{g}$ and its Levi decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$ corresponding to a subset $\Sigma \subset \Delta_0^+$ as above. If V is a finite dimensional irreducible representation space of $\mathfrak{p}$, then $\mathfrak{n}$ acts by nilpotent endomorphisms by the Engel's theorem, and so $\mathfrak{n}$ acts trivially. The reductive part $\mathfrak{l}$ decomposes into the semisimple factor $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ and the center $\mathfrak{z}$. We can always arrange $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{s}) \oplus \mathfrak{z}$. An irreducible representation of $\mathfrak{p}$ is determined by a dominant weight for $\mathfrak{s}$ and an element from $\mathfrak{z}^*$ and so the representation is specified by a weight λ for $\mathfrak{g}$ such that $\lambda(H_\alpha)$ is a non-negative integer for all $\alpha \in \Sigma$. Such a weight is called *dominant for* $\mathfrak{p}$. We shall denote by V_λ the irreducible $\mathfrak{p}$ module with highest weight λ. More precisely, λ decomposes into a dominant weight $\lambda_\mathfrak{s}$ for $\mathfrak{s}$ and an element from $\mathfrak{z}^*$, in the conformal case $\mathfrak{z}$ is one-dimensional and the negative of the latter element in $\mathfrak{z}^*$ is just the conformal weight, cf. 6.3. We shall describe how to get the proper coefficients in the examples below.

Notation. We shall express the representation determined by a dominant weight λ for $\mathfrak{p}$ by inscribing the values $(\lambda + \delta)(H_\alpha)$ on the fundamental coroots over the corresponding nodes.

10.14. Examples. Let us specify some important bundles in the conformal case. So we consider $\mathfrak{g} = \mathfrak{o}(m+2, \mathbb{C})$ and the Poincaré conformal (parabolic) subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Using the lists from 10.10 we can compute the values $\lambda(H_\alpha)$ for each highest weight λ. More explicitly, the first and the second coroots are $e_1 - e_2$ and $e_2 - e_3$, the last one $e_{l-1} + e_l$, in all dimensions $m \geq 4$. The conformal weight, as defined in 6.3 is determined by the coefficient a_1 at e^1 in the expression of λ as a sum of simple roots, see the explicit decomposition of $\mathfrak{o}(m+2, \mathbb{C})$ in 5.9 and notice the $-a$ entry in the first row corresponding to the multiple $a\mathbb{I}_m$ in the center. In order to get the coefficient over the omitted node, we have first to find the coefficient at e^1 in the combination of the fundamental weights indicated over the other nodes, to subtract this coefficient from the intended conformal weight and to place the result over the crossed node. The rest of the coefficients corresponds to the highest weight of the underlying representation of $\mathfrak{o}(m, \mathbb{C})$.

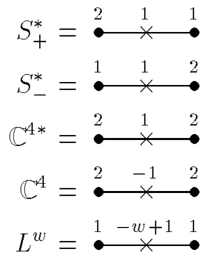
For example, we can write down the basic spin representations, the tangent space $\mathbb{C}^m$, the cotangent space $\mathbb{C}^{m*}$ and the conformal scalar densities L^w:





in even dimension

In the dimension $m + 2 = 6$, the diagrams are different:



Let us remember that the coefficients over the nodes are precisely the coefficients at the fundamental dominant forms in the expression of the weight, but these are the (possibly not dominant) weights of the whole algebra $\mathfrak{g}$. We know only that they are dominant for $\mathfrak{p}$. The usual 'raising and lowering of indices' effects the conformal weight only. With the spin representations, we increase the coefficient over the crossed node by one for each lowering of one subscript. In general, a spinor field $s^{(A'_1 \dots A'_p)(A_1 \dots A_r)}$ with r symmetric primed superscripts and p unprimed ones with conformal weight q is a section of the bundle corresponding to $\overset{p+1}{\bullet}\!\!-\!\!\!-\!\!\overset{q+1}{\times}\!\!-\!\!\!-\!\!\overset{r+1}{\bullet}$ (the weight is $\frac{1}{2}(p + r) + q$ if all indices are down).[30] The same diagrams are used also for the bundles corresponding to the dual (i.e. contragredient) representations. This strange notational convention is reasonable for the description of the operators since the corresponding morphism appear between modules corresponding to the dual representations.

