1. Conventions

The post-composition is a map $g_*\colon \operatorname{Hom}(C,D) \to \operatorname{Hom}(C,D')$, $f \mapsto gf$. If g is a chain map of degree k, the same is true of g_*. The pre-composition is a map $h^*\colon \operatorname{Hom}(C,D) \to \operatorname{Hom}(C',D)$, $f \mapsto (-1)^{|f||h|}fh$.[1] We have the expected $(gg')_* = g_*g'_*$ and $(h'h)^* = (-1)^{|h||h'|}h^*h'^*$.

We use these maps in the case of the suspension maps $s^k\colon C \to C[k]$. Thus, we obtain isomorphisms $(s^k)_*\colon \operatorname{Hom}(C,D) \cong \operatorname{Hom}(C,D[k])$, $f \mapsto s^kf$ and $(s^{-k})^*\colon \operatorname{Hom}(C,D) \cong \operatorname{Hom}(C[k],D)$, $f \mapsto (-1)^{k|f|}fs^{-k}$. As explained, we have $(s^k)^*(s^{-k})^* = (-1)^k$, i.e. in general, the suspension isomorphism $(s^{-k})^*$ is not an inverse of the suspension isomorphism $(s^k)^*$. Thus, when passing between the two chain complexes $\operatorname{Hom}(C,D)$ and $\operatorname{Hom}(C[k],D)$, we have to choose a direction.

The general isomorphism will be $(s^l)_*(s^{-k})^*\colon \operatorname{Hom}(C,D) \cong \operatorname{Hom}(C[k],D[l])$, in this order, so that $f \mapsto (-1)^{k|f|}s^lfs^{-k}$. This turns out to be compatible with the composition (unlike the other possibility $(s^{-k})^*(s^l)_*$ with the sign $(-1)^{(k+l)|f|}$).

2. Introduction

An A_∞-algebra is a homotopy version of an associative algebra. Thus, suppose that A is a dga and that $f\colon A \to B$ is a homotopy equivalence with a homotopy inverse g. Define a multiplication on B via

$$x \cdot y = f(gx \cdot gy).$$

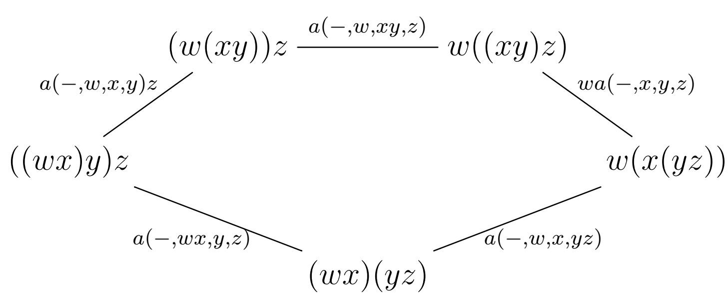
Now compare the two products

$$(x \cdot y) \cdot z = f(gf(gx \cdot gy) \cdot gz), \quad x \cdot (y \cdot z) = f(gx \cdot gf(gy \cdot gz)).$$

Both are homotopic to $f(gx \cdot gy \cdot gz)$ and in particular, the multiplication on B is associative up to homotopy. This is probably best explained in topological situation, where A is a topological space equipped with an associative operation. Denote

$$a\colon K_3 \times B^3 \to B$$

($K_3 = I$ is the unit interval) the homotopy; thus $a(-,x,y,z)$ is the corresponding path from $(xy)z$ to $x(yz)$. In fact, more is true: there are the following associativity homotopies:



[1] Think this way $R[k]\otimes\operatorname{Hom}(C,D) \xrightarrow{h\otimes 1} \operatorname{Hom}(C',C)\otimes\operatorname{Hom}(C,D) \xrightarrow{\text{twist}} \operatorname{Hom}(C,D)\otimes\operatorname{Hom}(C',C) \xrightarrow{\text{comp}} \operatorname{Hom}(C',D)$. The sign is due to the twist map.

It so happens that there is a second order homotopy

$$K_4 \times B^4 \to B$$

(K_4 is the pentagon) that fills in the above system of paths. Further, there is a third order homotopy

$$K_5 \times B^5 \to B$$

with K_5 a certain three-dimensional polytope called the associahedron whose boundary consists either of pentagons K_4 as above or squares (products $K_3 \times K_3$ of intervals):

$$\begin{array}{ccc} (((vw)x)y)z & \xrightarrow{a(-,(vw)x,y,z)} & ((vw)x)(yz) \\ {\scriptstyle (a(-,v,w,x)y)z}\Big| & {\scriptstyle a(-,a(-,v,w,x),y,z)} & \Big|{\scriptstyle a(-,v,w,x)(yz)} \\ ((v(wx))y)z & \xrightarrow[a(-,v(wx),y,z)]{} & (v(wx))(yz) \end{array}$$

This continues ad infinity.