Sometimes the notation $\mathcal{O}^{(A'_1 \dots A'_p)(A_1 \dots A_r)}[q]$ for the sheaf of all sections of the latter bundle is also used for the bundle. Lowering of all indices effects the weight, so that the same diagram can denote $\mathcal{O}_{(A'_1 \dots A'_p)(A_1 \dots A_r)}[p + q + r]$. For example the tangent bundle $TM \sim \mathcal{O}^{AA'} \sim \overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{-1}{\times}\!\!-\!\!\!-\!\!\overset{2}{\bullet}$ while $\Omega^1 \sim \mathcal{O}_{AA'}$. Some further

[30]The convention for the usage of primed and unprimed indices varies by different authors, we use that one from [Baston 90].

important bundles on four-dimensional manifolds are expressed below

$$\begin{aligned}
\Omega^2 &= \mathcal{O}_{(A'B')}[-1] \oplus \mathcal{O}_{(AB)}[-1] &&= \overset{3}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{1}{\bullet} \oplus \overset{1}{\bullet}\!-\!\overset{-2}{\times}\!-\!\overset{3}{\bullet}\\
\Omega^3 &= \mathcal{O}_{AA'}[-2] &&= \overset{2}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{2}{\bullet}\\
\Omega^4 &= \mathcal{O}[-4] &&= \overset{1}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{1}{\bullet}\\
S^2T^* &= \mathcal{O}_{(AB)(A'B')} \oplus \mathcal{O}[-2] &&= \overset{3}{\bullet}\!-\!\overset{-3}{\times}\!-\!\overset{3}{\bullet} \oplus \overset{1}{\bullet}\!-\!\overset{-1}{\times}\!-\!\overset{1}{\bullet}
\end{aligned}$$

10.15. The directed graph structure on the Weyl group. The number of positive roots in Δ which are transformed to negative ones by an element $S \in W$ is called the *length of* S, we write $|S|$. Equivalently, the length of S is the minimal number of the reflections corresponding to simple roots the composition of which gives S. We define the sign of S as $\operatorname{sgn} S = (-1)^{|S|}$.

We connect two elements w, w' in the Weyl group W of some complex semisimple algebra $\mathfrak{g}$ by an arrow, $w \to w'$, if $w' = S_\alpha(w)$ for some root $\alpha \in \Delta$ of $\mathfrak{g}$ and $|w'| = |w| + 1$. This directed graph structure defines a partial order on W, $w \le w'$ if there is a directed path from w to w' or $w = w'$.[31] The whole Weyl group is generated by the reflections corresponding to the simple roots. If a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ corresponding to $\Sigma \subset \Delta_0^+$ is fixed, then there is the parabolic subgroup $W_\mathfrak{p} \subset W$ generated by the simple reflections S_α, $\alpha \in \Sigma$. We define $W^\mathfrak{p} = \{w \in W;\ |S_\alpha w| = |w| + 1 \text{ for all } \alpha \in \Sigma\}$. Equivalently, $W^\mathfrak{p}$ consists of elements $w \in W$ with the property that if $w^{-1}\alpha \in -\Delta^+$ and $\alpha \in \Delta^+$, then α belongs to the span of $\Delta_0^+ \setminus \Sigma$. Thus, $W^\mathfrak{p}$ consist just of elements from W whose reflections send weights dominant for $\mathfrak{g}$ into weights dominant for $\mathfrak{p}$.

It is possible to prove that each $w \in W$ admits a unique decomposition as $w = w^\mathfrak{p} w_\mathfrak{p}$, with $w^\mathfrak{p} \in W^\mathfrak{p}$, $w_\mathfrak{p} \in W_\mathfrak{p}$, and $|w| = |w^\mathfrak{p}| + |w_\mathfrak{p}|$.

By the definition, there is the subgraph structure on $W^\mathfrak{p}$ and one can prove that for each $w' \in W^\mathfrak{p}$ different from the identity, there is some $w \in W^\mathfrak{p}$ with $w \to w'$. These subgraphs are described explicitly for the conformal Poincaré subalgebras $\mathfrak{b} \subset \mathfrak{g} = \mathfrak{o}(m+2, \mathbb{C})$ in 8.7.

10.16. The enveloping algebra. For every finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, its *universal enveloping algebra* $\mathfrak{U}(\mathfrak{g})$ is defined as the quotient $T(\mathfrak{g})/I$ of the (real or complex) tensor algebra generated by the elements of $\mathfrak{g}$ with respect to the two-sided ideal I in $T(\mathfrak{g})$ generated by all $x \otimes y - y \otimes x - [x, y]$ for x, $y \in \mathfrak{g}$. There is the induced increasing filtration $\mathfrak{U}^k(\mathfrak{g})$ from that on $T(\mathfrak{g})$ and the inclusion $i \colon \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$. We have $i([x, y]) = i(x)i(y) - i(y)i(x)$ for all x, $y \in \mathfrak{g}$ and $\mathfrak{U}(\mathfrak{g})$ has the following universal property:

For each associative algebra A over $\mathbb{K}$ with identity and each linear mapping $\varphi \colon \mathfrak{g} \to A$ satisfying $\varphi([x, y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$ for all x, $y \in \mathfrak{g}$, there is a unique algebra homomorphism $\bar\varphi \colon \mathfrak{U}(\mathfrak{g}) \to A$ such that $\bar\varphi \circ i = \varphi$ and $\bar\varphi(1) = 1$.

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[31] This graph structure is defined in the same way on much more general groups, the so called Coxeter groups, which are generated by a (finite) set of idempotents S_α like the Weyl groups. The strong partial order defined above is called the Bruhat order. The parabolic subgroups and subgraphs are also defined in the same way using the subsets of the generators. A detailed treatment can be found in [Hiller, 82].