The definition is simplified in the algebraic situation by the fact that we do not need to specify the polyhedra and their various faces since the boundary of a polyhedron is a *sum* of the faces (with correct signs). Thus, we will have

$$\begin{aligned} m_2 &\colon B \otimes B \to B \\ m_3 &\colon B \otimes B \otimes B \to B \\ m_4 &\colon B \otimes B \otimes B \otimes B \to B \end{aligned}$$

and these will satisfy

$$\begin{aligned} [d, m_2] &= 0, \\ [d, m_3] &= \pm m_2(m_2 \otimes 1) \pm m_2(1 \otimes m_2) \\ [d, m_4] &= \cdots \end{aligned}$$

3. Differential graded coalgebras

The formal definition is conveniently phrased using the language of coalgebras. A (coassociative) coalgebra is a module C together with a comultiplication $\Delta\colon C \to C \otimes C$ satisfying the following coassociativity rule:

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ {\scriptstyle \Delta}\big\downarrow & & \big\downarrow{\scriptstyle \Delta\otimes 1} \\ C \otimes C & \xrightarrow[1\otimes\Delta]{} & C \otimes C \otimes C \end{array}$$

Phrased differently, all possible iterations $\Delta^{(n)}\colon C \to C^{\otimes n}$ are equal (this corresponds to the associativity: all possible bracketings are equal).

A graded coalgebra is one in which C is a graded module and Δ preserves grading.

Example 1. Let V be a (graded) module. Define the tensor coalgebra T^cV to be

$$T^cV = \bigoplus_{n\geq 1} V^{\otimes n}$$

together with the following comultiplication:

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{0<i<n} \underbrace{(v_1 \otimes \cdots \otimes v_i)}_{\in T^cV} \otimes \underbrace{(v_{i+1} \otimes \cdots \otimes v_n)}_{\in T^cV} \in T^cV \otimes T^cV.$$

A *differential graded coalgebra (dgc)* is a chain complex and a coalgebra in such a way that the comultiplication

$$\Delta\colon C \to C \otimes C$$

is a chain map (of degree 0), i.e. such that the following square commutes:

$$\begin{CD} C @>{\Delta}>> C \otimes C \\ @V{d}VV @VV{d\otimes 1 + 1 \otimes d}V \\ C @>>{\Delta}> C \otimes C \end{CD}$$

Here $d \otimes 1 + 1 \otimes d$ is the differential on the tensor product with the usual Koszul sign convention $(f \otimes g)(x \otimes y) = (-1)^{|g|\cdot|x|} fx \otimes gy$. We call this the coLeibniz rule since it is exactly the dual of the compatibility condition between the multiplication and the differential in a differential graded algebra.

4. A_∞-algebras

An A_∞-algebra is a graded module A together with a differential graded coalgebra structure on $T^cA[1]$, where $A[1]$ denotes the suspension of A, i.e. $A[1]_n = A_{n-1}$. Since $T^cA[1]$ is already a graded coalgebra, this amounts to the specification of a differential $d\colon T^cA[1] \to T^cA[1]$ of degree -1 that satisfies the coLeibniz rule and $d \circ d = 0$.

We denote by $q\colon T^cA[1] \to A[1]$ the projection onto tensors of length 1. Then the projection onto tensors of length k is

$$T^cA[1] \xrightarrow{\Delta^{(k)}} (T^cA[1])^{\otimes k} \xrightarrow{q^{\otimes k}} A[1]^{\otimes k}.$$

The coLeibniz rule unfolds to

$$\begin{array}{ccccc} T^cA[1] & \xrightarrow{\Delta^{(k)}} & (T^cA[1])^{\otimes k} & & \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \sum_{i+1+j=k} 1^{\otimes i}\otimes d \otimes 1^{\otimes j}} & & \\ T^cA[1] & \xrightarrow[\Delta^{(k)}]{} & (T^cA[1])^{\otimes k} & \xrightarrow[q^{\otimes k}]{} & A[1]^{\otimes k}, \end{array}$$

i.e. the component of $d(sx_1 \otimes \cdots \otimes sx_n)$ of length k is

$$q^{\otimes k}\Delta^{(k)} d(sx_1 \otimes \cdots \otimes sx_n) = \sum q^{\otimes k}(1^{\otimes i} \otimes d \otimes 1^{\otimes j})\Delta^{(k)}(sx_1 \otimes \cdots \otimes sx_n)$$

The only non-zero terms come from the part of $\Delta^{(k)}$ that splits $sx_1 \otimes \cdots \otimes sx_n$ into i tensors of length one at the beginning and j tensors of length one at the end, i.e.

$$(sx_1) \otimes \cdots \otimes (sx_i) \otimes (sx_{i+1} \otimes \cdots \otimes sx_{n-j}) \otimes (sx_{n-j+1}) \otimes \cdots \otimes (sx_n)$$

Denoting the composition

$$d_\ell \colon A[1]^{\otimes \ell} \subseteq T^c A[1] \xrightarrow{d} T^c A[1] \xrightarrow{q} A[1],$$

we thus have

$$d = \sum 1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes j}.$$

This is equivalent to the coLeibniz rule. The fact that d is of degree -1 translates into all d_ℓ's being of degree -1. It remains to study the condition $d \circ d = 0$. There are various terms involved in $d \circ d$: when one composes

$$(1^{\otimes i'} \otimes d_{\ell'} \otimes 1^{\otimes j'}) \circ (1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes j})$$

and $j < j'$ then this equals $1^{\otimes i'} \otimes d_{\ell'} \otimes 1^{\otimes m} \otimes d_\ell \otimes 1^{\otimes j}$ while for $i < i'$, the result will be $-1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes m} \otimes d_{\ell'} \otimes 1^{\otimes j'}$ – the minus sign comes from the Koszul sign $(f \otimes g)(h \otimes k) = (-1)^{|g| \cdot |h|} fh \otimes gk$ that comes from swapping the $d_{\ell'}$ and d_ℓ. Thus, such terms cancel out. The remaining terms look like $1^{\otimes i'} \otimes d_{\ell'}(1^{\otimes i''} \otimes d_\ell \otimes 1^{\otimes j''}) \otimes 1^{\otimes j'}$ (where $i'' = i - i'$ and $j'' = j - j'$) and thus, the square zero condition only needs to be verified in the case $i' = j' = 0$ and for the number of inputs $i + \ell + j$ fixed and reads

$$\sum_{i+\ell+j=n} d_{i+1+j}(1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes j}) = 0.$$

Before working out small dimensions concretely, we will use the following diagram to translate d_ℓ's into more traditional operations

$$\begin{array}{ccc} A^{\otimes \ell} & \xrightarrow{m_\ell} & A \\ {\scriptstyle s^{\otimes \ell}} \downarrow \cong & (-1)^\ell & \cong \downarrow {\scriptstyle s} \\ A[1]^{\otimes \ell} & \xrightarrow[d_\ell]{} & A[1] \end{array}$$

i.e. we use[2] $m_\ell = (-1)^\ell s^{-1} d_\ell s^{\otimes \ell}$. Since s is an isomorphism of degree 1, the operation m_ℓ has degree $\ell - 2$. One then obtains from the formula for the d_ℓ's the corresponding formula

[2]The m_ℓ is the image of d_ℓ under the isomorphism $(s^{-1})_*(s^{\otimes \ell})^* \colon \operatorname{Hom}(A[1]^{\otimes \ell}, A[1]) \cong \operatorname{Hom}(A^{\otimes \ell}, A)$.

for the m_ℓ's:

$$\begin{aligned}
0 &= \sum s^{-1} d_{i+1+j}(1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes j}) s^{\otimes(i+\ell+j)} \\
&= \sum (-1)^i \cdot s^{-1} d_{i+1+j}(s^{\otimes i} \otimes d_\ell s^{\otimes \ell} \otimes s^{\otimes j}) \\
&= \sum (-1)^{i+\ell} \cdot s^{-1} d_{i+1+j}(s^{\otimes i} \otimes s m_\ell \otimes s^{\otimes j}) \\
&= \sum (-1)^{i+\ell+j(\ell-2)} \cdot s^{-1} d_{i+1+j} s^{\otimes(i+1+j)}(1^{\otimes i} \otimes m_\ell \otimes 1^{\otimes j}) \\
&= \sum (-1)^{i+\ell+j\ell+i+1+j} \cdot m_{i+1+j}(1^{\otimes i} \otimes m_\ell \otimes 1^{\otimes j}).
\end{aligned}$$

Since $i+\ell+j+1 = n+1$ is fixed, we may finally write the condition as

$$\sum_{i+\ell+j=n} (-1)^{i+j\ell} \cdot m_{i+1+j}(1^{\otimes i} \otimes m_\ell \otimes 1^{\otimes j}) = 0$$

The square zero condition for $n = 1$ then reads:

$$m_1 m_1 = 0,$$

thus m_1 is a differential; we will denote it by ∂. The square zero condition for $n = 2$ reads:

$$m_1 m_2 - m_2(m_1 \otimes 1) - m_2(1 \otimes m_1) = [\partial, m_2] = 0,$$

i.e. m_2 is a chain map. The square zero condition for $n = 3$ reads:

$$[\partial, m_3] + m_2(m_2 \otimes 1 - 1 \otimes m_2) = 0,$$

i.e. m_3 is a homotopy between $m_2(m_2 \otimes 1)$ and $m_2(1 \otimes m_2)$. When $\ell = 1$ the sign equals $(-1)^{i+j\ell} = (-1)^{i+j} = (-1)^{n-1}$ and thus, we always obtain terms

$$m_1 m_n - (-1)^n \sum_{i+1+j=n} m_n(1^{\otimes i} \otimes m_1 \otimes 1^{\otimes j}) = [\partial, m_n].$$

Consequently, the relation exhibits m_n as a null-homotopy of the remaining terms – these contain only m_ℓ's for $\ell < n$ and constitute the boundary of the polyhedron from the motivation. This easily implies the following proposition