According to the Birkhoff-Witt theorem, the canonical inclusion i extends to vector space isomorphisms $\sum_0^k S^k(\mathfrak{g}) = \mathfrak{U}^k(\mathfrak{g})$. These isomorphisms build an algebra isomorphism $S(\mathfrak{g}) = \sum_k S^k(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g})$ if and only if $\mathfrak{g}$ is abelian.

As a consequence of the Birkhoff-Witt theorem we get some canonical identifications. Given a vector space basis x_i of $\mathfrak{g}$, the vector space $\mathfrak{U}^k(\mathfrak{g})$ is generated by the expressions $x_{i_1} \dots x_{i_l}$, $i_1 \le i_2 \le \dots \le i_l$, $l \le k$. If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of vector spaces, then $\mathfrak{U}(\mathfrak{g}) = U(\mathfrak{a})U(\mathfrak{b}) = U(\mathfrak{a}) \otimes U(\mathfrak{b})$ where $U(\mathfrak{a})$ means the linear span of the elements $x_1 \dots x_l$ with $x_i \in \mathfrak{a}$ and similarly for $U(\mathfrak{b})$.

The real universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of a Lie algebra of a connected Lie group G is isomorphic to the algebra of left invariant vector fields (or right invariant vector fields) on G, i.e. to the algebra of left-invariant (or right-invariant) differential operators on the smooth functions on G.

The adjoint representation $\mathrm{ad}_x \colon \mathfrak{g} \to \mathfrak{g}$, $x \in \mathfrak{g}$ extends into a derivation on $\mathfrak{U}(\mathfrak{g})$. If $\mathfrak{g}$ is semisimple, then this representation is completely reducible. The subset $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$ of elements y with $\mathrm{ad}_x(y) = 0$ for all $x \in \mathfrak{g}$ is called the center of $\mathfrak{U}(\mathfrak{g})$. This is equivalent to the usual requirement that y commutes with all elements in $\mathfrak{U}(\mathfrak{g})$.

10.17. $\mathfrak{U}(\mathfrak{g})$-modules. Given a representation of a complex Lie algebra $\mathfrak{g}$, i.e. an algebra homomorphism $\varphi \colon \mathfrak{g} \to \mathrm{End} V$ for some complex vector space V, there is the uniquely defined algebra homomorphism $\bar{\varphi} \colon \mathfrak{U}(\mathfrak{g}) \to \mathrm{End} V$. If the representation is irreducible, then the actions of the elements from the center $\mathfrak{Z}(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$ of the complex algebra must be multiplications by scalars. This can be viewed as an algebra homomorphism $\xi \colon \mathfrak{Z}(\mathfrak{g}) \to \mathbb{C}$, the so called *infinitesimal character* of the representation φ.

Suppose now, we have two irreducible representation V_λ, V_ρ corresponding to two dominant weights λ and ρ for a semisimple complex Lie algebra $\mathfrak{g}$ and an intertwining linear mapping $D \colon V_\lambda \to V_\rho$, i.e. a $\mathfrak{U}(\mathfrak{g})$-module homomorphism. Let us write ξ_λ and ξ_ρ for the infinitesimal characters of V_λ and V_ρ. For every $v \in V_\lambda$, $z \in \mathfrak{Z}(\mathfrak{g})$ we have $zD(v) = D(zv) = D(\xi_\lambda(z)v) = \xi_\lambda(z)D(v)$ and so either $\xi_\lambda = \xi_\rho$ or $D = 0$. The same conclusion is true if both representations are generated by a single highest weight vector.

10.18. Verma modules. Let us consider first an arbitrary complex Lie algebra $\mathfrak{g}$ and its subalgebra $\mathfrak{p}$. Given a representation of $\mathfrak{p}$ in a finite dimensional vector space V, we define the *induced representation*

$$\mathrm{Ind}(\mathfrak{g}, V) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} V.$$

The representation space V is canonically embedded into the induced representation $\mathrm{Ind}(\mathfrak{g}, V)$ via $V \mapsto 1 \otimes_{\mathbb{C}} V \simeq \mathfrak{U}(\mathfrak{p}) \otimes_{\mathfrak{U}(\mathfrak{p})} V$.

In particular, if $\mathfrak{g}$ is semisimple, $\mathfrak{p}$ is a Borel subalgebra and if we consider the one-dimensional characters λ of the Borel subalgebra $\mathfrak{p}$, then the induced representations are called the *Verma modules* and denoted by M_λ (sometimes a shift in the weight is used in the notation for symmetry reasons: $\lambda - \delta$ instead of λ, δ being the lowest form). They always have the highest weight vector $1 \otimes 1$ which generates the whole $\mathfrak{U}(\mathfrak{g})$-module M_λ. The theory of Verma modules is well developed, in particular there is a complete classification of their homomorphisms.