Proposition 2. *Every dga can be thought of as an A_∞-algebra by setting m_1 to be the differential, m_2 to be the algebra multiplication and $m_\ell = 0$ for $\ell > 2$.*

Proof. The three special cases $n = 1, 2, 3$ above are clearly satisfied and the higher ones have either $\ell > 2$ or $i+1+j > 2$. Thus, they are satisfied trivially. □

There is a way of "truncating" an A_∞-algebra A to a dga, essentially by killing all the higher operations m_ℓ, $\ell > 2$. As in universal algebra, we may generate from the m_ℓ's operations by composition, i.e. we consider terms involving the m_ℓ's. Now consider the collection of terms that involve at least one m_ℓ, $\ell > 2$, and form a graded submodule $I \subseteq A$ generated by the images of such operations. This contains for example the following $m_2(w \otimes m_1 m_3(x \otimes y \otimes m_1 z))$. In particular, it is closed under $m_1 = \partial$ and thus forms a subcomplex. Form the quotient A/I; this possesses a multiplication induced from m_2 and since $m_2(1 \otimes m_2) - m_2(m_2 \otimes 1) = [\partial, m_3] \in I$, this multiplication is indeed associative.

5. A_∞-coalgebras

Almost dually, we define an A_∞-coalgebra C to be a graded module together with a structure of a differential graded algebra on the tensor algebra $TC[-1]$ on the desuspension of C. Again, this translates into $d\colon TC[-1] \to TC[-1]$ being a derivation (which amounts to the Leibniz rule) and $d \circ d = 0$. The differential is uniquely determined by its components

$$d_\ell\colon C[-1] \subseteq TC[-1] \xrightarrow{d} TC[-1] \twoheadrightarrow C[-1]^{\otimes \ell}$$

or alternatively $w_\ell\colon C \to C^{\otimes \ell}$ of degree $\ell - 2$, given by

$$w_\ell = -s^{\otimes \ell} d_\ell s^{-1}.$$

The square zero condition then becomes

$$\sum_{i+\ell+j=n} (-1)^{j+i\ell} \cdot (1^{\otimes i} \otimes w_\ell \otimes 1^{\otimes j}) w_{i+1+j} = 0$$

(As it stands, for each $c \in C$, only a finite number of the $w_\ell c$'s may be non-zero but this will not matter to us – we use coalgebras mainly to study algebras.) Most importantly, every dgc is an A_∞-coalgebra by setting w_1 to be the differential, w_2 to be the comultiplication and $w_\ell = 0$ for $\ell > 2$.

For an A_∞-algebra A, we denote the tensor coalgebra $T^c A[1]$ with the given differential d by BA, the bar complex of A. For an A_∞-coalgebra C, we denote the tensor algebra $TC[-1]$ with the given differential d by ΩC, the cobar complex of C. Thus, we have two functors

$$\Omega\colon \mathsf{DGC}_{\mathsf{nil}} \rightleftarrows \mathsf{DGA} : B$$

We will now show that these are in fact inverse equivalences up to a quasi-isomorphism (assuming that the underlying ring is a field), thus showing that the concept of a dga and a dgc is essentially the same. We will then refine this to the following: given an *A_∞-algebra* A, the dga ΩBA is equivalent to A and thus every A_∞-algebra "rectifies" to a dga.

6. Relation between algebras and coalgebras

We start by observing that Ω is left adjoint to B. To do that, we describe $\mathsf{DGA}(\Omega C, A)$ in a more elementary fashion. Since ΩC is a free graded algebra on $C[-1]$, such a homomorphism is uniquely specified by a graded map $\varphi_1\colon C[-1] \to A$, i.e. a map $t\colon C \to A$ of degree -1. Here we decide to use $\varphi_1 = s^* t$ so that the resulting algebra map $\varphi\colon \Omega C \to A$ satisfies $\varphi s^{-1} c = \varphi_1 s^{-1} c = -tc$. We will now rephrase the condition on φ to be a chain map. This amounts to the commutativity of

$$\begin{array}{ccc} \Omega C & \xrightarrow{\varphi} & A \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \partial} \\ \Omega C & \xrightarrow[\varphi]{} & A \end{array}$$

and it is easily verified that this is sufficient to be checked on the generating $C[-1] \subseteq \Omega C$. Thus, for $s^{-1}c$, we need

$$\partial t = -\partial \varphi s^{-1} = -\varphi d s^{-1} = -\varphi(-s^{-1}\partial - (s \otimes s)^{-1}\Delta) = \varphi s^{-1}\partial - \varphi(s^{-1} \otimes s^{-1})\Delta$$

and since the first term is of length 1 while the second of length 2, this is computed as

$$-t\partial - m(\varphi \otimes \varphi)(s^{-1} \otimes s^{-1})\Delta = -t\partial - m(t \otimes t)\Delta.$$

The chain condition is then

$$[\partial, t] + m(t \otimes t)\Delta = 0.$$

Diagramatically, the differential of t must equal minus the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{t \otimes t} A \otimes A \xrightarrow{m} A.$$

Such a map t is usually called a *twisting cochain* (denoting $t \cup t = m(t \otimes t)\Delta$, the equation becomes $[\partial, t] + t \cup t = 0$ and is called the Maurer-Cartan equation, due to its similarity with the famous equation in differential geometry – is there more?) and has various applications.

Since this condition is self-dual, it should be rather clear[3] that this should also amount to a dgc-map $C \to BA$, although for this to work correctly, we need the nilpotency condition on C: for a map $C \to A[1]$ to uniquely induce an algebra map $C \to BA$, we need that for each element $c \in C$, sufficiently high iteration $\Delta^{(n)}c$, $n \gg 0$, vanishes. This is clearly the case for a tensor coalgebra and thus, B really takes values in $\mathsf{DGC}_{\mathsf{nil}}$.

Theorem 3. *Let A be a dga, whose underlying graded module is free, i.e. each A_k is free. Then the counit $\varepsilon\colon \Omega BA \to A$ is a quasi-isomorphism.*

The counit is the dga-map corresponding to the dgc-map $\mathrm{id}\colon BA \to BA$ or alternatively to the twisting cochain $BA \to A$,

$$sx \mapsto x,\ sx_1 \otimes \cdots \otimes sx_n \mapsto 0 \text{ for } n \geq 2$$

We will denote $(x_1|\cdots|x_n) = s^{-1}(sx_1 \otimes \cdots \otimes sx_n)$. Thus, the induced dga-map ε is given on generators by

$$\varepsilon x = x,\ \varepsilon(x_1 \mid \cdots \mid x_n) = 0 \text{ for } n \geq 2$$

Abbreviating $\varepsilon_i = (-1)^{i-1+|x_1|+\cdots+|x_{i-1}|}$, the differential on ΩBA is given by

$$\begin{aligned} d(x_1 \mid \cdots \mid x_n) = & \sum \varepsilon_i \cdot (x_1 \mid \cdots \mid dx_i \mid \cdots \mid x_n) \\ & - \sum \varepsilon_i \cdot (x_1 \mid \cdots \mid x_{i-1}x_i \mid \cdots \mid x_n) \\ & + \sum \varepsilon_i \cdot (x_1 \mid \cdots \mid x_{i-1})(x_i \mid \cdots \mid x_n) \end{aligned}$$

Proof. Using the basis of each A_k, it is simple to verify that ΩBA as an algebra has a basis formed by $(x_1|\cdots|x_n)$ with each x_i a basis element and thus, as a graded module, the basis is formed by products of such. We will restrict to such elements. We consider the following

[3] In detail, t induces a coalgebra map $\psi\colon C \to BA$ with $q\psi = st$, i.e. $\psi = \sum (st)^{\otimes n}\Delta^{(n)}$. Then $st\partial = q\psi\partial = qd\psi = \sum qd(st)^{\otimes n}\Delta^{(n)} = d_1 st - d_2(s \otimes s)(t \otimes t)\Delta = -s\partial t - sm(t \otimes t)\Delta$.

filtration of ΩBA: let F_r be the subcomplex formed by elements of length $\le r$, where the length of $(x_1|\cdots|x_n)$ is n and the length of a product is the sum of the lengths.

We will now show that each F_r/F_{r-1} is contractible. The differential on this quotient is given by

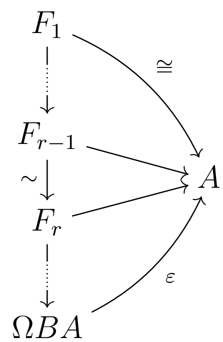
$$d(x_1|\cdots|x_n) = \sum \varepsilon_i \cdot (x_1|\cdots|dx_i|\cdots|x_n) + \sum \varepsilon_i \cdot (x_1|\cdots|x_{i-1})(x_i|\cdots|x_n)$$

The contraction is given by

$$h((x)(x_1|\cdots|x_n)\xi) = (-1)^{|x|+1} \cdot (x|x_1|\cdots|x_n)\xi$$

on products that start with an element of length 1 and by zero on other products.

The long exact sequence of homology groups associated with $0 \to F_{r-1} \to F_r \to F_r/F_{r-1} \to 0$ shows that the inclusion $F_{r-1} \to F_r$ is a quasi-isomorphism. We have



Since $F_1 \to A$ is clearly an isomorphism, each $F_d \to A$ must be a quasi-isomorphism. As also $H_*(\Omega BA) = \operatorname{colim} H_*(F_d)$, the map induced by ε in homology is a colimit of isomorphisms and thus an isomorphism. We remark that in fact, ε is a homotopy equivalence. □

7. Relation of dga's and A_∞-algebras

We will now describe a strictification of A_∞-algebras. We say that an A_∞-algebra A is *cellular* if, as a graded algebra, it is free with respect to the operations m_ℓ, $\ell \ge 2$. In this case, for a generator x, the differential dx must be expressible uniquely as a term $t(y_1 \otimes \cdots \otimes y_\ell)$ in generators y_i of lower degree. In this way, one may obtain A together with its differential by successively adding generators, ordered by their degree, subject to the relations $dx = t(y_1 \otimes \cdots \otimes y_\ell)$ as above. These are to be thought of as cells, attached by their boundaries.