In general, it is difficult to work with the induced representations since the structure of $\mathfrak{U}(\mathfrak{g})$ is complicated. However, if $\mathfrak{g}$ is semisimple and $\mathfrak{p}$ parabolic, the whole situation is much more similar to the theory of Verma modules. Let us recall $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{n}^-$ as a vector space direct sum of Lie subalgebras. Thus, given a finite dimensional representation of $\mathfrak{p}$ in V, we have $\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{p})} V \simeq \mathfrak{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} V$ (as vector spaces) by virtue of the Birkhoff-Witt theorem. We shall denote this $\mathfrak{U}(\mathfrak{g})$-module $M_{\mathfrak{p}}(V)$ and call it the *generalized Verma module*. If the representation is irreducible and corresponds to a dominant form λ for $\mathfrak{p}$, then the $\mathfrak{U}(\mathfrak{g})$-module $M_{\mathfrak{p}}(V_\lambda)$ is generated by the highest weight vector $1 \otimes v$ where v is the highest weight vector in V_λ.

In particular, if the subalgebra $\mathfrak{n}^-$ is abelian, then $\mathfrak{U}(\mathfrak{n}^-) = S(\mathfrak{n}^-)$, the symmetric algebra and the latter is equal to the algebra $S((\mathfrak{n}^-)^*)$ of polynomials on $\mathfrak{n}^-$. In the conformal case we deal with the Poincaré conformal parabolic subalgebra $\mathfrak{b} \subset \mathfrak{o}(m+2, \mathbb{C})$ and $\mathfrak{n}^- = \mathbb{C}^m$, the 'subalgebra of translations' which is abelian, cf. 5.9.

10.19. Homomorphisms of Verma modules. Consider dominant weights λ and ρ for complex parabolic $\mathfrak{p} \subset \mathfrak{g}$ and a homomorphism $D\colon M_{\mathfrak{p}}(V_\lambda) \to M_{\mathfrak{p}}(V_\rho)$ of $\mathfrak{U}(\mathfrak{g})$-modules. The whole modules are generated by the highest weight vectors $1 \otimes v_\lambda$ and $1 \otimes v_\rho$. Each element $z \in \mathfrak{Z}(\mathfrak{g})$ from the center must preserve the highest weight vectors and acts by scalar multiplication by $\xi_\lambda(z)$ and $\xi_\rho(z)$, the infinitesimal characters of the representations. Hence a non-zero morphism can exist only if the infinitesimal characters coincide, cf. 10.17. A classical theorem by Harish-Chandra states that $\xi_\lambda = \xi_\rho$ if and only if $\lambda + \delta$ and $\rho + \delta$ are conjugate under the action of the Weyl group W of $\mathfrak{g}$, here δ is the lowest form (half the sum of all positive roots). The *affine action of* W on the weights λ is defined for each $w \in W$ by $w.\lambda = w(\lambda + \delta) - \delta$. Thus, the above mentioned condition states: *If there is a non-zero* $\mathfrak{U}(\mathfrak{g})$*-module homomorphism* $M_{\mathfrak{p}}(V_\lambda) \to M_{\mathfrak{p}}(V_\rho)$ *then there is some* w *in the Weyl group of* $\mathfrak{g}$ *such that* $w.\lambda = \rho$.

If λ is dominant for $\mathfrak{g}$, then all weights ρ dominant for $\mathfrak{p}$ with the same infinitesimal character $\xi_\lambda = \xi_\rho$ are given by $\{w.\lambda \; ; \; w \in W^{\mathfrak{p}}\}$.

10.20. Action of the Weyl group on weights. Let us recall that a weight is denoted by inscribing its values on fundamental coroots over the corresponding nodes in the Dynkin diagram increased by 1. The action of the simple reflections on the weights can be described as follows, cf. [Baston, 90]. For each root $\alpha \in \Delta$, the reflection S_α acts on the weight λ by $\mathcal{S}_\alpha(\lambda) = \lambda - \langle \lambda, H_\alpha \rangle \alpha$ where H_α is the coroot corresponding to α. Hence the coefficients over the nodes are given by $\langle S_\alpha(\lambda), H_i \rangle + 1 = \langle \lambda, H_i \rangle - \langle \lambda, H_\alpha \rangle \langle \alpha, H_i \rangle + 1$ where H_i are the simple coroots. If α is a simple root, then $\langle \alpha, H_i \rangle$ is the Cartan integer which is obtainable directly form the Dynkin diagram. This yields the procedure for getting the new coefficients over the nodes after the affine action of a simple reflection:

Let a *be the coefficient of the* i*-th node corresponding to* λ*. In order to get the coefficients over the nodes corresponding to* $S_{\alpha_i}(\lambda + \delta)$*, add* a *to the adjacent coefficients, with multiplicity if there is a multiple edge directed towards the adjacent node, and replace* a *by* $-a$*.*

For example, if λ is $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}$ and we act by the middle simple reflection, we get the weight $\overset{a+b}{\bullet}\!\!-\!\!\!-\!\!\overset{-b}{\bullet}\!\!-\!\!\!-\!\!\overset{b+c}{\bullet}$. Similarly $\overset{a}{\bullet}\!\Rightarrow\!\overset{b}{\bullet}$ transforms under the action of the first simple reflection into $\overset{-a\quad 2a+b}{\bullet\!\Rightarrow\!\bullet}$, while the second simple reflection yields $\overset{a+b\quad -b}{\bullet\!\Rightarrow\!\bullet}$.