In the case of a cellular A_∞-algebra A, the strictification is simply the truncation $\operatorname{tr} A$. The truncation map $p\colon A \to \operatorname{tr} A$ is a quasi-isomorphism since its kernel admits a contracting vector field given by

$$t_0 \cdot m_\ell(t_1 \otimes \cdots \otimes t_\ell) \cdot x_1 \cdots x_k \Rightarrow m_{\ell+1}(t_0, t_1, \ldots, t_\ell) \cdot x_1 \cdots x_k$$

where the dot denotes the product m_2 with the usual bracketing convention $x{\cdot}y{\cdot}z = (x{\cdot}y){\cdot}z$. Here the term $m_\ell(t_1 \otimes \cdots \otimes t_\ell)$ is the right-most term that is *not* a variable and t_0 is the term in front of it – it may well be a product again, but this is not important. (There is a filtration given by $\deg(m_\ell(t_1 \otimes \cdots \otimes t_\ell) \cdot x_1 \cdots x_k) = (k, \ell, |t_1|)$. This gets increased by the vector field.) The only critical cells are the full products $x_1 \cdots x_k$ that form exactly the truncation $\operatorname{tr} A$.

To strictify a fairly general A_∞-algebra, we will first replace it by a quasi-isomorphic cellular A_∞-algebra. The replacement will somewhat stronger than a quasi-isomorphism.

A *strong deformation retractions (sdr)*

$$(f, g, h)\colon C \Rightarrow D$$

consists of the following data: two chain complexes C, D, two chain maps $f\colon C \to D$ and $g\colon D \to C$ together with a homotopy $h\colon C \to C$ of degree 1 satisfying

$$1 - fg = 0,\ 1 - gf = [d, h],\ fh = 0,\ hg = 0,\ hh = 0.$$

We will describe a cellular replacement of an A_∞algebra A only in the case that the underlying graded module is free (this happens always when k is a field). The replacement is again a sort of bar construction.

We will construct it as a universal strong deformation retraction $(f, g, h)\colon QA \Rightarrow A$ onto A in which f is a map of A_∞-algebras while g and h are only $\Bbbk$-linear. Thus, we

- start with a generating set $\{gx \mid x \in A \text{ a generator}\}$; here $|gx| = |x|$,
- close freely under operations m_ℓ, $\ell \geq 2$, and h
- subject to the relations $hg = 0$ and $hh = 0$.

Next we specify the projection f by

- $fg = 1$,
- $fm_\ell = m_\ell f^{\otimes \ell}$, i.e. f is a map of A_∞-algebras,
- $fh = 0$.

The differential on QA is then given by

- $dg = gd$, i.e. g is a chain map,
- $d(m_\ell(t_1 \otimes \cdots \otimes t_\ell)) = [d, m_\ell](t_1 \otimes \cdots \otimes t_\ell) + \sum_{i=1}^{\ell} (-1)^{\ell + |t_1| + \cdots + |t_{i-1}|} m_\ell(t_1 \otimes \cdots \otimes dt_i \otimes \cdots \otimes t_\ell)$,
- $dh = 1 - gf - hd$, i.e. $[d, h] = 1 - gf$.

It is simple to verify that

- $dhg = -hgd = 0$, $dhh = hhd = 0$ so that d is indeed defined on QA,
- $df = fd$ so that f is a chain map, and consequently
- $dd = 0$ so that d is indeed a differential.

By the second defining formula for d, it makes QA into an A_∞-algebra. By construction, (f, g, h) is indeed a sdr $QA \Rightarrow A$ whose projection f is a map of A_∞-algebras. In addition, QA is cellular with generators formed by the gx and the ht. Put together, it is a *cellular replacement*. Finally, we have a span of quasi-isomorphisms of A_∞-algebras $A \leftarrow QA \to \operatorname{tr} QA$, where $\operatorname{tr} QA$ is in fact a dga.

Remark. We remark that the above construction can also be made for $\Bbbk G$-modules where only f is required to be $\Bbbk G$-linear while g and h are only $\Bbbk$-linear. In this case, one obtains the usual bar construction. Denoting the elements of G by a, a generator of QM looks

$$a_0 h a_1 h \cdots h a_n g x = a_0 | a_1 | \cdots | a_n \otimes x.$$

8. Transfer of the structure

In this section, we will describe how to transfer the structure of an A_∞-algebra along homotopy equivalences. It is true, although not completely trivial,[4] that an arbitrary homotopy equivalence can be replaced by a span of sdr's, $C \Leftarrow S \Rightarrow D$. For this reason, it is possible to restrict to sdr's.

Before going into the transfer stuff, we observe that sdr's are closed under direct sums (easy) and also tensor products: if $(f, g, h)\colon C_i \Rightarrow D_i$ is a finite number of sdr's then so is

$$(f^{\otimes k}, g^{\otimes k}, \sum_{i+1+j=k} (gf)^{\otimes i} \otimes h \otimes 1^{\otimes j})\colon C_1 \otimes \cdots \otimes C_k \Rightarrow D_1 \otimes \cdots \otimes D_k.$$

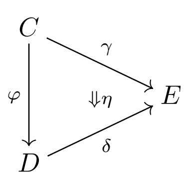
It is worth pointing out that this tensor product of sdr's is associative and thus it does not matter in which order we tensor sdr's.

The advantage of sdr's over general homotopy equivalences is the following theorem.

Theorem 4 (Basic Perturbation Lemma)**.** *Let there be given a sdr $(f_0, g_0, h_0)\colon C_0 \Rightarrow D_0$ and a perturbation δ, i.e. a map of degree -1 such that $d = d_0 + \delta$ is also a differential. Suppose that $h_0\delta$ (equivalently δh_0) is pointwise nilpotent. Then there exists a perturbation of the differential of D_0 and of the maps f_0, g_0, h_0 such that $(f, g, h)\colon C \Rightarrow D$ is another sdr. These perturbations are functorial in a suitable sense.*

Given a sdr $A \Rightarrow A'$, consider both A, A' equipped with trivial A_∞-structures, i.e. $m_\ell = 0$, $\ell \geq 2$, we denote the resulting bar complex $B_0A = \bigoplus_n A[1]^{\otimes n}$. Then there is a sdr $B_0A \Rightarrow B_0A'$ since sdr's are closed under tensor products and direct sums. Now

[4] Let $\varphi\colon C \to D$ be a homotopy equivalence with a homotopy inverse ψ and a homotopy $[d, \eta] = 1 - \psi\varphi$. Let $\mathrm{cyl}(\varphi)$ be the mapping cylinder. It has the following universal property: maps $\mathrm{cyl}(\varphi) \to E$ are in bijection with triples (γ, η, δ) where $\gamma\colon C \to E$ and $\delta\colon D \to E$ are chain maps and $[d, \eta] = \delta\varphi - \gamma$ is a homotopy; in othe words, such maps correspond to homotopy commutative triangles



We denote the components of the universal such triangle by $g\colon C \to \mathrm{cyl}(\varphi)$, $i\colon D \to \mathrm{cyl}(\varphi)$ and κ. There is also a projection $p = (\varphi, 0, 1)\colon \mathrm{cyl}(\varphi) \to D$ that fits into a sdr (p, i, μ) with $\mu = (-\kappa, 0, 0)$. Then $\varphi = pg$. It is easily verified that p is a projection of a sdr with injection $iz = (0, 0, z)$. Thus, g is a homotopy equivalence. Moreover, it admits a retraction $f = (1, -\eta, \psi)$ that is homotopic to $\psi p = (\psi\varphi, 0, \psi)$ via the homotopy $\lambda = (-\eta, 0, 0)$. The retraction condition means $fg = 1$ and it remains to ensure the side conditions $fh = 0$, $hg = 0$, $hh = 0$.

Let h denote the concatenation of homotopies $gf \sim_{\kappa f + i\varphi\lambda} (i\varphi)(\psi p) \sim_{i\theta p} ip \sim_\mu 1$, where $[d, \theta] = 1 - \varphi\psi$. The resulting homotopy is $h(x, y, z) = (0, -\eta y + \psi z, (\theta\varphi - \varphi\eta)x + \theta z)$. The map $1 - gf$ is the projection onto the summand C' of $\mathrm{cyl}(\varphi)$ complementary to C. Restricting the homotopy h to C' produces a contraction $h' = (1 - gf)h$ of C'. It is simple to produce a contraction that squares to zero, namely $h'dh'$. Then one reconstructs a homotopy with the required properties by extending to C by the zero homotopy, i.e. the resulting homotopy operator is $h'dh'(1-gf) = (1-gf)hd(1-gf)h(1-gf) = (1-gf)h(1-gf)dh(1-gf)$. In fact, since $hg = 0$ in our case, we may use $(1 - gf)hdh(1 - gf)$.

let A be equipped with an A_∞-structure, i.e. a coderivation on T^cA, and think of it as a perturbation of the differential on B_0A. Then the basic perturbation lemma gives a sdr $BA \Rightarrow BA'$ where BA' is a perturbation of B_0A' and is obtained from the lemma. Thus, the lemma will produce an A_∞-structure on A' once we check that the perturbed differential is indeed a coderivation. This follows from the functoriality of the basic perturabation lemma: consider the commutative square

$$\begin{array}{ccc} B_0A & \xrightarrow{f_0} & B_0A' \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle\Delta} \\ B_0A \otimes B_0A & \xrightarrow[f_0\otimes f_0]{} & B_0A' \otimes B_0A' \end{array}$$

and observe that the two maps Δ in fact form a map of sdr's that even commute with the perturbations $\delta = d - d_0$ at the top and $\delta \otimes 1 + 1 \otimes \delta$ at the bottom – since both A with the trivial A_∞-structure and A with the given A_∞-structure are A_∞-algebras, Δ commutes with d_0 and $d = d_0 + \delta$, hence also with δ.