10.21. The Lie algebra cohomologies. Consider an arbitrary Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module A. The degree q cochains with coefficients in A are defined as the space $C^q(\mathfrak{g}; A)$ of all (continuous) skew-symmetric q-linear A-valued forms on $\mathfrak{g}$. By the definition, $C^q(\mathfrak{g}, A) = \operatorname{Hom}(\Lambda^q(\mathfrak{g}); A)$ carries a natural $\mathfrak{g}$-module structure. We define the differential $\partial\colon C^q(\mathfrak{g}; A) \to C^{q+1}(\mathfrak{g}; A)$ by the formula

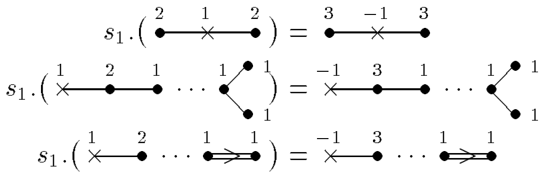
(1)
$$\begin{aligned}\partial c(X_1, \dots, X_{q+1}) = &\sum_{1\le s<t\le q+1} (-1)^{s+t-1} c([X_s, X_t], X_1, \dots \hat{s} \dots \hat{t} \dots, X_{q+1})\\ &+ \sum_{1\le s\le q+1} (-1)^s X_s.c(X_1, \dots \hat{s} \dots X_{q+1})\end{aligned}$$

One verifies easily $\partial^2 = 0$ and we obtain a complex by setting $C^q(\mathfrak{g}; A) = 0$ and $\partial(C^q(\mathfrak{g}; A)) = 0$ if $q < 0$. This complex is denoted by $C^*(\mathfrak{g}; A)$ and the corresponding cohomologies are denoted by $H^q(\mathfrak{g}; A)$ and called the cohomologies of $\mathfrak{g}$ with coefficients in A.

We need this general definition in a special case. Let us consider an algebra with grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots$. Then $\mathfrak{g}_{-1}$ is an abelian Lie subalgebra and $\mathfrak{g}_0$ is a Lie subalgebra acting on all homogeneous components $\mathfrak{g}_p$ turning them into $\mathfrak{g}_0$-modules. The whole $\mathfrak{g}$ is a $\mathfrak{g}_{-1}$-module via the adjoint action. The Lie algebra cohomology $H^*(\mathfrak{g}_{-1}; \mathfrak{g})$ is called the Spencer cohomology. The grading of $\mathfrak{g}$ induces a natural grading on the cochains, $C^*(\mathfrak{g}_{-1}; \mathfrak{g}) = \sum_{p,q} C^{p,q}(\mathfrak{g}_{-1}; \mathfrak{g})$ where $C^{p,q}(\mathfrak{g}_{-1}; \mathfrak{g}) \subset C^q(\mathfrak{g}_{-1}; \mathfrak{g})$ is the subset of $\mathfrak{g}_{p-1}$-valued forms. Since the Lie algebra $\mathfrak{g}_{-1}$ is abelian, only the second term remains in (1) and we get a differential $\partial C^{p,q}(\mathfrak{g}_{-1}; \mathfrak{g}) \to C^{p-1,q+1}(\mathfrak{g}_{-1}; \mathfrak{g})$. The *Spencer bigraded cohomology* $H^{p,q}(\mathfrak{g}_{-1}; \mathfrak{g})$ is the cohomology of this complex, $H^{p,q}(\mathfrak{g}_{-1}; \mathfrak{g}) := \partial^{-1}(0) \cap C^{p,q}(\mathfrak{g}_{-1}, \mathfrak{g})/\partial(C^{p+1,q-1}(\mathfrak{g}_{-1}; \mathfrak{g}))$. The action of $\mathfrak{g}_0$ on the homogeneous components induces an action on the cochains which intertwines the differential and so there is a distinguished $\mathfrak{g}_0$-module structure on $H^{*,*}(\mathfrak{g}_{-1}; \mathfrak{g})$.

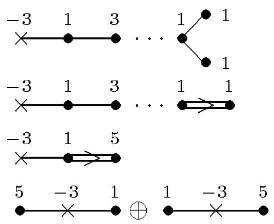
In the main text, we need the conformal case where $\mathfrak{g} = \mathfrak{b}_{-1} \oplus \mathfrak{b}_0 \oplus \mathfrak{b}_1 = \mathfrak{o}(m+2, \mathbb{C})$, $\mathfrak{b}_0$ is the reductive part of the parabolic subalgebra $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{b}_1$. All irreducible representations of $\mathfrak{b}_0 = \mathfrak{co}(m, \mathbb{C})$ in $H^*(\mathfrak{b}_{-1}; \mathfrak{g})$ can be established by the Kostant's theory (developed for general parabolic subalgebras in complex reductive algebras), see [Vogan, 81, p. 123]: *If A is a finite dimensional $\mathfrak{b}$-module of highest weight λ, then the irreducible finite dimensional representations of $\mathfrak{g}_0$ with highest weight μ occur in $H^*(\mathfrak{b}_{-1}; A)$ if and only if there is a $w \in W^{\mathfrak{b}} \subset W$ such that $\mu = w.\lambda = w(\lambda+\delta) - \delta$ and in that case it occurs in degree $|w|$ with multiplicity one*, (see 10.15 and 10.19 for the notation).

In our situation, λ is the maximal root ($e^1 + e^2$, see 10.10) and the affine action of $W^{\mathfrak{b}}$ is described in 10.20. In particular, if we want to compute $H^{*,1}(\mathfrak{b}_{-1}; \mathfrak{g})$, we have to evaluate the affine action of s_1 if $m \ge 4$ (this is the only elements of length one in $W^{\mathfrak{b}}$, see 10.15 and 8.7)



Since $H^{0,1}(\mathfrak{b}_{-1};\mathfrak{g}) = \mathfrak{b}_{-1} \otimes (\mathfrak{b}_{-1})^*/\mathfrak{b}_0$ by the definition, this cohomology must be non-zero. Since there is only one irreducible representation available, the other two first order cohomologies must be zero. Hence $H^{1,1}(\mathfrak{b}_{-1};\mathfrak{g}) = H^{2,1}(\mathfrak{b}_{-1};\mathfrak{g}) = 0$, see also [Baston, 90] or [Ochiai, 70].

Similarly, we can compute the second cohomologies. In dimensions $m > 4$ we have to compute $(s_1 s_2).\lambda$, in dimension $m = 4$, the second cohomologies have two summands, $(s_1 s_2).\lambda$ and $(s_1 s_3).\lambda$. We get the representations



$m = 2n > 4$

$m = 2n + 1 > 5$

$m = 5$

$m = 4$

The conformal weights show that all these representations must occur in the cohomology space $H^{1,2}(\mathfrak{b}_{-1};\mathfrak{g})$ and so $H^{0,2}(\mathfrak{b}_{-1};\mathfrak{g}) = H^{2,2}(\mathfrak{b}_{-1};\mathfrak{g}) = 0$.

The cohomologies of the complexified algebras $\mathfrak{g}^{\mathbb{C}}$ are the complexifications of the real cohomologies. Hence the vanishing of the above cohomology spaces in the complex case implies the vanishing of the same ones for the real conformal case as well.

10.22. The Hodge theory. Given a general Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module, the chains $C_q(\mathfrak{g}; A)$ are defined as the space $A \otimes \Lambda^q \mathfrak{g}$ and the differential is defined by $\partial(a \otimes (X_1 \wedge \dots \wedge X_q)) = \sum_{1 \le s < t \le q} (-1)^{s+t-1} a \otimes ([X_s, X_t] \wedge X_1 \wedge \dots \hat{s} \dots \hat{t} \dots \wedge X_q) + \sum_{1 \le s < t \le q} (-1)^s X_s . a \otimes (X_1 \wedge \dots \hat{s} \dots \wedge X_q)$. Since $\partial^2 = 0$ we obtain the homology $H_q(\mathfrak{g}; A)$. If both the algebra $\mathfrak{g}$ and the $\mathfrak{g}$-module A are finite dimensional, then $H^q(\mathfrak{g}; A^*) = (H_q(\mathfrak{g}; A))^*$. Let us assume that $\mathfrak{g}$ and A are moreover graded and that there is a distinguished Hermitian metric in each homogeneous component $\mathfrak{g}_q$. Then we can identify the cochains with their duals, i.e. $C^q(\mathfrak{g}; A) \simeq C_q(\mathfrak{g}; A)$ and the differential on the chains gives rise to $\partial^* : C^q(\mathfrak{g}; A) \to C^{q+1}(\mathfrak{g}, A)$. The operator $\Box = \partial^* \partial + \partial \partial^* : C^q(\mathfrak{g}; A) \to C^q(\mathfrak{g}; A)$ is called the *Laplace operator*. The cochains with $\Box(c) = 0$ are called *harmonic*.

In the conformal case we can express the adjoint differential using arbitrary basis x_i of $\mathfrak{b}_{-1}$ and the dual bases y_i of $\mathfrak{b}_1$ ($\mathfrak{b}_1$ is dual to $\mathfrak{b}_{-1}$ with the contragredient representation of $\mathfrak{b}_0$, see 5.9)

$$\partial^* c(X_1, \dots, X_{q-1}) = \sum_{j=1}^{m} [y_j, c(x_j, X_1, \dots, X_{q-1})]$$

which is a linear mapping $C^{p,q}(\mathfrak{b}_{-1};\mathfrak{g}) \to C^{p+1,q-1}(\mathfrak{b}_{-1};\mathfrak{g})$. In each cohomology class of $H^{p,q}(\mathfrak{b}_{-1};\mathfrak{g})$ there is a unique harmonic representative $f \in C^{p,q}(\mathfrak{b}_{-1};\mathfrak{g})$. The Laplace operator acts by scalar multiplication on irreducible representations of $\mathfrak{b}_0$ occurring in $H^{*,*}(\mathfrak{b}_{-1};\mathfrak{g})$. More explicitly, if the irreducible representation has the highest weight μ then $\square$ acts by

$$\frac{1}{2}(\langle\Lambda+\delta,\Lambda+\delta\rangle - \langle\mu+\delta,\mu+\delta\rangle)$$

where Λ is the maximal root of $\mathfrak{g}$ and δ is the lowest form.