In fact, we obtain more than just an A_∞-algebra structure on A', namely the chain maps $f\colon BA \to BA'$ and $g\colon BA' \to BA$ (and also h). As observed in the above diagram, f is a coalgebra map.

Definition 5. An A_∞*-map* $A \to A'$ is defined to be a dgc-map $BA \to BA'$.

As in the case of coderivations, such a map must be of the form

$$f(sx_1 \otimes \cdots \otimes sx_n) = \sum_{\ell_1+\cdots+\ell_k=n} f_{\ell_1}(sx_1 \otimes \cdots \otimes sx_{\ell_1}) \otimes \cdots \otimes f_{\ell_k}(sx_{n-\ell_k+1} \otimes \cdots \otimes sx_n)$$

where $f_\ell\colon (sA)^{\otimes \ell} \to sA'$. For f to be a chain map, it must satisfy the following:

$$\sum_{\ell_1+\cdots+\ell_k=n} d_k(f_{\ell_1} \otimes \cdots \otimes f_{\ell_k}) = \sum_{i+\ell+j=n} f_{i+1+j}(1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes j})$$

(in particular, f_2 measures the extent to which f_1 does not respect the multiplication, $[d_1, f_2] = f_1 d_2 - d_2(f_1 \otimes f_1)$, etc.) Thus, both f and g are A_∞-maps.

Remark: another point of view is that $f\colon BA \to BA'$ is a "twisting cochain" $BA \to A'$ (however, A' is an A_∞-algebra and thus a twisting cochain is not equivalent to $\Omega_\infty BA \to A'$ since the left-hand side does not exist) – this has components $(sA)^{\otimes \ell} \to A'$ of degree -1 and these are precisely the f_ℓ's.

The basic perturbation lemma in fact provides formulas for the respective perturbations. The differential on BA' is given by the formula

$$d = d_0 + f_0(\delta - \delta h_0 \delta + \delta h_0 \delta h_0 \delta - \cdots)g_0.$$

Recalling that $\delta = \sum_{\ell \geq 2} 1^{\otimes i} \otimes d_\ell \otimes 1^{\otimes j}$, it is simple to see that the application of h_0 to $\delta(h_0\delta)^r g_0$, $r \geq 0$, produces a non-zero term only when h_0 is applied to the d_ℓ-term, so that one may replace the appearance of each term $h_0\delta$ by $\sum (g_0 f_0)^{\otimes i} \otimes h_0 d_\ell \otimes 1^{\otimes j}$. A succesive application

$$((g_0 f_0)^{\otimes i'} \otimes h_0 d_{\ell'} \otimes 1^{\otimes j'})((g_0 f_0)^{\otimes i} \otimes h_0 d_\ell \otimes 1^{\otimes j})$$

is zero unless $i' \le i$. In this case, one may replace each $(g_0 f_0)^{\otimes i} \otimes h_0 d_\ell \otimes 1^{\otimes j}$ by $1^{\otimes i} \otimes h_0 d_\ell \otimes 1^{\otimes j}$. Putting together, we obtain formulas

$$d_\ell = \sum_{\substack{r\geq 0,\\ i_1\leq\cdots\leq i_r}} (-1)^r \cdot f_0 d_{\ell_0}(1^{\otimes i_1} \otimes h_0 d_{\ell_1} \otimes 1^{\otimes j_1}) \cdots (1^{\otimes i_r} \otimes h_0 d_{\ell_r} \otimes 1^{\otimes j_r}) {g_0}^{\otimes \ell}$$

Since we have

$$1^{\otimes i} \otimes h_0 d_\ell \otimes 1^{\otimes j} = (-1)^{(\ell-1)(j+1)} \cdot s^{\otimes(i+1+j)}(1^{\otimes i} \otimes h_0 m_\ell \otimes 1^{\otimes j})(s^{\otimes(i+\ell+j)})^{-1},$$

we may also rewrite the formula as

$$m_\ell = \sum_{\substack{r\geq 0,\\ i_1\leq\cdots\leq i_r}} (-1)^{r+(\ell_1-1)j_1+\cdots+(\ell_r-1)j_r} \cdot f_0 m_{\ell_0}(1^{\otimes i_1} \otimes h_0 m_{\ell_1} \otimes 1^{\otimes j_1}) \cdots (1^{\otimes i_r} \otimes h_0 m_{\ell_r} \otimes 1^{\otimes j_r}) {g_0}^{\otimes \ell}.$$

The iterations as above may be depicted in terms of trees, whose vertices are decorated by the m_ℓ's, whose inner edges are decorated by h_0, whose incoming leaves are decorated by g_0 and whose outgoing leaf is decorated f_0.

There are similar formulas for f, g and h.

References