References

Atiyah, M.; Bott, R.; Patodi, V.K., *On the heat equation and the index theorem*, Inventiones Math. **19** (1973), 279–330.

Atiyah, M.; Bott, R.; Shapiro, A., *Clifford modules*, Topology **3** (1964), 3–38.

Bailey, T. N,; Eastwood, M. G.; Graham, C. R., *Invariant theory for conformal and CR geometry*, preprint 1992.

Baston, R. J., *Verma modules and differential conformal invariants*, J. Differential Geometry **32** (1990), 851–898.

Baston, R. J., *Almost Hermitian symmetric manifolds, I: Local twistor theory; II: Differential invariants*, Preprints (1990).

Baston, R.J.; Eastwood, M.G., *The Penrose Transform, Its Interaction with Representation Theory*, Clarenden press, Oxford, 1989.

Baston, R.J.; Eastwood, M.G., *Invariant operators*, Twistors in mathematics and physics, Lecture Notes in Mathematics 156, Cambridge University Press, 1990.

Bernstein, I. N.; Gelfand, I. M.; Gelfand, S. I., *Structure of representations generated by vectors of highest weight*, Funct. Anal. Appl. **5** (1971), 1–8.

Boe, B. D.; Collingwood, D. H., *A comparison theory for the structure of induced representations I.*, J. of Algebra **94** (1985), 511-545.

Boe, B. D.; Collingwood, D. H., *A comparison theory for the structure of induced representations II.*, Math. Z. **190** (1985), 1-11.

Boerner, H., *Darstellungen von Gruppen*, 2nd ed., Springer, Grundlehren der math. Wissenschaften 74, Berlin Heidelberg New York, 1967.

Branson, T. P., *Differential operators canonically associated to a conformal structure*, Math. Scand. **57** (1985), 293–345.

Branson, T. P., *Conformal transformations, conformal change, and conformal covariants*, Proceedings of the Winter School on Geometry and Physics, Srni 1988, Suppl. Rendiconti Circolo Mat. Palermo, Serie II **21** (1989), 115–134.

Branson T. P., *Second-order conformal covariants I., II.*, Kobenhavns universitet matematisk institut, Preprint Series, No. 2, 3, (1989).

Budinich, P.; Trautman, A., *The spinorial chessboard*, Trieste Notes in Physics, Springer-Verlag, 1988.

Cap, A.; Slovák, J., *Infinitesimally natural operators are natural*, J. Diff. Geom. and Appl. **2** (1992), 45–55.

Eastwood, M. G., *On the weights of conformally invariant operators*, Twistor Newsl. **24** (1987), 20–23.

Eastwood, M. G.; Graham, C. R., *Invariants of CR Densities*, Proccedings of Symposia in Pure Mathematics, Part 2 **52** (1991), 117–133.

Eastwood, M. G.; Graham, C. R., *Invariants of conformal densities*, Duke Math. J. **63** (1991), 633–671.

Eastwood, M. G.; Rice, J. W., *Conformally invariant differential operators on Minkowski space and their curved analogues*, Commun. Math. Phys. **109** (1987), 207–228.

Eastwood, M. G.; Tod, P., *Edth – a differential operator on the sphere*, Math. Proc. Cambr. Phil. Soc. **92** (1982), 317–330.

Epstein, D.B.A., *Natural tensors on Riemannian manifolds*, J. Diff. Geom. **10** (1975), 631–645.

Epstein, D. B. A.; Thurston W. P., *Transformation groups and natural bundles*, Proc. London Math. Soc. **38** (1979), 219–236.

Fefferman, C., *Parabolic invariant theory in complex analysis*, Adv. Math. **31** (1979), 131–262.

Fefferman, C.; Graham, R., *Conformal invariants*, Élie Cartan et les Mathématiques d'Aujourd'hui, Astérisque, 1985, pp. 95–116.

Fegan, H. D., *Conformally invariant first order differential operators*, Quart. J. Math. **27** (1976), 371–378.

Friedrich, H., *Twistor connection and normal conformal Cartan connection*, Gen. Rel. Grav. **8** (1977), 303–312.

Gilkey, P. B., *Curvature and the eigenvalues of the Laplacian for elliptic complexes*, Advances in Math. **10** (1973), 344–382.

Gilkey, P. B., *Local invariants of a pseudo-Riemannian manifold*, Math. Scand. **36** (1975), 109–130.

Gilkey, P. B., *Invariance Theory, The Heat Equation, And the Atiyah-Singer Index Theorem*, Mathematics Lecture Series 11, Publish or Perish Inc., Wilmington, Delaware, 1984.

Gover, A. R., *Conformally invariant operators of standard type*, Quart. J. Math. **40** (1989), 197–208.

Graham, C. R., *Non-existence of curved conformally invariant operators*, Preprint (1990).

Grozman, P. Ya., *Classification of bilinear invariant operators on tensor fields*, (Russian), Funct. Anal. and its Appl. **14 No 2** (1980), 58–59.

Gurevich, G. B., *Foundations of the theory of algebraic invariants*, (Russian), OGIZ, Moscow-Leningrad, 1948.

Hiller, H, *Geometry of Coxeter groups*, Research Notes in Mathematics, Pitman, Boston, London, Melbourne, 1982.

Hitchin, N. J., *Linear field equations on self-dual spaces*, Proc. R. Soc. Lond. **A 370** (1980), 173–191.

Jacobson, N., *Lie algebras*, Interscience Publishers, New York – London, 1962.

Jakobsen, H. P., *Conformal invariants*, Publ. RIMS, Kyoto Univ **22** (1986), 345–364.

Jakobsen H. P.; Vergne, M., *Wave and Dirac operators, and representations of the conformal group*, Jour. of Funct. Anal. **24** (1977), 52–106.

Kirillov, A. A., *Introduction to representation theory*, (Russian), Nauka, Moscow, 1972.

Kirillov, A. A., *On invariant differential operations over geometric quantities*, (Russian), Func. Anal. and its Appl. **11 No 2** (1977), 39–44.

Knapp, W., *Representation theory of semisimple groups*, Princeton university press, Princeton, New Jersey, 1986.

Kobayashi, S., *Transformation groups in differential geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

Kolář, I.; Vadovičová, I., *On the structure function of a G-structure*, Math. Slovaca **35** (1985), 277–282.

Kolář, I.; Michor, P. W.; Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin Heidelberg New York, 1993.

Krupka D.; Janyška J., *Lectures on differential invariants*, Univerzita J. E. Purkyně, Brno, 1990.

Lawson, H. B.;Michelsohn M. L., *Spin geometry*, Princeton University Press, Princeton, 1989.

Lepowsky, J., *A generalization of the Bernstein-Gelfand-Gelfand resolution*, J. of Algebra **49** (1977), 496–511.

Lubczonok, G., *On the reduction theorems*, Ann. Polon. Math. **26** (1972), 125–133.

Naymark, M. A., *Group representation theory*, (Russian), Nauka, Moscow, 76.

Ochiai, T., *Geometry associated with semisimple flat homogeneous spaces*, Trans. Amer. Math. Soc. **152** (1970), 159–193.

Ørsted, B., *Conformally invariant differential equations and projective geometry*, J. Funct. Anal. **44** (1981), 1–23.

Peetre, J., *Rectifications à l'article "Une caractérisation abstraite des opérateurs différentiels"*, Math. Scand. **8** (1960), 116–120.

Penrose, R.; Rindler, W., *Spinors and Space-time, vol. I. Two-spinor calculus and relativistic fields; vol. II. Spinor and twistor methods in space-time geometry*, Cambridge University Press, 1984; 1986.
Rudakov, A. N., *Irreducible representations of infinite dimensional Lie algebras of Cartan type*, (Russian), Izv. AN SSSR, Ser. Mat. **38** (1974), 835–866.
Rudakov, A. N., *Irreducible representations of infinite dimensional Lie algebras of types S and H*, (Russian), Izv. AN SSSR, Ser. Mat. **39** (1975), 496–511.
Samelson, H., *Notes on Lie algebras*, Universitext, Springer-Verlag, 1989.
Slovák, J., *Peetre theorem for nonlinear operators*, Annals of Global Analysis and Geometry **6** (1988), 273–283.
Slovák, J., *On invariant operations on a manifold with connection or metric*, J. Diff. Geom. **36** (1992), 633–650.
Slovák, J., *On invariant opeators on pseudo-Riemannian manifolds*, Comm. Math. Univ. Carolinae **33** (1992), 269–276.
Stredder, P., *Natural differential operators on Riemannian manifolds and representations of the orthogonal and special orthogonal group*, J. Diff. Geom. **10** (1975), 647–660.
Terng C. L., *Natural vector bundles and natural differential operators*, American J. of Math. **100** (1978), 775–828.
Verma, D. N., *Structure of certain induced representations of complex semisimple Lie algebras*, Bull. Amer. Math. Soc. **74** (1968), 160–166.
Vogan, D. A. jr., *Representations of real reductive Lie groups*, Progress in Math., No. 15, Birkhäuser, Boston, 1981.
Weyl, H., *The classical groups*, Princeton University Press, Princeton, 1939.
Wünsch, V., *On conformally invariant differential operators*, Math. Nachr. **129** (1986), 269–281.
Zhelobenko, D. P., *Compact Lie groups and their representations*, (Russian), Nauka, Moscow, 1970.
Zuckerman, G., *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. Math. **106** (1977), 295–308.

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